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On stable matchings and flows

Dedicated to András Frank on the occasion of his 60th birthday

Tamás Fleiner

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Abstract

We describe a flow model that generalizes ordinary network flows the same way as stable matchings generalize the bipartite matching problem. We prove that there always exists a stable flow, generalize the lattice structure of stable marriages to stable flows and prove a flow extension of Pym's linking theorem.

Keywords: stable marriages; stable allocations; network flows

1 Introduction

In the stable marriage problem of Gale and Shapley [6], there are n men and n women and each person ranks the members of the opposite gender by an arbitrary strict, individual preference order. A *marriage scheme* in this model is a set of marriages between different men and women. Such a scheme is *unstable* if there exists a *blocking pair*, that is, a man m and a woman w in such a way that m is either unmarried or m prefers w to his wife, and at the same time, w is either unmarried or prefers m to her partner. A marriage scheme is *stable* if it is not unstable, that is, not blocked by any pair. It is a natural problem to find a stable marriage scheme if it exists at all. Nowadays, it is already folklore that for any preference rankings of the n men and n women, a stable marriage scheme exists. This theorem was proved first by Gale and Shapley in [6]. They constructed a special stable marriage scheme with the help of a finite procedure, the so-called deferred acceptance algorithm. It also turned out that for the existence of a stable scheme it is not necessary that the number of men is the same as the number of women or that for each person, all members of the opposite group are acceptable: the deferred acceptance algorithm is so robust that it works properly in these more general settings.

Several interesting properties about the structure of stable marriage schemes are known. Donald Knuth [7] attributes to John Conway the observation that stable

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marriages have a lattice structure: if each man picks the better assignment out of two stable marriage schemes then another stable marriage scheme is created in which each women gets the worse husband out of the two.

There are further known extensions of the stable marriage problem. Baïou and Balinski proved in [1] that if each edge of the underlying bipartite graph has a nonnegative capacity and each vertex has a nonnegative quota then the accordingly modified deferred acceptance algorithm shows that there always exists a so called stable allocation. An allocation is an assignment of nonnegative values to the edges that do not exceed the corresponding capacities such that the total allocation of no vertex exceeds its capacity. (That is, a "marriage" can be formed with an "intensity" different from 0 and 1 and each participant has an individual upper bound on his/her total "marriage intensity".) An allocation is stable if any unsaturated edge e has a saturated end vertex v such that no edge e' incident with v and preferred by v less than e has a positive value. Beyond proving the existence of stable assignments, Baïou and Balinski used flow-type arguments to speed up the deferred acceptance algorithm in [1]. Later, Dean and Munshi came up with an even faster algorithm for the same problem [4] that also has to do a with network flows.

It is fairly well-known that the bipartite matching problem can be formulated in the more general network flow model, and the alternating path algorithm for maximum bipartite matchings is a special case of the augmenting path algorithm of Ford and Fulkerson for maximum flows. However, it seems that the question whether there exists a flow generalization of the stable marriage theorem has not been addressed so far. This very problem is in the focus of our present work. In section 2, we formulate the stable flow problem and give an alternative fixed point proof of the result of [1] by Baïou and Balinski on stable allocations. Section 3 contains the stable flow theorem, a generalization of the Gale-Shapley theorem to flows. Our reduction of the stable flow problem to the stable allocation problem is similar to the reduction the maximum flow problem to the maximum b-matching problem. Actually, our construction has to do also with the one that Cechlárová and Fleiner used in [3] to extend the stable roommates model to a multiple partner model. Section 4 is devoted to certain structural results on stable flows, in particular we generalize the lattice structure of stable marriages. To achieve this, we lean on the construction we used for the reduction. The last section contains an extension of Pym's path linking theorem to flows that we prove with the help of our stability concept. We conclude in the last section by describing some generalizations of stable flows that can be handled with our method and by asking some open problems.

It seems that our work opens more questions than it answers. Perhaps the most challenging open problem is to find theoretical or real word applications of the framework that we shall describe.

2 Preliminaries

Recall that by a *network* we mean a quadruple (D, s, t, c), where $D = (V, A)$ is a digraph, s and t are different nodes of D and $c : A \to \mathbb{R}_+$ is a function that determines the capacity $c(a)$ of each arc a of A. (Sometimes it is assumed that no arc enters vertex s and no arc leaves vertex t. Though this assumption would simplify a lot on our results, we do not require it for the reason that the result we prove is significantly more general. Still, if the reader finds it difficult to follow the arguments, it might be convenient to consider the source-sink case and skip the irrelevant parts.) A *flow* of network (D, s, t, c) is a function $f : A \to \mathbb{R}$ such that capacity condition $0 \le f(a) \le c(a)$ holds for each arc a of A and each vertex v of D different from s and t satisfies the Kirchhoff law: $\sum_{uv\in A} f(uv) = \sum_{vu\in A} f(vu)$, that is, the amount of the incoming flow equals the amount of the outgoing flow for v.

A *network with preferences* is a network (D, s, t, c) along with a preference order $\le_v$ for each vertex v of V, such that $\le_v$ is a linear order on the arcs that are incident with v. For a given network with preferences, it is convenient to think that vertices of D are "players" that trade with a certain product. An arc uv of D from player u to player v with capacity $c(uv)$ represents the possibility that player u can supply at most $c(uv)$ units of products to player v. A "trading scheme" is described by a flow f of the network, as for any two players u and v, flow $f(uv)$ determines the amount of product that u sells to v. Everybody in the market would like to trade as much as possible, that is, each player v strives to maximize the amount of flow through v. In particular, if flow f allows player v to receive some more flow (that is, there are products on the market that v can buy) and v can also send some more flow (i.e. some player would be happy to by more products from v) then flow f does not correspond to a stable market situation.

Another instability occurs when $vw \le_v vu$ (player v prefers to sell to w rather than to u) and flow f is such that w would be happy to buy more product from v (that is $f(vw) < c(vw)$ and w has some extra selling capacity), moreover $f(vu) > 0$ (v sells a positive amount of products to u). In this situation, v would send flow rather to w than to u, hence a stable market situation does not allow the above situation. A similar instability can be described if we talk about outgointg arcs instead of entering ones, that is, if we exchange the roles of buying and selling.

To formalize our concept of stability we need a few definitions. For a network (D, s, t, c) and flow f we say that arc a is *f-unsaturated* if $f(a) < c(a)$, that is, if it is possible to send some extra flow thorough P. A *blocking path of flow f* is an alternating sequence of incident vertices and arcs $P = (v_1, a_1, v_2, a_2, \dots, a_{k-1}, v_k)$ such that all the following properties hold.

$$\text{arc } a_i \text{ points from } v_i \text{ to } v_{i+1} \text{ for } i = 1, 2, \dots, k-1 \text{ and} \tag{1}$$

$$\begin{gathered}\text{vertices } v_1, \dots, v_k \text{ are different}\\ \text{with the possible exception that } v_1 = v_k \text{ may occur and}\end{gathered} \tag{2}$$

$$\text{vertices } v_2, v_3, \dots, v_{k-1} \text{ are different from } s \text{ and } t \tag{3}$$

$$\text{each arc } a_i \text{ is } f\text{-unsaturated and} \tag{4}$$

$$v_1 = s \text{ or } v_1 = t \text{ or there is an arc } a' = v_1 u \text{ such that} \\ f(a') > 0 \text{ and } a_1 <_{v_1} a' \text{ and} \tag{5}$$

$$v_k = s \text{ or } v_k = t \text{ or there is an arc } a'' = w v_k \text{ such that} \\ f(a'') > 0 \text{ and } a_{k-1} <_{v_k} a'' \ . \tag{6}$$

(Note that (2) allows that a blocking path can in fact be a cycle.) So directed path (or cycle) P is blocking if each player that corresponds to an inner vertex of P is happy and capable to increase the flow along P, moreover v_1 can send extra flow either because $v_1 = s$ or $v_1 = t$ is the "source" or "target" node or because v_1 may decrease the flow toward some vertex u that v_1 prefers less than v_2, and at last, v_k can receive some extra flow either because either $v_k \in \{s, t\}$ or v_k can refuse flow from w whom v_k ranks below v_{k-1}. (In a network, there is no difference between the roles of s and t: as none of them have to obey the Kirchhoff law, both of them can send or receive flow. If the reader is uncomfortable with the idea that the target node sends flow to the source then consider the case where no arc enters s and no arc leaves t. This assumption simplifies some of the proofs.) We say that an f-unsaturated path $P = (v_1, v_2, \ldots, v_k)$ is *f-dominated at* v_1 if (5) does not hold, and P is f-dominated at v_k if (6) does not hold.

