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Restricted b-matchings in degree-bounded graphs

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Abstract

We present a min-max formula and a polynomial time algorithm for a slight generalization of the following problem: in a simple undirected graph in which the degree of each node is at most $t+1$, find a maximum t-matching containing no member of a list $\mathcal{K}$ of forbidden $K_{t,t}$ and K_{t+1} subgraphs. An analogous problem for bipartite graphs without degree bounds was solved by Makai [15], while the special case of finding a maximum square-free 2-matching in a subcubic graph was solved in [1].

1 Introduction

Let $G = (V, E)$ be an undirected graph and let $b : V \to \mathbb{Z}_+$ be an upper bound on the nodes. An edge set $F \subseteq E$ is called a *b-matching* if $d_F(v)$, the number of edges in F incident to v is at most $b(v)$ for each node v. (This is often called *simple* b-matching in the literature.) For some integer $t \geq 2$, by a *t-matching* we mean a b-matching with $b(v) = t$ for every $v \in V$. Let $\mathcal{K}$ be a set consisting of $K_{t,t}$'s, complete bipartite subgraphs of G on two colour classes of size t, and K_{t+1}'s, complete subgraphs of G on $t+1$ nodes. By *$\mathcal{K}$-free b-matching* we mean a b-matching not containing any member of $\mathcal{K}$. In this paper, we give a min-max formula on the size of $\mathcal{K}$-free b-matchings and a polynomial time algorithm for finding one with maximum size (that is, a $\mathcal{K}$-free b-matching $F \subseteq E$ with maximum cardinality) under the assumptions that for any $K \in \mathcal{K}$ and any node v of K,

$$V_K \text{ spans no parallel edges} \tag{1}$$

$$b(v) = t \tag{2}$$

$$d_G(v) \leq t + 1. \tag{3}$$

Note that this is a generalization of the problem mentioned in the abstract. The most important special case of $\mathcal{K}$-free b-matching is to find a maximum C_3-free or C_4-free 2-matching in a graph. The motivation for these problems is twofold. On the

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one hand, a natural relaxation of the Hamiltonian cycle problem is to find a $C_{\leq k}$-free 2-factor, that is, a 2-factor containing no cycle of length at most k. Cornuéjols and Pulleyblank [2] showed this problem to be NP-complete for $k \geq 5$. In his Ph.D. thesis [6], Hartvigsen proposed a solution for the case $k = 3$. Hence the remaining question is to find a maximum $C_{\leq 4}$-free 2-matching, and another natural question is to find a maximum C_4-free 2-matching (possibly containing triangles).

The other motivation comes from connectivity-augmentation, that is, when one would like to make a graph $G = (V, E)$ k-node-connected by the addition of a minimum number of new edges. It is easy to see that for $k = n - 2$ ($n = |V|$), this problem is equivalent to finding a maximum matching in the complement graph of G. For $k = n - 3$ the problem is equivalent to finding a maximum C_4-free 2-matching. C_4-free 2-matching admits two natural generalizations. The first one is $K_{t,t}$-free t-matchings considered in this paper, while the second is t-matchings containing no complete bipartite graph $K_{a,b}$ with $a + b = t + 2$. This latter problem is equivalent to connectivity augmentation for $k = n - t - 1$. The complexity of connectivity augmentation for general k is yet open, while connectivity augmentation by one, that is, when the input graph is already $(k - 1)$-connected was recently solved in [20] (this corresponds to the case when the graph contains no $K_{a,b}$ with $a + b = t + 3$, in particular, $d(v) \leq t + 1$).

The weighted version of these problems are also of interest. The weighted $C_{\leq k}$-free 2-matching problem asks for a $C_{\leq k}$-free 2-matching with maximum weight in point of a weight function defined on the edge set. For $k = 2$ the problem is just to find a 2-matching with maximum weight, while Király showed [11] that the problem is NP-complete for $k = 4$ even in bipartite graphs with $0 - 1$ weights on the edges. However, if the weight function is "node induced" on every square then the problem becomes polynomially solvable in bipartite graphs ([15, 19]). The case of $k = 3$ in general graphs is still open. Hartvigsen and Li [9], and recently Kobayashi [12] gave polynomial-time algorithm for the weighted C_3-free 2-matching problem in subcubic graphs with for arbitrary weight functions.

Let us now consider the special case of C_4-free 2-matchings in bipartite graphs. This problem was solved by Hartvigsen [7, 8] and Király [10]. A generalization of the problem to maximum $K_{t,t}$-free t-matchings was given by Frank [3] who observed that this is a special case of covering positively crossing supermodular functions on set pairs solved by him and Jordán in [4]. Makai [15] generalized Frank's theorem for the case when a list $\mathcal{K}$ of forbidden $K_{t,t}$'s is given (that is, a t-matching may contain $K_{t,t}$'s not in $\mathcal{K}$.) He gave a min-max formula based on a polyhedral description for the minimum cost version for node-induced cost functions. Pap [16] gave a further generalization of the maximum cardinality version for excluded complete bipartite subgraphs and developed a simple, purely combinatorial algorithm. For node induced cost functions, such an algorithm was given by Takazawa [19] for $K_{t,t}$-free t-matching.

Much less is known when the underlying graph is not assumed to be bipartite and finding a maximum C_4-free 2-matching is still open. The special case when the graph is subcubic was solved by the first author and Kobayashi [1]. In terms of connectivity augmentation, the equivalent problem is augmenting an $(n - 4)$-connected graph to $(n - 3)$ connected. Our theorem is a generalization of this result.

It is worth mentioning that the polynomial solvability of the above problems seems to show a strong connection with jump systems. In [18], Szabó proved that for a list $\mathcal{K}$ of forbidden $K_{t,t}$ and K_{t+1} subgraphs the degree sequences of $\mathcal{K}$-free t-matchings form a jump system in any graph. Concerning bipartite graphs, Kobayashi and Takazawa showed [14] that the degree sequences of $C_{\leq k}$-free 2-matchings do not always form a jump system for $k \geq 6$. These results are consistent with the polynomial solvability of the $C_{\leq k}$-free 2-matching problem, even when restricting it to bipartite graphs. Similar results are known about even factors due to [13]. Hence Szabó's result suggests that a maximum $\mathcal{K}$-free t-matching should be solvable in polynomial time.