A flow f of a network with preferences is *stable* if no blocking path exists for f. In the *stable flow problem* we have given a network with preferences and our task is to find a stable flow if such exists.

A special case of the stable flow problem is the stable allocation problem of Baïou and Balinski [1]. The *stable allocation problem* is defined by finite disjoint sets W and F of workers and firms, a map $q : W \cup F \to \mathbb{R}$, a subset E of $W \times F$ along with a map $p : E \to \mathbb{R}$ and for each worker or firm $v \in W \cup F$ a linear order $<_v$ on those pairs of E that contain v. We shall refer to pairs of E as "edges" and hopefully it will not cause ambiguity. Quota $q(v)$ denotes the maximum of total assignment that worker or firm v can accept and capacity $p(wf)$ of edge $e = wf$ means the maximum allocation that worker w can be assigned to firm f along e. An *allocation* is a nonnegative map $g : E \to \mathbb{R}$ such that $g(e) \leq p(e)$ holds for each $e \in E$ and for any $v \in W \cup F$ we have

$$g(v) := \sum_{vx \in E} g(vx) \leq q(v) \ , \tag{7}$$

that is the total assignment $g(v)$ of player v cannot exceed quota $q(v)$ of v. If (7) holds with equality then we say that player v is *g-saturated.* An allocation is *stable* if for any edge wf of E at least one of the following properties hold:

$$\text{either } g(wf) = p(wf) \text{ (the particular employment is realized with full capacity)} \tag{8}$$

$$\text{or worker } w \text{ is } g\text{-saturated and } w \text{ does not prefer } f \text{ to any of his employers} \\ \text{(we say that } wf \text{ is } g\textit{-dominated at } w\text{)} \tag{9}$$

$$\text{or firm } f \text{ is } g\text{-saturated and } f \text{ does not prefer } w \text{ to any of its employees,} \\ \text{that is, edge } wf \text{ is } g\textit{-dominated at firm } f. \tag{10}$$

If g_1 and g_2 are allocations and $w \in W$ is a worker then we say that *allocation g_1 dominates allocation g_2 for worker w* (in notation $g_1 \leq_w g_2$) if one of the following properties is true:

$$\text{either } g_1(wf) = g_2(wf) \text{ for each } f \in F \tag{11}$$

or

$$\begin{aligned} \textstyle\sum_{f' \in F} g_1(wf') = \sum_{f' \in F} g_2(wf') = q(w), \text{ and} \\ g_1(wf) < g_2(wf) \text{ and } g_1(wf') > 0 \text{ implies that } wf' <_w wf. \end{aligned} \tag{12}$$

That is, if w can freely choose his allocation from $\max(g_1, g_2)$ then w would choose g_1 either because g_1 and g_2 are identical for w or because w is saturated in both allocations and g_1 represents w's choice out of $\max(g_1, g_2)$. By exchanging the roles of workers and firms one can define domination relation $\leq_f$ for any firm f, as well.

For a stable allocation problem one can design a network (D, s, t, c) such that $V(D) = \{s, t\} \cup W \cup F$, $A(D) = \{sw : w \in W\} \cup \{ft : f \in F\} \cup \{wf : wf \in E\}$ and $c(sw) = q(w)$, $c(ft) = q(f)$ and $c(wf) = p(wf)$ for any worker w and firm f. That is, we consider the underlying bipartite graph, orient its edges from W to F, add new vertices s and t, with an arc from s to each worker-node and an arc from each firm-node to t, and capacities are given by the original edge-capacities and the corresponding quotas. It is straightforward to see from the definitions that g is a stable allocation if and only if there exists a stable flow f such that $g(e) = f(\vec{e})$ holds for each edge $e \in E$, where $\vec{e}$ is the arc that corresponds to edge e. The stable allocation problem was introduced by Baïou and Balinski as a certain "continuous" version of the stable marriage problem in [1]. It turned out that a natural extension of the deferred acceptance algorithm of Gale and Shapley [6] works for stable allocation problem and the structure of stable allocations is similar to that of stable marriages. Beyond stating the existence of stable allocations, the theorem below describes some structural properties of them.

Theorem 2.1 (See Baïou and Balinski [1]). *1. If stable allocation problem is described by W, F, E, p and q then there always exists a stable allocation g. Moreover, if p and q are integral, then there exists an integral stable allocation g.*

2. If g_1 and g_2 are stable allocations and $v \in W \cup F$ then $g_1 \leq_v g_2$ or $g_2 \leq_v g_1$ holds.

3. Moreover, stable allocations have a natural lattice structure. Namely, if g_1 and g_2 are stable allocations then $g_1 \vee g_2$ and $g_1 \wedge g_2$ are stable allocations, where

$$(g_1 \vee g_2)(wf) = \begin{cases} g_1(wf) & \text{if } g_1 \leq_w g_2 \\ g_2(wf) & \text{if } g_2 \leq_w g_1 \end{cases} \tag{13}$$

and

$$(g_1 \wedge g_2)(wf) = \begin{cases} g_1(wf) & \text{if } g_1 \leq_f g_2 \\ g_2(wf) & \text{if } g_2 \leq_f g_1 \end{cases} \tag{14}$$

In other words, if workers choose from two stable allocations then we get another *stable allocation, and this is also true for the firms' choices. Moreover, it is true that*

$$(g_1 \vee g_2)(wf) = \begin{cases} g_1(wf) & \text{if } g_1 \geq_f g_2 \\ g_2(wf) & \text{if } g_2 \geq_f g_1 \end{cases} \tag{15}$$

and

$$(g_1 \wedge g_2)(wf) = \begin{cases} g_1(wf) & \text{if } g_1 \geq_w g_2 \\ g_2(wf) & \text{if } g_2 \geq_w g_1 \end{cases} \tag{16}$$

That is, in stable allocation $g_1 \vee g_2$ where each worker picks his better assignment, each firm receives the worse out of the two. Similarly, in $g_1 \wedge g_2$ the choice of the firms means the less preferred situation to the workers.

Our proof of Theorem 2.1 is based on fixed points. Actually, we prove an infinite version where graph G may have infinitely many vertices, vertices can be incident with infinitely many edges as far as the preference orders are well-orders. Our proof is closely related to the one we had in [5]. There, the studied matching model was "discrete", thus the method described there cannot be directly applied to the "continuous" problem. The reader who is familiar with the Baïou-Balinski result and knows Theorem 2.1 well might want to skip the rest of this section.