Among our assumptions, (1) and (2) may be considered as natural ones as they hold for the maximum $K_{t,t}$-free t-matching problem in a simple graph. We exclude parallel edges on the node sets of members of $\mathcal{K}$ in order to avoid having two different $K_{t,t}$'s on the same two colour classes or two K_{t+1}'s on the same ground set. However, the degree bound (3) is a restrictive assumption and dissipates essential difficulties. Our proof strongly relies on this and the theorem cannot be straightforwardly generalized for the general case, as it can be shown by using the example in Chapter 6 of [20].

The proof and algorithm use the contraction technique of [11], [16] and [1]. Our contribution on the one hand is the extension of this technique for $t \geq 3$ and forbidding K_{t+1}'s as well, while on the other hand we give a significantly simpler argument as in [1].

Throughout the paper we use the following notation. For an undirected graph $G = (V, E)$, the set of edges induced by $X \subseteq V$ is denoted by $E[X]$. For disjoint subsets X, Y of V, $E[X, Y]$ denotes the set of edges between X and Y. The set of nodes in $V - X$ adjacent to X by some edge from $F \subseteq E$ is denoted by $\Gamma_F(X)$. The degree of a node v regarding to F is denoted by $d_F(v)$ in which loops are counted twice, while $d_F(X, Y)$ stands for the number of edges going between disjoint subsets X and Y. For a node $v \in V$, we sometimes abbreviate the set $\{v\}$ by v, e.g. $d_F(v, X)$ is the number of edges between v and X. For a set $X \subseteq V$, let $h_F(X) = \sum_{v \in X} d_F(v)$, the sum of the number of edges incident to X and twice the number of edges spanned by X. We use $b(U) = \sum_{v \in U} b(v)$ for a function $b : V \to \mathbb{Z}_+$ and a set $X \subseteq V$.

Let $\mathcal{K}$ be the list of forbidden $K_{t,t}$ and K_{t+1} subgraphs. For disjoint subsets X, Y of V we denote by $\mathcal{K}[X]$ and $\mathcal{K}[X, Y]$ the members of $\mathcal{K}$ contained in X and having edges only between X and Y, respectively. That is, $\mathcal{K}[X, Y]$ stands for forbidden $K_{t,t}$'s whose colour classes are subsets of X and Y. V_K and E_K denotes the node-set and edge-set of the forbidden graph $K \in \mathcal{K}$, respectively.

The rest of the paper is organized as follows. In Section 2 we formalize the theorem and prove the trivial max $\leq$ min direction. Two shrinking operations are introduced in Section 3, and Section 4 contains the proofs of the max $\geq$ min direction. Finally, the algorithm is presented in Section 5.

2 Main theorem

Before stating our theorem, let us recall the well-known min-max formula on the maximum size of a b-matching (see e.g. [17, Vol A, p. 562.]).

Theorem 2.1 (Maximum size of a b-matching)**.** *Let $G = (V, E)$ be a graph with an upper bound $b : V \to \mathbb{Z}_+$. The maximum size of a b-matching is equal to the minimum value of*

$$b(U) + |E[W]| + \sum_T \left\lfloor \tfrac{1}{2}(b(T) + |E[T, W]|) \right\rfloor \tag{4}$$

where U and W are disjoint subsets of V, and T ranges over the connected components of $G - U - W$.

Let us now formulate our theorem. There are minor technical difficulties in case of $t = 2$ that do not occur for larger t. In order to make the formulation and the proof simpler it is worth introducing the following definitions. We refer to forbidden $K_{2,2}$ and K_3 subgraphs as squares and triangles, respectively.

Definition 2.2. For $t = 2$, we call a complete subgraph on four nodes *square-full* if it contains three forbidden squares.

Note that, by assumption (3), every square-full subgraph is a connected component of G. We denote the number of square-full components of G by $\mathcal{S}(G)$ for $t = 2$, while $\mathcal{S}(G) = 0$ for $t > 2$. It is easy to see that a $\mathcal{K}$-free b-matching contains at most three edges from each square-full component of G. The following definition will be used in the proof of the theorem.

Definition 2.3. For $t = 2$, a forbidden triangle is called *square-covered* if its node set is contained in the node set of a forbidden square, otherwise *uncovered*.

The theorem is as follows.

Theorem 2.4. *Let $G = (V, E)$ be a graph with an upper bound $b : V \to \mathbb{Z}_+$ and $\mathcal{K}$ be a list of forbidden $K_{t,t}$ and K_{t+1} subgraphs of G so that (1), (2) and (3) hold. Then the maximum size of a $\mathcal{K}$-free b-matching is equal to the minimum value of*

$$\tau(U, W, \mathcal{P}, \dot{\mathcal{K}}) = b(U) + |E[W]| - |\dot{\mathcal{K}}[W]| + \sum_{T \in \mathcal{P}} \left\lfloor \tfrac{1}{2}(b(T) + |E[T, W]| - |\dot{\mathcal{K}}[T, W]|) \right\rfloor - \mathcal{S}(G) \tag{5}$$

where U and W are disjoint subsets of V, $\mathcal{P}$ is a partition of the connected components of $G - U - W$ and $\dot{\mathcal{K}} \subseteq \mathcal{K}$ is a collection of node-disjoint forbidden subgraphs.

It is easy to see that the contribution of a square-full component to (5) is always 3 and a maximum $\mathcal{K}$-free b-matching contains exactly 3 of its edges. Hence we may count these components of G separately, so the following theorem immediately implies the general one.

Theorem 2.5. *Let $G = (V, E)$ be a graph with an upper bound $b : V \to \mathbb{Z}_+$ and $\mathcal{K}$ be a list of forbidden $K_{t,t}$ and K_{t+1} subgraphs of G so that (1), (2) and (3) hold. Furthermore, if $t = 2$, assume that G has no square-full component. Then the maximum size of a $\mathcal{K}$-free b-matching is equal to the minimum value of*

$$\tau(U, W, \mathcal{P}, \dot{\mathcal{K}}) = b(U) + |E[W]| - |\dot{\mathcal{K}}[W]| + \sum_{T \in \mathcal{P}} \left\lfloor \tfrac{1}{2}(b(T) + |E[T, W]| - |\dot{\mathcal{K}}[T, W]|) \right\rfloor \quad (6)$$

where U and W are disjoint subsets of V, $\mathcal{P}$ is a partition of the connected components of $G - U - W$ and $\dot{\mathcal{K}} \subseteq \mathcal{K}$ is a collection of node-disjoint forbidden subgraphs.

Proof of max $\leq$ min in Theorem 2.5. Let M be a $\mathcal{K}$-free b-matching. Then clearly $|M \cap (E[U] \cup E[U, V - U])| \leq b(U)$ and $|M \cap E[W]| \leq |E[W]| - |\dot{\mathcal{K}}[W]|$. Moreover, for each $T \in \mathcal{P}$ we have

$$\begin{aligned} 2 \cdot |M \cap (E[T] \cup E[T, W])| &= 2 \cdot |M \cap E[T]| + 2 \cdot |M \cap E[T, W]| \\ &\leq 2 \cdot |M \cap E[T]| + |M \cap E[T, W]| + |E[T, W]| - |\dot{\mathcal{K}}[T, W]| \\ &\leq b(T) + |E[T, W]| - |\dot{\mathcal{K}}[T, W]|. \end{aligned}$$

These together prove the inequality. □

3 Shrinking

In the proof of max $\geq$ min we use two shrinking operations to get rid of the $K_{t,t}$ and K_{t+1} subgraphs in $\mathcal{K}$.

Definition 3.1 (Shrinking a $K_{t,t}$ subgraph)**.** Let K be a $K_{t,t}$ subgraph of $G = (V, E)$ with colour classes K_A and K_B. *Shrinking* of K in G consists of the following operations:

- identify the nodes in K_A, and denote the corresponding node by k_a,
- identify the nodes in K_B, and denote the corresponding node by k_b, and
- replace the edges between K_A and K_B with $t - 1$ parallel edges between k_a and k_b (we call the set of these edges a *shrunk bundle between k_a and k_b*).

When identifying the nodes in K_A and K_B, the edges (and also loops) spanned by K_A and K_B are replaced by loops on k_a and k_b, respectively. Each edge $e \in E - E_K$ is denoted by e again after shrinking a $K_{t,t}$ subgraph and is called the *image* of the original edge. By abuse of notation, for an edge set $F \subseteq E - E_K$, the corresponding subset of edges in the contracted graph is also denoted by F. Hence for an edge set $F \subseteq E - E_K$ we have $h_F(K_A) = d_F(k_a)$, $h_F(K_B) = d_F(k_b)$.

Definition 3.2 (Shrinking a K_{t+1} subgraph)**.** Let K be a K_{t+1} subgraph of $G = (V, E)$. *Shrinking* of K in G consists of the following operations:

- identify the nodes in V_K, and denote the corresponding node by k,

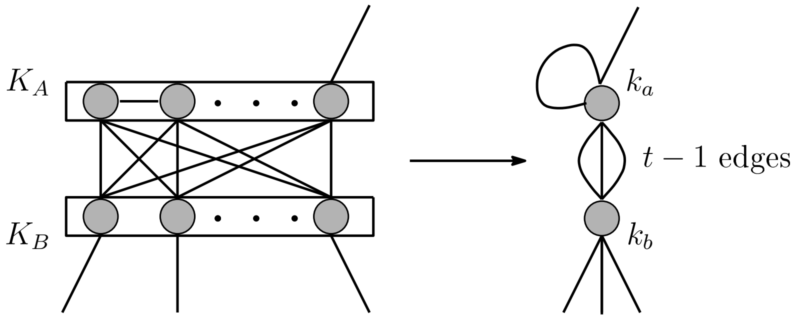


Figure 1: Shrinking a $K_{t,t}$ subgraph

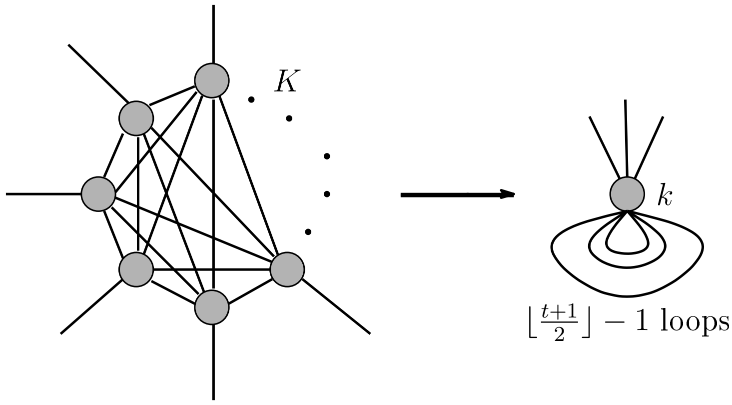


Figure 2: Shrinking a K_{t+1} subgraph

- replace the edges in E_K by $\left\lfloor \frac{t+1}{2} \right\rfloor - 1$ loops on the new node k.

Again, for an edge set $F \subseteq E - E_K$, the corresponding subset of edges in the contracted graph is also denoted by F.

We usually denote the graph obtained by applying one of the shrinking operations by $G^\circ = (V^\circ, E^\circ)$. Throughout the section, the graph G, the function b and the list $\mathcal{K}$ of forbidden subgraphs are supposed to satisfy the conditions of Theorem 2.5. It is easy to see, by using (3), that two members of $\mathcal{K}$ are edge-disjoint if and only if they are also node-disjoint, hence we simply call such pairs *disjoint*.

The following two lemmas give the connection between the maximum size of a $\mathcal{K}$-free b-matching in G and a b°-matching in G° where b° is a properly defined upper bound on V°.