Our main tool for the proof is Tarski's fixed point theorem [13] on complete lattices. A *lattice* is a paritally ordered set $(L, \preceq)$ where any two elements of L (say x and y) have a least upper bound (denoted by $x \vee y$) and a greatest lower bound (denoted by $x \wedge y$). A lattice $(L, \preceq)$ is *complete* if any subset X of P has a least upper bound (denoted by $\bigvee X$) and a greatest lower bound (denoted by $\bigwedge X$). (In paricular, any complete lattice has a unique minimal and a unique maximal element that is not necessarily true for a lattice like $(\mathbb{R}, \leq)$.) For a poset $(P, \preceq)$, mapping $f : P \to P$ is *monotone* if for any elements x, y of P, $x \preceq y$ implies $f(x) \preceq f(y)$.

Lemma 2.2 (Tarski's fixed point theorem [13])**.** *If $(L, \preceq)$ is a complete lattice and map $f : L \to L$ is monotone then there exists a fixed point $x \in L$ of f, that is $f(x) = x$. Moreover, fixed points of f form a complete lattice under the restricted partial order of $\preceq$.* □

Note that each finite lattice L is complete hence it has a unique minimal element, say 0. It is easy to see that $0 \preceq f(0) \preceq f(f(0)) \preceq f(f(f(0)))\ldots$, hence after some iteration we find a fixed point $f^{(k)}(0) = f(f^{(k)}(0))$. Actually, it is not difficult to see that the iteration of monotone function f in the next proof gives a variant of the Gale-Shapley algorithm that finds a stable allocation.

Proof of Theorem 2.1. 1. Let L be the set of nonnegative mappings $l : E \to \mathbb{R}$ such that $0 \leq l(e) \leq p(e)$ holds for any edge e of E. Observe that L forms a complete lattice under partial order $\leq$. (As usual, we say for functions l and l' that $l \leq l'$ iff their domain is the same and $l(x) \leq l'(x)$ holds for any element x of the common domain.) This is because if $l_i : E \to \mathbb{R}$ are elements of L for $i \in I$ then $\bigvee_{i \in I} l(e) := \sup\{l_i(e) : i \in I\}$ and $\bigwedge_{i \in I} l(e) := \inf\{l_i(e) : i \in I\}$ defines other elements of L that are the least upper bound and greatest lower bounds of the l_i's, respectively. We define two choice functions on L, one for the workers, and one for the firms. For any element l of L let C_W denote the the choice of the workers, where each worker w freely chooses his assigment the best possible way obeying his quota but disregarding the quotas of the firms in such a way that l is an upper bound on the choice. (For example, if there is one worker w with quota 8 and e_i is the ith choice of w and $l(e_i) = i$ for each edge then $C_W(l)(e_i) = i$ for $i = 1, 2, 3$, $C_W(l)(e_4) = 2$ and $C_W(l)(e_i) = 0$ for $i > 4$.) We can define choice function C_F of the firms on L by switching the role of workers and firms in the definition of C_W.

For any map $l \in L$ let $\overline{C_W}(l)$ denote those assignments that the workers ignore, and let $\overline{C_F}(l)$ be the same for the firms. That is, $\overline{C_W}(l)(e) := l(e) - C_W(l)(e)$ and $\overline{C_F}(l)(e) := l(e) - C_F(l)(e)$ for each edge e of E. As $0 \leq \overline{C_W}(l) \leq l$ and $0 \leq \overline{C_F}(l) \leq l$ is true for any $l \in L$, this follows that $\overline{C_W}(l), \overline{C_F}(l) \in L$ holds. Moreover it is not difficult to see that both mappings $\overline{C_W}$ and $\overline{C_F}$ are monotone: if some assignment is ignored by the workers (firms) then this very assignment is also ignored when a wider choice is offered to the workers (firms).

Define mappings on L by $C^*_W(l) := p - \overline{C_W}(l)$ and $C^*_F(l) := p - \overline{C_F}(l)$, that is for each edge e, $C^*_W(l)(e) = p(e) - \overline{C_W}(l)(e)$ and $C^*_F(l)(e) = p(e) - \overline{C_F}(l)(e)$. It is easy to see that for any element l of L, we have $C^*_W(l), C^*_F(l) \in L$, moreover if $l \leq l'$ are two elements of L then $C^*_W(l) \geq C^*_W(l')$ and $C^*_F(l) \geq C^*_F(l')$ holds. (As this property is the opposite of monotonicity in some sense, sometimes it is called “antitone”.) So if we compose these maps, then we get monotone function $f := C^*_F \circ C^*_W : L \to L$ that must have a fixed point by Theorem 2.2. That is, there is some function l of L such that

$$p(e) - \overline{C_F}\left(p - \overline{C_W}(l)\right)(e) = l(e) \tag{17}$$

holds for any edge e of E.

We shall prove that $g := C_W(l)$ is a stable allocation. To justify that g is an allocation it is enough to show that all quotas are observed as $g \in L$ ensures that no capacity is exceeded. No worker's quota is exceeded because $g = C_W(l)$ so we only have to check that no firm has more allocation than its quota. Define $g_W := l - g$ and $g_F := p - l$, that is, $p = g_W + g + g_F$. Property (17) implies that

$$g_F = p - l = \overline{C_F}(p - \overline{C_W}(l)) = \overline{C_F}(g + g_F)$$

and this is equivalent to $g = C_F(g + g_F)$, showing that no firm quota is exceeded by g. So g is an allocation, indeed. To prove the stability of g, let e be an arbitrary edge of E. If $g(e) = p(e)$ then property (8) holds. Otherwise $g(e) < p(e)$. As $p(e) = g_W(e) + g(e) + g_F(e)$, either $g_W(e) > 0$ or $g_F(e) > 0$ (or both hold). In the first case property (9) follows from $g = C_W(g + g_W)$. If $g_F(e) > 0$ then $g = C_F(g + g_F)$ implies (10). So g is a stable allocation as the theorem claims.

The above proof also justifies the integrality property if instead of complete lattice L we use complete lattice L_{int} of the integral elements of L.

2. For given stable allocations g_1 and g_2 construct digraph D on the vertices of G by orienting those edges e of E where $g_1(e) \neq g_2(e)$ the following way. The arc set of D consists of two disjoint subsets A_1 and A_2. If $g_1(e) < g_2(e)$ then property (8) cannot hold for e with stable allocation g_1. Hence either (9) or (10) must hold for $g = g_1$. In the first case $\vec{e} = fw$, in the second case we have $\vec{e} = wf$ for the oriented version of

e that belongs to A_1. That is, we orient e to that vertex where it is g_1-dominated. Similarly, if $g_1(e) > g_2(e)$ then $g_2(e) < p(e)$, so e has to be g_2-dominated at f or at w. We orient e to that vertex where it is g_2-dominated and arc $\vec{e}$ belongs to A_2. Observe that arc $\vec{e_1}$ of A_1 and and arc $\vec{e_2}$ of A_2 cannot be oriented towards the same vertex, because if an edge with positive g_2-value is g_1-dominated at vertex v then no edge with positive g_1 value can be g_2-dominated at v.