Lemma 3.3. *Let $G^\circ = (V^\circ, E^\circ)$ be the graph obtained by shrinking a $K_{t,t}$ subgraph K. Let $\mathcal{K}^\circ$ be the set of forbidden subgraphs disjoint from K and define b° as $b^\circ(v) = b(v)$ for $v \in V - V_K$ and $b^\circ(k_a) = b^\circ(k_b) = t$. Then the difference between the maximum size of a $\mathcal{K}$-free b-matching in G and the maximum size of a $\mathcal{K}^\circ$-free b°-matching in G° is exactly $t^2 - t$.*

Lemma 3.4. *Let $G^\circ = (V^\circ, E^\circ)$ be the graph obtained by shrinking a K_{t+1} subgraph $K \in \mathcal{K}$ where K is uncovered if $t = 2$. Let $\mathcal{K}^\circ$ be the set of forbidden subgraphs disjoint from K and define b° as $b^\circ(v) = b(v)$ for $v \in V - V_K$, $b^\circ(k) = t$ if t is even and $b^\circ(k) = t+1$ if t is odd. Then the difference between the maximum size of a $\mathcal{K}$-free b-matching in G and the maximum size of a $\mathcal{K}^\circ$-free b°-matching in G° is exactly $\left\lfloor \frac{t^2}{2} \right\rfloor$.*

The proof of Lemma 3.3 is based on the following claim.

Claim 3.5. *Assume that $K \in \mathcal{K}$ is a $K_{t,t}$ subgraph with colour classes K_A and K_B and M' is a $\mathcal{K}$-free b-matching of $G - E_K$. Then M' can be extended to a $\mathcal{K}$-free b-matching M of G with $|M| = |M'| + t^2 - \max\{1, h_{M'}(K_A), h_{M'}(K_B)\}$.*

Proof. First we consider the case $t \geq 3$. Let P be a minimum size matching of K covering each node $v \in V_K$ with $d_{M'}(v) = 1$ (note that $d_{M'}(v) \leq 1$ for $v \in V_K$ as $d(v) \leq t+1$). If there is no such node, then let P consist of an arbitrary edge in E_K. We claim that $M = M' \cup (E_K - P)$ satisfies the above conditions. Indeed, M is a b-matching and $|M \cap E_K| = t^2 - \max\{1, h_{M'}(K_A), h_{M'}(K_B)\}$ clearly holds, so we only have to verify that it is also $\mathcal{K}$-free.

Assume that there is a forbidden $K_{t,t}$ subgraph K' in M with colour classes K'_A, K'_B. $E_{K'}$ must contain an edge $uv \in E_K \cap M$ with $u \in K'_A$ and $v \in K'_B$. By symmetry, we may assume that $u \in K_A$. As $b(u) = t$, $\Gamma_M(u) = K'_B$ and also $|\Gamma_M(u) \cap K_B| \geq t-1$. Hence $|K'_B \cap K_B| \geq t-1$. Consider a node $z \in K_A$. Since $d_M(z, K_B) \geq t-1$ and $t \geq 3$, we get $d_M(z, K'_B) > 0$, thus $K_A \subseteq \Gamma_M(K'_B)$. Because of $\Gamma_M(K'_B) = K'_A$, this gives $K_A = K'_A$. $K_B = K'_B$ follows similarly, giving a contradiction.

If there is a forbidden K_{t+1} subgraph K' in M, then $E_{K'}$ must contain an edge $uv \in E_K \cap M$, $u \in K_A$. As above, $|V_{K'} \cap K_B| \geq t-1$. Using $t \geq 3$ again, $K_A \subseteq \Gamma_M(V_{K'} \cap K_B) \subseteq V_{K'}$. But $K_A \subseteq V_{K'}$ is a contradiction since $t+1 = |V_{K'}| \geq |V_{K'} \cap K_A| + |V_{K'} \cap K_B| \geq t + t - 1 = 2t-1 > t+1$.

Now let $t = 2$ and let $K_A = \{v_1, v_3\}$, $K_B = \{v_2, v_4\}$. If $\max\{h_{M'}(K_A), h_{M'}(K_B)\} \leq 1$, then we may assume by symmetry that $d_{M'}(v_1) = d_{M'}(v_2) = 0$. Now $M = M' \cup \{v_1v_2, v_1v_4, v_2v_3\}$ is clearly a $\mathcal{K}$-free 2-matching. If $\max\{h_{M'}(K_A), h_{M'}(K_B)\} = 2$, we claim that at least one of $M_1 = M' \cup \{v_1v_2, v_3v_4\}$ and $M_2 = M' \cup \{v_1v_4, v_2v_3\}$ is $\mathcal{K}$-free. Assume M_1 contains a forbidden square or triangle K'; by symmetry assume it contains the edge v_1v_2. If K' contains v_3v_4 as well, then K' is the square $v_1v_3v_4v_2$. Otherwise, it consists of v_1v_2 and a path L of length 2 or 3 between v_1 and v_2, not containing v_3 and v_4. In the first case, the only forbidden subgraph possibly contained in M_2 is the square $v_1v_3v_2v_4$, implying that $\{v_1, v_2, v_3, v_4\}$ is a square-full component, a contradiction. In the latter case, it is easy to see that M_2 cannot contain a forbidden subgraph. □

Proof of Lemma 3.3. First we show that if M is a $\mathcal{K}$-free b-matching in G then there is a $\mathcal{K}^\circ$-free b°-matching M° in G° with $|M^\circ| \geq |M| - (t^2 - t)$. Let $M' = M - E_K$. Clearly, $|M \cap E_K| \leq t^2 - \max\{1, h_{M'}(K_A), h_{M'}(K_B)\}$. In G°, let M° be the union of M' and $t - \max\{1, d_{M'}(k_a), d_{M'}(k_b)\}$ parallel edges from the shrunk bundle between k_a and k_b. Is is easy to see that M° is a $\mathcal{K}^\circ$-free b°-matching in G° with $|M^\circ| \geq |M| - (t^2 - t)$.

The proof is completed by showing that for an arbitrary $\mathcal{K}^\circ$-free b°-matching M° in G° there exists a $\mathcal{K}$-free b-matching M in G with $|M| \geq |M^\circ| + (t^2 - t)$. Let H denote the set of parallel edges in the shrunk bundle between k_a and k_b, and let $M' = M^\circ - H$. Now $|M^\circ \cap H| \leq t - \max\{1, d_{M'}(k_a), d_{M'}(k_b)\}$ and, by Claim 3.5, M' may be extended to a $\mathcal{K}$-free b-matching in G with $|M \cap E_K| = t^2 - \max\{1, h_{M'}(K_A), h_{M'}(K_B)\}$, that is

$$\begin{aligned} |M| &= |M^\circ| - |M^\circ \cap H| + |M \cap E_K| \\ &\geq |M^\circ| - (t - \max\{1, d_{M'}(k_a), d_{M'}(k_b)\}) + (t^2 - \max\{1, h_{M'}(K_A), h_{M'}(K_B)\}) \\ &\geq |M^\circ| + (t^2 - t). \end{aligned}$$

□

Lemma 3.4 can be proved in a similar way by using the following claim.

Claim 3.6. *Assume that $K \in \mathcal{K}$ is a K_{t+1} subgraph and M' is a $\mathcal{K}$-free b-matching of $G - E_K$. If $t = 2$, then assume that K is uncovered. Then there exists a $\mathcal{K}$-free b-matching M of G with $|M| = |M'| + \binom{t+1}{2} - \left\lceil \frac{\max\{1, h_{M'}(V_K)\}}{2} \right\rceil$.*

Proof. Let P be a minimum size subgraph of K covering each node $v \in V_K$ with $d_{M'}(v) = 1$ (so P is a matching or a matching and one more edge in E_K). If there is no such node, then let P consist of an arbitrary edge in E_K. For $t = 2$ and 3, we will choose P in a specific way, as given later in the proof. We show that $M = M' \cup (E_K - P)$ satisfies the above conditions. Indeed, M is a b-matching and $|M \cap E_K| = \binom{t+1}{2} - \left\lceil \frac{\max\{1, h_{M'}(K)\}}{2} \right\rceil$ clearly holds, so we only have to show that it is also $\mathcal{K}$-free.

Assume that there is a forbidden K_{t+1} subgraph K' in M. $E_{K'}$ must contain an edge $uv \in E_K \cap M$. By the minimal choice of P at least one of $|\Gamma_M(u) \cap V_K| \geq t - 1$ and $|\Gamma_M(v) \cap V_K| \geq t - 1$ is satisfied which implies $|V_{K'} \cap V_K| \geq t - 1$. For $t \geq 3$ this immediately implies $V_K \subseteq \Gamma_M(V_{K'} \cap V_K) \subseteq V_{K'}$, a contradiction.

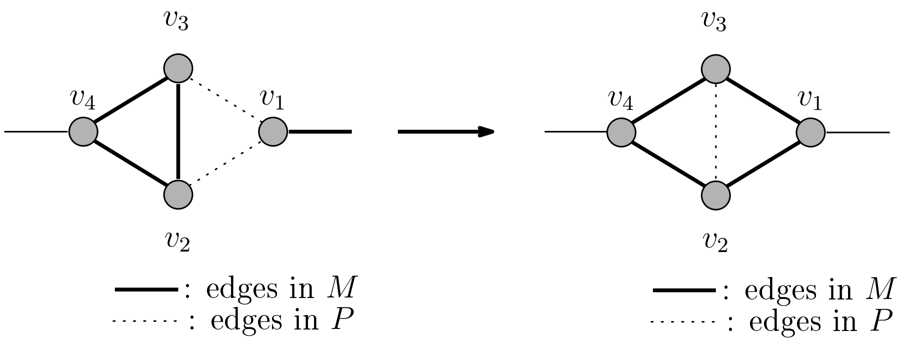


Figure 3: Choice of P for $t = 2$ in the proof of Claim 3.6

If $t = 2$, then $|V_{K'} \cap V_K| \geq 1$ does not imply $V_K \subseteq V_{K'}$ and an improper choice of P may enable M to contain a forbidden K_3. The only such case is when $h_{M'}(V_K) = 3$, $V_K = \{v_1, v_2, v_3\}$, $V_{K'} = \{v_2, v_3, v_4\}$, $v_2v_4, v_3v_4 \in M'$ and $P = \{v_1v_2, v_1v_3\}$ (Figure 3).

In this case, we may leave the edge incident to v_1 from M' and then $P = \{v_2v_3\}$ is a good choice. Indeed, the only problem could be that $v_1v_2v_3v_4$ is a forbidden square, contradicting K being uncovered.

Otherwise $h_{M'}(V_K) \leq 2$ implies $|P| \leq 1$. Hence at least one of $|\Gamma_M(u) \cap V_K| = 2$ and $|\Gamma_M(v) \cap V_K| = 2$ is satisfied meaning $K' = K$, a contradiction again.