Define capacities for the arcs of D by

$$c(\vec{e}) = \begin{cases} g_2(e) - g_1(e) & \text{if } g_1(e) < g_2(e) \\ g_1(e) - g_2(e) & \text{if } g_1(e) > g_2(e) \ . \end{cases}$$

Observe that if some arc $\vec{e} = uv$ of A_1 is oriented to vertex v (hence edge e is g_1-dominated at v) then vertex v is g_1-saturated, hence the total capacity of those arcs of A_2 that leave v is at least as much as the total capacity of those arcs of A_1 that enter v. Hence the total capacity of the A_1 arcs is not more than the total capacity of the arcs of A_2. Exchanging the roles of A_1 and A_2 and of g_1 and g_2 in the above argument we get that the total capacity of A_2 cannot exceed the total capacity of A_1, showing that we have equality everywhere. In particular, if v is a non-isolated vertex of D then all arcs entering v belong to either A_1 or A_2 and all arcs leaving v belong to the other arcset. Moreover, the total capacity of incoming arcs equals the total capacity of outgoing arcs and v is both g_1-saturated and g_2-saturated. This follows that depending on the type of the arcs leaving v, either $g_1 \leq_v g_2$ or $g_2 \leq_v g_1$ holds. To finish the proof of the second part of the theorem, observe that for isolated vertices v of D the two allocations are the same on the edges incident v, so both $g_1 \leq_v g_2$ and $g_2 \leq_v g_1$ are true.

3. Using the lattice property of fixed points in Lemma 2.2 is an attracting idea to prove the third part of Theorem 2.1. To carry this over, we would need that fixed points form a sublattice (that is, for the restricted lattice operations $\vee, \wedge$) rather than a lattice subset for the restricted partial order. However, this is not true in general. Moreover, we should prove that each stable allocation is coming from some fixed point. Though this approach would achieve the goal, we choose the direct proof with the help of digraph D defined in the proof of the second part above.

We shall prove only (13) and (15), as the proof of (14) and (16) is the same except for the roles of workers and firms are interchanged. The proof of the second part of Theorem 2.1 shows that if an arc $\vec{e}$ of A_1 (A_2) enters v then $g_1 \leq_v g_2$ ($g_2 \leq_v g_1$) and if arc $\vec{e}$ of A_1 (of A_2) leaves v then there is an arc of A_2 (of A_1) that enters v hence $g_2 \leq_v g_1$ ($g_1 \leq_v g_2$) holds. This means that (13) can be reformulated with the help of our digraph D in such a way that for any edge $e = wf$ we have

$$(g_1 \vee g_2)(e) = \begin{cases} g_1(e) & \text{if no arc of } D \text{ corresponds to } e, \\ & \text{that is, if } g_1(e) = g_2(e) \\ g_1(e) & \text{if } \vec{e} = fw \in A_1 \text{ or } \vec{e} = wf \in A_2 \\ g_2(e) & \text{if } \vec{e} = wf \in A_1 \text{ or } \vec{e} = fw \in A_2 \ . \end{cases} \quad (18)$$

It is straigthforward to see that the above reformulation is equivalent to (15). So (13) and (15) define the same function $g_1 \vee g_2$ on E. Moreover, (13) shows that $g_1 \vee g_2$

neither exceeds any edge capacity nor any worker-quota, while (15) proves that each firm-quota is observed, so $g_1 \vee g_2$ is an allocation, indeed.

To prove stability, we show that for any edge $e = wf$ of E (8) or (9) or (10) holds. If (8) is not true for e then $(g_1 \vee g_2)(e) < p(e)$. If $g_1(e) < p(e) > g_2(e)$ then e must be g_1-dominated and g_2-dominated at w or at f. If e is g_1-dominated or g_2-dominated at w then e is $(g_1 \vee g_2)$-dominated at w by (13). Otherwise e is g_1-dominated and g_2-dominated at f hence e is $(g_1 \vee g_2)$-dominated at f by (15).

The remaining case is when $(g_1 \vee g_2)(e) < p(e)$ and (say) $g_1(e) = p(e)$, hence $g_2(e) < p(e)$ by (18). This means that there is an arc $\vec{e}$ of digraph D. If $\vec{e}$ enters w then e is g_2-dominated at w, hence e is $(g_1 \vee g_2)$-dominated at w by (13). If $\vec{e}$ leaves w then $g_2 \leq_f g_1$. Moreover, by the structure of D we described in the proof of the second part of Theorem 2.1, there must be some other arc $\vec{e'}$ of D that enters w, and hence $g_1 \leq_w g_2$. It follows from (13) that $p(e) > (g_1 \vee g_2)(e) = g_1(e) = p(e)$, a contradiction. □

3 Stable flows

Our goal here is to prove a generalization of Theorem 2.1. The difficulty is that though the deferred acceptance algorithm works well with quota function q, somehow it has difficulties with the Kirchhoff law.

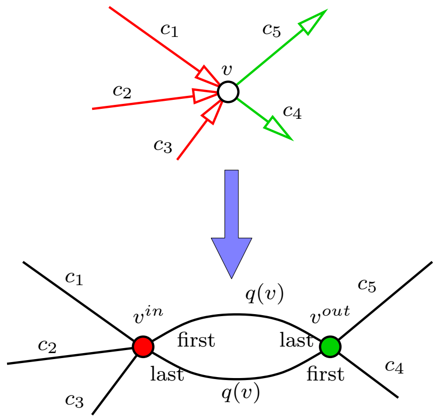
Theorem 3.1. *If network (D, s, t, c) and preference orders $<_v$ describe a stable flow problem then there always exists a stable flow f. If capacity function c is integral then there exists an integral stable flow.*

It is possible to prove Theorem 3.1 by a mixture of the deferred acceptance algorithm and the augmenting path algorithm. That is, starting from s, we follow "first choice paths" until they arrive to t and augment along them with observing the capacity constraints. If a new path collides with an earlier one then some amount of flow is refused by the receiving vertex and we try to reroute the flow excess from the starting point of the refused arc. We have a stable flow as soon as we cannot find an augmenting path from s to t.

Our proof of Theorem 3.1 follows a different approach for two reasons. On one hand, it seems that in the area of stable matchings neither the reduction of one problem to another one nor the use of graph terminology is common. We demonstrate here that these methods may be fruitful. On the other hand, the "deferred augmentation" algorithm we sketched above does not give much information about the rich structure of stable flows that follows from the lattice property of stable allocations.

With the help of the given stable flow problem we shall define a stable allocation problem. For each vertex v of D calculate

$$M(v) := \min \left(\sum_{xv \in A(D)} c(xv), \sum_{vx \in A(D)} c(vx) \right) ,$$



that is, $M(v)$ is the minimum of total capacity of those arcs of D that enter and leave v. So $M(v)$ is an upper bound on the amount of flow that can flow through vertex v. Choose $q(v) := M(v) + 1$. Construct graph G_D as follows. Split each vertex v of D into two distinct vertices v^{in} and v^{out}, and for each arc uv of D add edge $u^{out}v^{in}$ to G_D.

For each vertex v of D different from s and t add two parallel edges between v^{in} and v^{out}: to distinguish between them we will refer them as $v^{in}v^{out}$ and $v^{out}v^{in}$. Let $p(v^{in}v^{out}) = p(v^{out}v^{in}) := q(v)$, $p(u^{out}v^{in}) := c(uv)$ and $q(v^{in}) = q(v^{out}) := q(v)$. To finish the construction of the stable allocation problem, we need to fix a linear preference order for each vertex of G_D. For vertex v^{in} let $v^{in}v^{out}$ be the most preferred and $v^{out}v^{in}$ be the least preferred edge, and the order of the other edges incident with v^{in} are coming from the preference oder of v on the corresponding arcs. For vertex v^{out} the most preferred vertex is $v^{out}v^{in}$ and the least preferred one is $v^{in}v^{out}$, and the other preferences are coming from $<_v$.

The proof of Theorem 3.1 is a consequence of the following Lemma that describes a close relationship between stable flows and stable allocations.

Lemma 3.2. *If network (D, s, t, c) and preference orders $<_v$ describe a stable flow problem then $f : A(D) \to \mathbb{R}$ is a stable flow if and only if there is a stable allocation g of G_D such that $f(uv) = g(u^{out}v^{in})$ holds for each arc uv of D.*

Proof. Assume first that g is a stable allocation in G_D. This means that none of the $v^{in}v^{out}$ edges is blocking, so either $g(v^{in}v^{out}) = p(v^{in}v^{out}) = q(v)$ or $v^{in}v^{out}$ must be g-dominated at at v^{out}, hence v^{out} is assigned to $q(v^{out}) = q(v)$ amount of allocation. As $q(v)$ is more than the total capacity of arcs leaving v, $g(v^{in}v^{out}) > 0$ or $g(v^{out}v^{in}) > 0$ must hold. So v^{out} must have exactly $q(v)$ amount of allocation whenever $v^{in}v^{out}$ is present. An exchange of *in* and *out* shows that the presence of $v^{out}v^{in}$ implies that v^{in} has exactly $q(v^{in}) = q(v)$ allocation. These observations directly imply that the Kirchhoff law holds for f at each node different from s and t. The capacity condition is also trivial for f, hence f is a flow of D. Observe that by the choice of q, neither s nor t is g-saturated hence no edge is g-dominated at s or at t.

Assume that path $P = (v_1, v_2, \ldots, v_k)$ blocks flow f. As P is f-unsaturated, each edge $v_i^{out}v_{i+1}^{in}$ of G_D must be g-dominated at v_i^{out} or at v_{i+1}^{in}. Path P is blocking, hence either $v_1 \in \{s, t\}$, and hence $v_1^{out}v_2^{in}$ cannot be dominated at v_1 or there is a v_1u arc with positive flow value such that $v_1u > v_1v_2$. In both cases, edge $v_1^{out}v_2^{in}$ has to be g-dominated at v_2^{in}. It means that $g(v_2^{in}v_2^{out}) > 0$. As arc v_2v_3 is f-unsaturated, this follows that edge $v_2^{out}v_3^{in}$ must be g-dominated at v_3^{in}. This yields that $g(v_3^{in}v_3^{out}) > 0$. Again, arc v_3v_4 is f-unsaturated, hence edge $v_3^{out}v_4^{in}$ has to be g-dominated at v_4^{in}, and so on. At the end we get that $v_{k-1}^{out}v_k^{in}$ is g-dominated at v_k^{in}. If $v_k \in \{s,t\}$ then it is impossible as both these vertices are g-unsaturated. Otherwise by the blocking property of P there is an arc wv_k with positive flow and $v_{k-1}v_k <_{v_k} wv_k$, hence again, $v_{k-1}^{out}v_k^{in}$ cannot be g-dominated at v_k^{in}. The contradiction shows that no path can block f.

Assume now that f is a stable flow of D. We have to exhibit a stable allocation g of G_D such that f is the "restriction" of g. To determine g, our real task is to find the $g(v^{in}v^{out})$ and $g(v^{out}v^{in})$ values, as all other values of g are determined directly by f: $g(u^{out}v^{in}) = f(uv)$. The stable allocation we look for might not be unique. In what follows, we shall construct the *canonical representation* g_f of f.

Let S be the set of those vertices u of D such that there exists an f-unsaturated directed path $P = (v_1, v_2, \ldots, v_k = u)$ that is not f-dominated at v_1. As no path can block f, neither s, nor t belongs to S. To determine g_f, for each vertex $v \neq s,t$ allocate the remaining quota of v to $v^{in}v^{out}$ or to $v^{out}v^{in}$ depending on whether $v \in S$ or $v \notin S$ holds. More precisely, define

$$g_f(v^{in}v^{out}) = \begin{cases} q(v) - \sum_{x\in V(D)} f(vx) & \text{if } v \in S \\ 0 & \text{if } v \notin S \end{cases} \quad \text{and} \tag{19}$$

$$g_f(v^{out}v^{in}) = \begin{cases} q(v) - \sum_{x\in V(D)} f(xv) & \text{if } v \notin S \\ 0 & \text{if } v \in S\ . \end{cases} \tag{20}$$

By the definition of q, both $g_f(v^{in}v^{out})$ and $g_f(v^{out}v^{in})$ are nonnegative. If $v \in S$ then the amount of total allocation of v^{out} is $q(v) = q(v^{out})$ by (19), and for $v \notin S$ the amount of total allocation of v^{in} is $q(v) = q(v^{in})$ by (20). So if $v \neq s,t$ then the total allocation of v^{in} and v^{out} is $q(v)$ by the Kirchhoff law. The total allocations of s^{in}, s^{out} and t^{in}, t^{out} is less than $q(s)$ and $q(t)$ respectively, by the choice of q. That is, g_f is an allocation on G_D.

To justify the stability of g_f, we have to show that no blocking edge exists. We have seen earlier, that the presence of $v^{in}v^{out}$ in G_D means that v^{out} g-dominates $v^{in}v^{out}$. Similarly, each edge $v^{out}v^{in}$ is g_f-dominated at v^{in}. Assume now that $g_f(v^{out}u^{in}) < p(v^{out}u^{in}) = c(vu)$ holds.

If there is an f-unsaturated path P that is not f-dominated at its starting node and ends with arc vu then $u \in S$ by the definition of S, hence $g_f(u^{out}u^{in}) = 0$. Moreover, if some edge $w^{out}u^{in}$ with $v^{out}u^{in} <_{u^{in}} w^{out}u^{in}$ would have positive allocation then path P would block f, a contradiction. As u^{in} has $q(u^{in})$ amount of total allocation, edge $v^{out}u^{in}$ is g_f-dominated at u^{in}.

The last case is when any f-unsaturated path that ends with arc vu is f-dominated at its starting vertex. In particular, $v \notin S$, so $g_f(v^{in}v^{out}) = 0$. Moreover, f-unsaturated path (v,u) must be f-dominated at v, hence $v \notin \{s,t\}$ and $v^{out}u^{in}$ is g_f-dominated at v^{out} as v^{out} has $q(v) = q(v^{out})$ amount of allocation. The conclusion is that $g := g_f$ is a stable allocation, just as we claimed. □

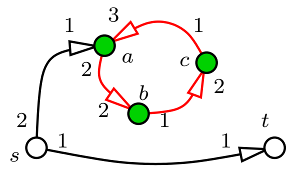
We can finish the proof of the existence of a stable flow now.

Proof of Theorem 3.1. There is a stable allocation for G_D by Theorem 2.1, hence there is a stable flow for D due to the first part of Theorem 3.2. If c is integral then

$q(v)$ is an integer for each vertex v of D hence p is integral for G_D. The integrality property of stable allocations in the first part of Theorem 2.1 shows that there is an integral stable allocation g of G_D that describes an integral stable flow f of D. □

At the end of this section let us point out a weakness of the stability concept of flows. The motivation behind the notion is that we look for a flow that corresponds to an equilibrium situation where the players represented by the vertices of the network act in a selfish way. This equilibrium situation occurs if no coalition of the players can block the underlying flow f, and this blocking is described by the definition as a certain f-unsaturated path (or cycle through s or t) along which the players are capable and prefer to increase the flow. However, in some sense an f-unsaturated cycle C per se causes instability because the players of C mutually agree to send some extra flow along C. So it is natural to define flow f of network (D, s, t, c) with preferences to be *strongly stable* if f is stable and there exists no f-unsaturated cycle in D whatsoever. If f is a stable flow then we can "augment" along f-unsaturated cycles, and hence we can construct a flow $f' \geq f$ such that there no longer exists an f'-unsaturated cycle. But unfortunately flow f' might not be stable any more because we might have created a blocking path by the cycle-augmentations.