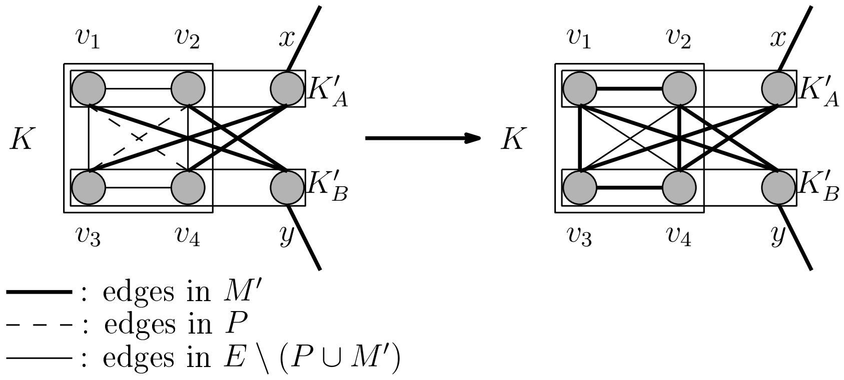


Figure 4: Choice of P for $t = 3$ in the proof of Claim 3.6

Now assume that there is a forbidden $K_{t,t}$ subgraph K' in M with colour classes K'_A, K'_B. The same argument gives a contradiction for $t \geq 4$. However, in case of $t = 3$, choosing P arbitrarily may enable M to contain a forbidden $K_{3,3}$ in the following single configuration: $V_K = \{v_1, v_2, v_3, v_4\}$, $K'_A = \{v_1, v_2, x\}$, $K'_B = \{v_3, v_4, y\}$, $xv_3, xv_4, yv_1, yv_2, xy \in M'$ and $P = \{v_1v_2, v_3v_4\}$ (Figure 4). In this case, $P = \{v_1v_4, v_2v_3\}$ is a good choice.

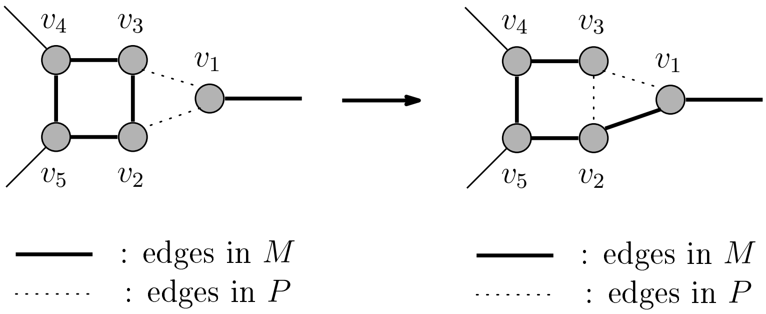


Figure 5: Choice of P for $t = 2$ in the proof of Claim 3.6

Finally, for $t = 2$ no forbidden square appears if $h_{M'}(K) \leq 2$ as otherwise K would be a square-covered triangle. If $h_{M'}(K) = 3$, then such a square K' may appear only if $V_K = \{v_1, v_2, v_3\}$, $V_{K'} = \{v_2, v_3, v_4, v_5\}$, $v_3v_4, v_4v_5, v_5v_2 \in M'$, $P = \{v_1v_2, v_1v_3\}$ ($v_1 \neq v_4, v_5$ as K is uncovered). In this case both $P = \{v_1v_2, v_2v_3\}$ and $P = \{v_1v_3, v_2v_3\}$ give a proper M (Figure 5). $\square$

Proof of Lemma 3.4. First we show that if M is a $\mathcal{K}$-free b-matching in G then there is a $\mathcal{K}^\circ$-free b°-matching M° in G° with $|M^\circ| \geq |M| - \left\lfloor \frac{t^2}{2} \right\rfloor$. Let $M' = M - E_K$.

Clearly, $|M \cap E_K| \leq \binom{t+1}{2} - \left\lceil \frac{\max\{1,h_{M'}(V_K)\}}{2} \right\rceil$. In G°, let M° be the union of M' and $\left\lfloor \frac{t-\max\{1,d_{M'}(k)\}}{2} \right\rfloor$ or $\left\lfloor \frac{t+1-\max\{1,d_{M'}(k)\}}{2} \right\rfloor$ loops on k depending on whether t is even or not, respectively. Is is easy to see that M° is a $\mathcal{K}^\circ$-free b°-matching in G° with $|M^\circ| \geq |M| - \left\lfloor \frac{t^2}{2} \right\rfloor$.

The proof is completed by showing that for an arbitrary $\mathcal{K}^\circ$-free b°-matching M° in G° there exists a $\mathcal{K}$-free b-matching M in G with $|M| \geq |M^\circ| + \left\lfloor \frac{t^2}{2} \right\rfloor$. Let H denote the set of loops on k obtained when shrinking K, and let $M' = M^\circ - H$. Now $|M^\circ \cap H| \leq \left\lfloor \frac{t-\max\{1,d_{M'}(k)\}}{2} \right\rfloor$ if t is even and $|M^\circ \cap H| \leq \left\lfloor \frac{t+1-\max\{1,d_{M'}(k)\}}{2} \right\rfloor$ if t is odd. By Claim 3.5, M' can be extended modified as to get a $\mathcal{K}$-free b-matching in G with $|M \cap E_K| = \binom{t+1}{2} - \left\lceil \frac{\max\{1,h_{M'}(V_K)\}}{2} \right\rceil$, that is

$$\begin{aligned}
|M| &= |M^\circ| - |M^\circ \cap H| + |M \cap E_K| \\
&\geq |M^\circ| - \left\lfloor \frac{t-\max\{1,d_{M'}(k)\}}{2} \right\rfloor + \binom{t+1}{2} - \left\lceil \frac{\max\{1,h_{M'}(V_K)\}}{2} \right\rceil \\
&\geq |M^\circ| + \left\lfloor \frac{t^2}{2} \right\rfloor
\end{aligned}$$

if t is even and

$$\begin{aligned}
|M| &= |M^\circ| - |M^\circ \cap H| + |M \cap E_K| \\
&\geq |M^\circ| - \left\lfloor \frac{t+1-\max\{1,d_{M'}(k)\}}{2} \right\rfloor + \binom{t+1}{2} - \left\lceil \frac{\max\{1,h_{M'}(V_K)\}}{2} \right\rceil \\
&\geq |M^\circ| + \left\lfloor \frac{t^2}{2} \right\rfloor
\end{aligned}$$

if t is odd. □

4 Proof of Theorem 2.5

We prove max $\geq$ min by induction on $|\mathcal{K}|$. For $\mathcal{K} = \emptyset$, this is simply a consequence of Theorem 2.1.

Assume now that $\mathcal{K} \neq \emptyset$ and let K be a forbidden subgraph such that K is uncovered if $t = 2$. Let $G^\circ = (V^\circ, E^\circ)$ denote the graph obtained by shrinking K, let b° defined as in Lemma 3.3 or 3.4 depending on whether K is a $K_{t,t}$ or a K_{t+1}. We denote by $\mathcal{K}^\circ$ the list of forbidden subgraphs disjoint from K.