In fact, there exist networks with preferences that do not have a strongly stable flow. One example is on the figure: each arc has unit capacity, preferences are indicated around the vertices: lower rank is preferred to the higher. As no arc leaves subset $U := \{a, b, c\}$ of the vertices, no flow can leave U, hence no flow enters U. In particular, arc sa has zero flow. If we assume indirectly that f is a strongly stable flow then cycle abc cannot block, hence there must be a unit flow along it. But now path sa is blocking, a contradiction.

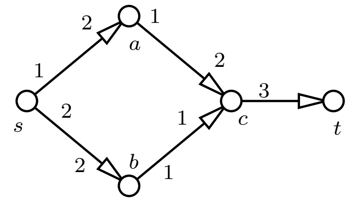


Stable flows have a blocking cycle

4 The structure of stable flows

It is well-known about the stable marriage problem that in each stable marriage scheme, the same set of participants get married. That is, if someone does not get a marriage partner in some stable scheme then this very person remains single in each stable marriage scheme. A generalization of this is the rural hospital theorem of Roth [11] (see also Theorem 5.13 in [12]). It is about the college model, where instead of men we work with colleges, women correspond to students and each college has a quota on the maximum number of students. In the college admission problem it is true that if a certain college c cannot fill up its quota in a stable admission scheme then c receives the same set of students in any stable admission scheme. (The phenomenon is named after hospitals because the assignment problem of medical interns to hospitals is the first well-known application of the college model.)

It seems that the rural hospital theorem cannot be generalized to the stable flow problem. It may happen in a network that a certain vertex transmits different amounts of flow in two stable flows. An example is shown in the figure where each arc has unit capacity. There are two stable flows: one is along path *sact* and the other follows path *sbct*. So in one stable flow, vertex a transmits some flow and no flow passes through a in another one.



Network for a stable flow

There is however a consequence of the rural hospital theorem that can be generalized, namely, that the size of a stable matching is always the same. We have seen that the stable allocation problem is a special case of the stable flow problem, and from the construction it is apparent that the size of a stable matching (more precisely the total amount of assignments in a stable allocation) equals the value of the corresponding flow.

Theorem 4.1. *If network (D, s, t, c) and preference orders $<_v$ describe a stable flow problem and f_1 and f_2 are stable flows then the value of f_1 and f_2 are the same. More generally, $f_1(a) = f_2(a)$ for any arc of D that is incident with s (or with t).*

Proof. Lemma 3.2 implies that there exist stable allocations g_1 and g_2 of G_D that correspond to stable flows f_1 and f_2, respectively. The value of a flow is the net amount that leaves s in D, or, in G_D one can calculate it as the difference of total allocation of s^{out} and s^{in}. This means that the second part of the theorem implies the first one.

As there is no edge between s^{out} and s^{in}, the choice of $q(s)$ implies that both s^{out} and s^{in} are g_1-unsaturated. Hence property (12) can hold neither for s^{in} nor for s^{out}. But Theorem 2.1 implies that g_1 and g_2 are $\leq_{s^{out}}$ and $\leq_{s^{in}}$-comparable. So property (11) must be true for both flows g_1 and g_2 for vertices $v = s^{out}$ and $v = s^{in}$. This shows the second part of the Theorem for s. The argument for t is analogous to the above one. $\square$

As we have seen in Theorem 2.1, stable allocations have a lattice structure. Based on the connection of stable allocations and stable flows described in Lemma 3.2, we shall prove that stable flows of a network with preferences also form a natural lattice. So assume that f is a stable flow in network $(D, s, t, c,)$ with preferences and let stable allocation g_f of G_D be the canonical representation of f as in the proof of Lemma 3.2.

Observe that any vertex $v \neq s, t$ of D, exactly one of $g_f(v^{in}v^{out})$ and $g_f(v^{out}v^{in})$ is positive by the choice of q and g_f. For stable flow f, we can classify the vertices of D different from s and t: v is an *f-vendor* if $g_f(v^{in}v^{out}) > 0$ and v is an *f-customer* if $g_f(v^{out}v^{in}) > 0$. If v is an f-vendor then no edge $v^{out}u^{in}$ can be g_f-dominated at v^{out} (as $g_f(v^{in}v^{out}) > 0$), hence player v sends that much flow to other vertices as much they accept. Similarly, if v is an f-customer then no edge $u^{out}v^{in}$ can be g_f-dominated at v^{out}, that is, player v receives as much flow as the others can supply her.

To explore the promised lattice structure of stable flows, let f_1 and f_2 two stable flows with canonical representations g_{f_1} and g_{f_2}, respectively. From Theorem 2.1 we know that stable allocations form a lattice, so $g_{f_1} \vee g_{f_2}$ and $g_{f_1} \wedge g_{f_2}$ are also stable allocations of G_D, and by Theorem 3.1, these stable allocations define stable flows $f_1 \vee f_2$ and $f_1 \wedge f_2$, respectively. How can we determine these latter flows directly, without the canonical representations? To answer this, we translate the lattice property of stable allocations on G_D to stable flows of D.

Theorem 4.1 shows that stable flows cannot differ on arcs incident with s or t, so on these arcs $f_1 \vee f_2$ and $f_1 \wedge f_2$ are determined. However, vertices different from s and t may have completely different situations in stable flows f_1 and f_2. The two colour classes of graph G_D are formed by the v^{in} and v^{out} type vertices, respectively. So, by Theorem 2.1, $g_{f_1} \vee g_{f_2}$ can be determined such that (say) each vertex v^{out} selects the better allocation and each vertex v^{in} receives the worse allocation out of the ones that g_{f_1} and g_{f_2} provides them. Similarly, for stable allocation $g_{f_1} \wedge g_{f_2}$ the "in"-type vertices choose according their preferences and the "out"-type ones are left with the less preferred allocations. In the language of flows this means the following. If we want to construct $f_1 \vee f_2$ and v is a vertex different from s and t then either all arcs entering v will have the same flow in $f_1 \vee f_2$ as in f_1, or for all arcs a entering v we have $(f_1 \vee f_2)(a) = f_2(a)$ holds. A similar statement is true for the arcs leaving v. To determine which of the two alternatives is the right one, the following rules apply:

- If v is an f_1-vendor and an f_2-customer then v chooses f_2. If v is an f_2-vendor and an f_1-customer then v chooses f_1. That is, each vertex strives to be a customer.

- If v is an f_1-vendor and an f_2-vendor and v transmits more flow in f_1 than in f_2 (i.e. $0 < g_{f_1}(v^{in}v^{out}) < g_{f_2}(v^{in}v^{out})$) then v chooses f_1. That is, vendors prefer to sell more.

- If v is an f_1-customer and an f_2-customer and v transmits more flow in f_1 than in f_2 (i.e. $0 < g_{f_1}(v^{out}v^{in}) < g_{f_2}(v^{out}v^{in})$) then v chooses f_2. That is, customers prefer to buy less.

- Otherwise v is a customer in both f_1 and f_2 or v is a vendor in both flows and v transmits the same amount in both flows (i.e. $g_{f_1}(v^{out}v^{in}) = g_{f_2}(v^{out}v^{in})$ and $g_{f_1}(v^{in}v^{out}) = g_{f_2}(v^{in}v^{out})$). In this situation, v chooses the better "selling position" and gets the worse "buying position" out of stable flows f_1 and f_2.

Clearly, for the construction of $f_1 \wedge f_2$, one always has to choose the "other" options than the above rules describe.

The lattice structure of stable flows defines a partial order on stable flows: $f_1 \preceq f_2$ if and only if $f_1 \vee f_2 = f_2$ holds, or equivalently, if $f_1 \wedge f_2 = f_1$ is true. According to the above rules this means that each f_1-customer v is an f_2-customer, such that v buys at least as much in f_1 as in f_2. Each f_2-vendor u is an f_1-vendor and u sells at most as much in f_1 as in f_2. If w plays the same role (vendor or customer) in both flows and transmits the same amount then v prefers the selling position of f_2 and the buying position of f_1.

5 Linking of flows

In this section, we shall prove a flow-extension of the linking property of paths. To introduce the linking property, perhaps it is best to start from the well-known Mendelsohn-Dulmage theorem [8]. It states that if there are two matchings (M_1 and M_2) in a bipartite graph G with colour classes A and B then there exists a matching M of G that covers each vertex of A that is covered by M_1 and all of the vertices of B that are covered by M_2. This result is true for infinite graphs as well, and one can easily deduce from it the famous Cantor-Bernstein theorem stating that relation $\leq$ is antisymmetric on cardinalities.

So it is possible to combine two matchings of a bipartite graph to get a third one. One can regard the edges of a matching as one-edge paths, and it is an interesting problem whether it is also possible to combine longer paths in a similar manner. The affirmative answer was proved by Pym [9, 10]. Pym's linking theorem deals with a directed graph D such that both $\mathcal{P}$ and $\mathcal{Q}$ are node disjoint directed paths of it, such that a path of $\mathcal{P}$ may share a vertex with a path of $\mathcal{Q}$. (In the Mendelsohn-Dulmage settings, we orient the edges of G from A to B and each matching becomes a collection of node disjoint directed one-edge paths by this.) The assertion is that in this situation there always exists a set $\mathcal{R}$ of node disjoint directed paths of D such that each path of $\mathcal{R}$ starts from a starting node of a path of $\mathcal{P} \cup \mathcal{Q}$, ends in a terminal node of $\mathcal{P} \cup \mathcal{Q}$, moreover from each starting node of $\mathcal{P}$ and at each terminal node of $\mathcal{Q}$ a path of $\mathcal{R}$ starts and terminates, respectively. In addition, Pym proved the so-called switching property: $\mathcal{R}$ can be chosen such that each path r of $\mathcal{R}$ starts with a (possibly empty) initial segment of some path of $\mathcal{P}$, at a vertex it switches to another path of $\mathcal{Q}$ that forms the terminal segment of r. Clearly, the Mendelsohn-Dulmage theorem is a special case of Pym's linking theorem. For an alternative proof of Pym's theorem based on stable matchings see [5].

Later on, Brualdi and Pym [2] gave a variant of the linking theorem in which the switching property does not hold in general, but each vertex and arc that is commonly used by a path of $\mathcal{P}$ and $\mathcal{Q}$ will be used by some path of $\mathcal{R}$, as well. For this, "generalized paths" have to be allowed, that is circular and (possibly doubly) infinite directed paths can be present in $\mathcal{R}$. The main result in this section is an extension of the linking property from paths to network flows. Our result generalizes neither the switching property in the original theorem of Pym nor the covering property of the Brualdi-Pym variant. Though we prove a fractional result, with the help of the integral version of the stable flow theorem, the same proof justifies an integral linking theorem for flows.

Theorem 5.1. *Let $D = (V, A)$ be a directed graph and $N = (D, s, t, c)$ be a network. If f_1 and f_2 are feasible flows of N then there is a feasible flow f of N such that*

$$f(sv) \geq f_1(sv) \text{ and } f(vs) \leq f_1(vs) \text{ moreover} \tag{21}$$

$$f(ut) \geq f_2(ut) \text{ and } f(tu) \leq f_2(tu) \tag{22}$$

holds for any vertices u, v of D. If flows f_1 and f_2 are integral then there is an integral feasible flow f of N with properties (21) and (22).

Proof. For each arc a of A let a' be a parallel copy of arc a and let A' denote the set of these copies a'. Furthermore, define

$$A'_1 := \{a' \in A' : f_1(a) > f_2(a)\} \text{ and } A'_2 := \{a' \in A' : f_2(a) > f_1(a)\}$$

as the sets of those parallel copies where f_1 and f_2, respectively is greater on the corresponding arc of A. Let $D' = (V, A \cup A'_1 \cup A'_2)$ be the directed graph that we get from D by adding a parallel arc for each arc a of D where the two flows f_1 and f_2 do not agree. Define capacity c' on $A(D')$ by

$$c'(x) := \begin{cases} \min(f_1(a), f_2(a)) & \text{if } x = a \in A \\ f_1(a) - f_2(a) & \text{if } x = a' \in A'_1 \\ f_2(a) - f_1(a) & \text{if } x = a' \in A'_2 \end{cases}$$

This way we get network $N' = (D', s, t, c')$. As $c'(a)+c'(a') = \max(f_1(a), f_2(a)) \le c(a)$ for $a \in A$, each feasible flow f' of N' corresponds to a feasible flow $\overline{f'}$ for N, where $\overline{f'}(a) := f'(a) + f'(a')$ for each arc a of D.

Choose linear preference orders $<_v$ arbitrarily on the arcs incident with v such that on the set of incoming arcs v prefers arcs of A'_1 to A, and any arc of A is better than any arc of A'_2. On the outgoing arcs the order is "opposite": outgoing arcs of A'_2 form the set of most preferred arcs (in an arbitrary order), then follow the arcs of A that leave v and the least preferred outgoing arcs are the ones in A'_1. With these linear orders, N' becomes a network with preferences. Let f' be an arbitrary stable flow of this network that exists by Theorem 3.1. We shall prove that the corresponding flow $f = \overline{f'}$ has properties (21) and (22).

Let S^{out} denote the set of those vertices v of D' such that there exists a directed path $P(s, v)$ of D' on f'-unsaturated arcs of $A \cup A_1$ from s to v and let S^{in} denote the set of those vertices v of D' such that there is a directed path $P(v, s)$ of D' on f'-unsaturated arcs of $A \cup A_1$ from v to s. It is easy to see that property (21) of $\overline{f'}$ is equivalent to $S^{out} = S^{in} = \emptyset$. Let T^{in} stand for the set of those vertices u of D' such that there exists a directed path $P(u, t)$ of D' on f'-unsaturated arcs of $A \cup A'_2$ from u to t. At last, T^{out} denotes the set of those vertices v of D' such that there exists a directed path $P(t, u)$ of D' on f'-unsaturated arcs of $A \cup A'_2$ from t to u. Again, it is straightforward that property (22) of $\overline{f'}$ is equivalent to $T^{in} = T^{out} = \emptyset$.

By the stability of f', neither s, nor t belongs to any of the sets S^{out}, S^{in}, T^{in} and T^{out}. This means that each vertex of these four sets obeys the Kirchhoff law. Hence for any of these four sets $X \in \{S^{out}, S^{in}, T^{in}, T^{out}\}$, the total amount of flow f' that enters X is the same as the total amount of flow f' that leaves X.

Indirectly, let us assume that S^{out} is nonempty. By definition, if arc $x = vw$ of $A \cup A'_1$ leaves S^{out} then $f'(x) = c'(x)$, as otherwise w would also belong to S^{out}. So the total amount of flow f' that leaves S^{out} is at least as much as the total amount of flow f_1 that leaves S^{out}. Observe that if least preferred arc $x = uv$ of A'_2 enters S^{out} then $f'(x) = 0$, as otherwise $P(s, v)$ would be a blocking path to stable flow f'. Moreover, for any vertex v of S^{out}, the first arc a of $P(s, v)$ is f'-unsaturated, hence the total amount of flow of f' that enters S^{out} is strictly less than that of f_1. As the total amount of f_1 that enters S^{out} is the same as the total amount of f_1 that leaves S^{out}, this is a contradiction and shows that $S^{out} = \emptyset$, as we claimed.