By induction, the maximum size of a $\mathcal{K}^\circ$-free b°-matching in G° is equal to the minimum value of $\tau(U^\circ, W^\circ, \mathcal{P}^\circ, \dot{\mathcal{K}}^\circ)$. Let us choose an optimal $U^\circ, W^\circ, \mathcal{P}^\circ, \dot{\mathcal{K}}^\circ$ so that $|U^\circ|$ is minimal. The following claim gives a useful property of U°.

Claim 4.1. *Assume that $v \in U$ is such that $d(v, W) + |\Gamma(v) \cap (V - W)| \leq b(v) + 1$. Then $\tau(U - v, W, \mathcal{P}', \dot{\mathcal{K}}) \leq \tau(U, W, \mathcal{P}, \dot{\mathcal{K}})$ where $\mathcal{P}'$ is obtained from $\mathcal{P}$ by replacing its members incident to v by their union plus v.*

Proof. By removing v from U, $b(U)$ decreases by $b(v)$. $|E[W]| - |\dot{\mathcal{K}}[W]|$ remains unchanged, while the bound on $d(v, W) + |\Gamma(v) \cap (V - W)|$ implies that the increment in the sum over the components of $G - U - W$ is at most $b(v)$. □

Case 1: K is a $K_{t,t}$ with colour classes K_A and K_B.

By Lemma 3.3, the difference between the maximum size of a $\mathcal{K}$-free b-matching in G and the maximum size of a $\mathcal{K}^\circ$-free b°-matching in G° is exactly t^2-t. We will define $U, W, \mathcal{P}$ and $\dot{\mathcal{K}}$ such that

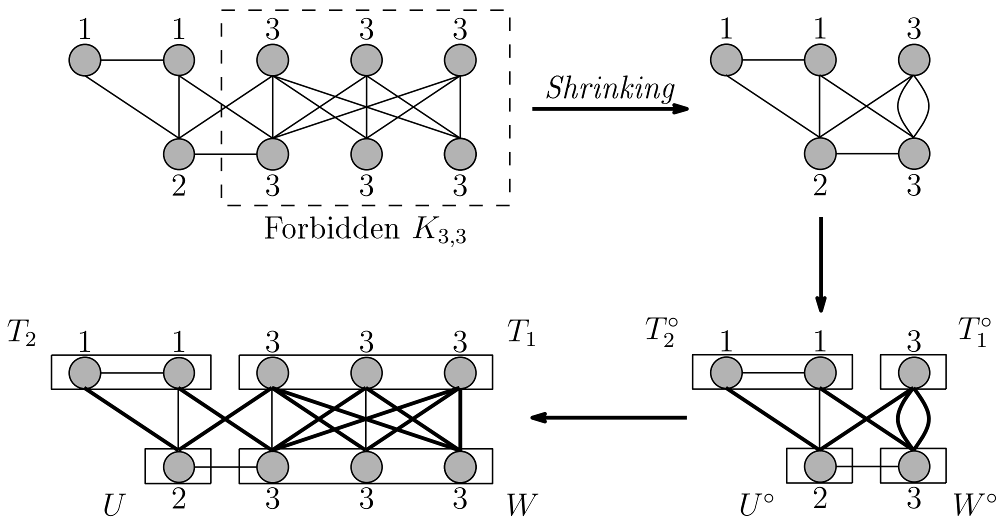
$$\tau(U,W,\mathcal{P},\dot{\mathcal{K}}) = \tau(U^\circ,W^\circ,\mathcal{P}^\circ,\dot{\mathcal{K}}^\circ) + t^2 - t. \tag{7}$$

The shrinking replaces K_A and K_B by two nodes k_a and k_b with $t-1$ parallel edges between them. Let U, W and $\mathcal{P}$ denote the pre-images of $U^\circ, W^\circ, \mathcal{P}^\circ$ in G, respectively and let $\dot{\mathcal{K}} = \dot{\mathcal{K}}^\circ \cup \{K\}$. By (3), $d_{G^\circ - k_b}(k_a), d_{G^\circ - k_a}(k_b) \le t$. Since $b^\circ(k_a) = b^\circ(k_b) = t$, Claim 4.1 and the minimal choice of $|U^\circ|$ implies that if $k_a \in U^\circ$, then $k_b \in W^\circ$.

Hence we have the following cases (T° denotes a member of $\mathcal{P}^\circ$). In each case we are only considering those terms in $\tau(U^\circ,W^\circ,\mathcal{P}^\circ,\dot{\mathcal{K}}^\circ)$ that change when taking $\tau(U,W,\mathcal{P},\dot{\mathcal{K}})$ instead.

- $k_a \in U^\circ,\ k_b \in W^\circ$: $b(U) = b^\circ(U^\circ) + t^2 - t$.
- $k_a, k_b \in W^\circ$: $|E[W]| = |E^\circ[W^\circ]| + t^2 - t + 1$ and $|\dot{\mathcal{K}}[W]| = |\dot{\mathcal{K}}^\circ[W^\circ]| + 1$.
- $k_a \in W^\circ,\ k_b \in T^\circ$: $|E[T,W]| = |E^\circ[T^\circ,W^\circ]| + t^2 - t + 1$, $b(T) = b^\circ(T^\circ) + t^2 - t$ and $|\dot{\mathcal{K}}[T,W]| = |\dot{\mathcal{K}}^\circ[T^\circ,W^\circ]| + 1$ (Figure 6).
- $k_a \in T^\circ,\ k_b \in W^\circ$: similar to the previous case.
- $k_a, k_b \in T^\circ$: $b(T) = b^\circ(T^\circ) + 2t^2 - 2t$.

(7) is satisfied in each of the above cases, hence we are done. Note that in the first and the last case we may leave out K from $\dot{\mathcal{K}}$ as it is not counted in any term.