Suppose next that S^{in} is nonvoid. Clearly, if arc $x = wv$ of $A \cup A'_1$ enters S^{in} then $f'(x) = c'(x)$, as otherwise $w \in S^{in}$ would hold. Thus the total amount of flow f' that leaves S^{in} is not less than the total amount of f_1 leaving S^{in}. Moreover, if arc $x = vu$ of A'_2 leaves S^{in} then $f'(x) = 0$, as otherwise $P(v, s)$ would block f'. For any $v \in S^{in}$, for the last arc of $P(v, s)$ is f'-unsaturated hence the total amount of flow of f' that leaves S^{in} is strictly less than that of f_1. This contradiction proves that $S^{in} = \emptyset$.

To finish the proof, unfortunately we have to bore the reader by repeating the two paragraphs above with the evident changes to prove the emptiness of T^{in} and T^{out}.

Assume that T^{in} is nonempty. By definition, if arc $x = wv$ of $A \cup A'_2$ enters T^{in} then $f'(x) = c'(x)$, as otherwise w would also belong to Tin. So the total amount of flow f' that enters T^{in} is at least as much as the total amount of flow f_2 that enters T^{in}. Observe that if least preferred arc $x = vu$ of A'_1 leaves T^{in} then $f'(x) = 0$, as otherwise $P(v, t)$ would be a blocking path to stable flow f'. Moreover, for any vertex v of S, the last arc a of $P(v, s)$ is f'-unsaturated, hence the total amount of flow of f' that leaves T^{in} is strictly less than that of f_2. As the total amount of f_2 that enters T^{in} is the same as the total amount of f_2 that leaves T^{in}, this is a contradiction and shows that $T^{in} = \emptyset$, as we claimed.

Suppose at last that T^{out} is nonvoid. Clearly, if arc $x = vw$ of $A \cup A'_2$ leaves T^{out} then $f'(x) = c'(x)$, as otherwise $w \in T^{out}$ would hold. Thus the total amount of flow f' that leaves T^{out} is not less than the total amount of f_2 leaving T^{out}. Moreover, if arc $x = uv$ of A'_1 enters T^{out} then $f'(x) = 0$, as otherwise $P(t, v)$ would block f'. If $v \in T^{out}$ then the first arc of $P(t, v)$ is f'-unsaturated hence the total amount of flow of f' that enters T^{out} is strictly less than that of f_1. This contradiction proves that $T^{out} = \emptyset$, hence flow $f = \overline{f'}$ satisfies (21) and (22) just as we claimed.

The integrality property in the last sentence of the theorem follows from the integrality property in the first part of Theorem 3.1. □

As we indicated, Theorem 5.1 does not generalize the switching property that Pym proved. It is folklore that with the help of the stable marriage theorem of Gale and Shapley [6], the linking theorem of path with this switching property can be proved. To do this, we regard paths of $\mathcal{P}$ as men, paths of $\mathcal{Q}$ as women and a common vertex of path p of $\mathcal{P}$ and path q of $\mathcal{Q}$ corresponds to an edge between man p and women q. (Note that parallel edges are possible.) Men prefer edges that correspond to vertices that are closer to the initial vertex of the corresponding path and women prefer those edges that correspond to a vertex closer to the terminal vertex of the path. It is straightforward to check that the edges of a stable matching correspond to a set of vertices that is a valid set of "switching vertices" to get node disjoint set $\mathcal{R}$ of paths.

Actually, the above proof can be generalized to prove a stronger form of Theorem 5.1. In the proof we need stable allocations instead of stable marriages. The sketch is as follows. Decompose flows f_1 and f_2 as a positive combination of unit flows along a directed path or along a directed cycle. Omit all cycles that do not contain s or t, and regard the remaining cycles as "paths". Let these paths be the vertices of our auxiliary graph G and let us have an edge between path p_1 of the decomposition of f_1 and path p_2 of the decomposition of f_2 if they have an inner vertex in common. Paths from f_1 prefer to switch earlier, and paths of f_2 prefer to switch later. Let the quota of each path be the linear coefficient from the decomposition, and let the capacity of the edges be infinity. A stable allocation for G tells us how should we reroute f_1 to f_2 to get flow f with a decomposition such that each path in the decomposing of f starts with some (possibly empty) path from the decomposition of f_1 and switches to some (possibly empty) path from the decomposition of f_2. The interested reader can work out the details for herself.

It is not clear whether there is a valid generalization of the Brualdi-Pym variant of the linking theorem to flows.

6 Conclusion

There are some well-known generalizations of network flows that are interesting from the point of view of stability. Luckily, some of these can be handled with the help of the construction we used to prove Theorem 3.1. If our "network" has unoriented edges, then the usual trick helps that substitutes each edge by two oppositely oriented arcs, each with the capacity of the corresponding edge. If some vertex v of the network has capacity then we can handle this by reducing the capacity of both vertices v^{out} and v^{in} from $q(v)$ to $c(v)$. The third usual generalization is where the network has more source and more terminal nodes. Here the usual trick works again, but there is an alternative approach. Namely, that the problem woth several sources and terminals can be regarded as a problem where there exist more than two vertices that does not obey the Kirchhoff law. As we have seen in the proof of Theorem 3.1, such vertices v can be modelled in such a way that in the graph G_D we do not have the edges $v^{out}v^{in}$ and $v^{in}v^{out}$. Any stable allocation of this sparser G_D determines a stable flow with more terminals. However, with this approach it is possible that we have a flow from some source node another source node. This weirdness can actually happen in our basic flow model as well: a stable flow might go from t to s and have a negative value. Can we avoid this? The answer is yes (up to some extent): one has to introduce an edge $s^{in}s^{out}$ for each source node s and edge $t^{out}t^{in}$ for each target node t. By this, we can make sure that each soruce node sends more (or equal amount of) flow than the amount it receives and no target node receives less flow than the amount it sends.

Summarizing the last observations we can say that if there are both edges $v^{out}v^{in}$ and $v^{in}v^{out}$ are present then vertex v obeys the Kirchoff law, if no edge is present then no Kirchoff law is required for v. If there is exactly one of these edges belong to G_D then depending on which edge is the one, v becomes a net sender or a net receiver of flow.

A circulation is a well-known notion closely related to flows. Rougly speaking, it is a flow without the two special nodes s and t, that is, beyond the capacity constraint, the Kirchhoff law has to hold for each vertex. Obviously, our approach can handle the stable circulation problem, if we have both parallel edges between v^{out} and v^{in} for each vertex v of D. However, it turns out that 0 is a stable circulation, which is somehow disturbing. It makes more sense to look for a strongly stable circulation that has no blocking path and no blocking cycle. But just like for flows, a strongly stable circulation might not exist. A natural problem is to find an efficient algorithm for constructing a strongly stable circulation (or flow) in a network.

Another direction of possible generalizations of stable flows is that we allow more complex preferences, e.g. ties in the preference lists. As the stable flow model is a genuine generalization of stable allocations that generalize stable matchings, each negative complexity result is valid for flows as well. However, it is interesting to observe that special exchange economy of housing markets can be formulated in our flow model with indifferences. If each vertex of D represents a player, each arc and each vertex has unit capacity, and arc uv means that player v prefers the house of player u to her own one, then we have preferences only on the arcs entering v and v is indifferent on the arcs leaving v. With this settings, core allocations of the housing market correspond bijectively to strongly stable circulations. This indicates that the top trading cycles algorithm of Gale might be useful to handle some stable flow problems where indifferences are present.

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