$\tau(U,W,\mathcal{P},\dot{\mathcal{K}}) = 5 + 3^2 - 3 = 11$ $\qquad$ $\tau(U^\circ,W^\circ,\mathcal{P}^\circ,\dot{\mathcal{K}}^\circ) = 5$

Figure 6: Extending M°

Case 2: K is a K_{t+1}.

By Lemma 3.4, the difference between the maximum size of a $\mathcal{K}$-free b-matching in G and the maximum size of a $\mathcal{K}^\circ$-free b°-matching in G° is $\left\lfloor \frac{t^2}{2} \right\rfloor$. We show that for the pre-images U, W and $\mathcal{P}$ of U°, W° and $\mathcal{P}^\circ$ with $\dot{\mathcal{K}} = \dot{\mathcal{K}}^\circ \cup \{K\}$ satisfy

$$\tau(U, W, \mathcal{P}, \dot{\mathcal{K}}) = \tau(U^\circ, W^\circ, \mathcal{P}^\circ, \dot{\mathcal{K}}^\circ) + \left\lfloor \frac{t^2}{2} \right\rfloor. \tag{8}$$

After shrinking $K = (V_K, E_K)$ we get a new node k with $\lfloor \frac{t+1}{2} \rfloor - 1$ loops on it. (3) implies that there are at most $t+1$ non-loop edges incident to k. Since $b^\circ(k) \geq t$, Claim 4.1 implies $k \notin U$. Hence we have the following two cases (T° denotes a member of $\mathcal{P}^\circ$).

- $k \in W^\circ$: $|E[W]| = |E^\circ[W^\circ]| + \binom{t+1}{2} - \lfloor \frac{t+1}{2} \rfloor + 1$ and $|\dot{\mathcal{K}}[W]| = |\dot{\mathcal{K}}^\circ[W^\circ]| + 1$.
- $k \in T^\circ$: $b(T) = b^\circ(T^\circ) + t^2$ if t is even and $b(T) = b^\circ(T^\circ) + t^2 - 1$ for an odd t.

(8) is satisfied in both cases, hence we are done. We may also leave out K from $\dot{\mathcal{K}}$ in the second case as it is not counted in any term.

5 Algorithm

In this section we show how the proof of Theorem 2.5 immediately yields an algorithm for finding a maximum $\mathcal{K}$-free b-matching in strongly polynomial time.

In such problems, an important question from an algorithmic point of view is how $\mathcal{K}$ is represented. For example, in the $\mathcal{K}$-free b-matching problem for bipartite graphs solved by Pap in [16], the set of excluded subgraphs may be exponentially large. Therefore Pap assumes that $\mathcal{K}$ is given by a membership oracle, that is, a subroutine is given for determining whether a given subgraph is a member of $\mathcal{K}$. However, with such an oracle there is no general method for determining whether $\mathcal{K} = \emptyset$. Fortunately, we do not have to tackle such problems: by the next claim, we may assume that $\mathcal{K}$ is given explicitly, as its size is linear in n. We use $n = |V|$, $m = |E|$ for the number of nodes and edges of the graph, respectively.

Claim 5.1. *If the graph $G = (V, E)$ satisfies (1) and (3), then the total number of $K_{t,t}$ and K_{t+1} subgraphs is bounded by $\frac{(t+3)n}{2}$.*

Proof. Assume that $v \in V$ is contained in a forbidden subgraph. If we leave an edge incident to v, the remaining t edges may be contained in at most one K_{t+1} subgraph hence the number of K_{t+1}'s containing v is at most $t+1$. However, these t edges also determine one of the colour classes of those $K_{t,t}$'s containing them. If we pick a node from this colour class and leave an edge incident to it (but not to v), then the remaining t edges, if they do so, exactly determine the other colour class of a $K_{t,t}$ subgraph. In point of v this means that the number of $K_{t,t}$ subgraphs containing v is bounded by $(t+1)t = t^2 + t$. Hence the total number of forbidden $K_{t,t}$ and K_{t+1} subgraphs is at most $\frac{(t^2+t)n}{2t} + \frac{(t+1)n}{t+1} = \frac{(t+3)n}{2}$. □

Let us now turn to the algorithm. First we choose an inclusionwise maximal subset $\mathcal{H} = \{H_1, \ldots, H_k\}$ of disjoint forbidden subgraphs greedily. For $t = 2$, let us first choose squares as long as possible and then continue with triangles. This can be done in $O(t^3 n)$ time as follows. Let us maintain an array of size m that encodes for each edge whether it is used in one of the selected forbidden subgraphs or not. When increasing $\mathcal{H}$, one only has to check whether any of the edges of the examined forbidden subgraph is already used. This may be done in t^2 or $\binom{t}{2}$ steps, assuming that the members of $\mathcal{K}$ are given by edge-lists. Together with Claim 5.1 we get the $O(t^3 n)$ bound.

Let us shrink the members of $\mathcal{H}$ simultaneously (this can be easily done since they are disjoint), resulting in a graph $G' = (V', E')$ with a bound $b' : V' \to \mathbb{Z}_+$ and no forbidden subgraphs since $\mathcal{H}$ was maximal. One can find a maximal b'-matching M' in G' in $O(|V'||E'| \log |V'|) = O(nm \log m)$ time as in [5]. Using the constructions described in Lemmas 3.3 and 3.4 for $H_k, H_{k-1}, \ldots, H_1$, this can be modified into a maximal $\mathcal{K}$-free b-matching M. Note that, for $t = 2$, H_i is always uncovered in the actual graph since we started including triangles in $\mathcal{H}$ only when there were no more forbidden squares disjoint from the previously chosen members of $\mathcal{H}$. A dual optimal solution $U, W, \mathcal{P}, \dot{\mathcal{K}}$ can be constructed simultaneously as in the proof of Theorem 2.5. The best time bound of the shrinking and extension steps may depend on the data structure used and the representation of the graph. In any case, one such step may be performed in $O(m)$ time and $|\mathcal{H}| = O(n)$, hence the total running time is $O(t^3 n + nm \log m)$.

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