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Inductive Constructions in the Analysis of Two-Dimensional Rigid Structures

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Abstract

We give an overview of how inductive constructions of certain families of graphs can be used to characterize and analyze the generic behaviour of two-dimensional frameworks with respect to rigidity and global rigidity.

We also give a different proof for (a slightly stronger version of) a result of Servatius and Whiteley on the construction of minimally rigid mixed graphs.

Keywords: rigid graphs; globally rigid graphs; direction-length frameworks

1 Introduction

One of the most useful tools in the characterization of (globally) rigid graphs is an inductive construction. Such a construction can be used to prove that a family of graphs is (globally) rigid, provided that the operations used in the construction are known to preserve (global) rigidity. In this paper we will describe some examples when this proof method works and yields necessary and sufficient conditions for (global) rigidity.

A d-dimensional *bar-and-joint framework* is a pair (G, p), where G is a graph and $p : V \to \mathbb{R}^d$ is a map. Intuitively, we can think of (G, p) as a collection of bars and joints where vertices correspond to joints and each edge to a rigid bar joining its end-points. The framework is *rigid* in $\mathbb{R}^d$ if it has no continuous deformations in $\mathbb{R}^d$ that respect the length contraints on the edges. It is *globally rigid* in $\mathbb{R}^d$ if all frameworks (G, q) in $\mathbb{R}^d$ in which the edge lengths are the same as in (G, p) are congruent to (G, p).

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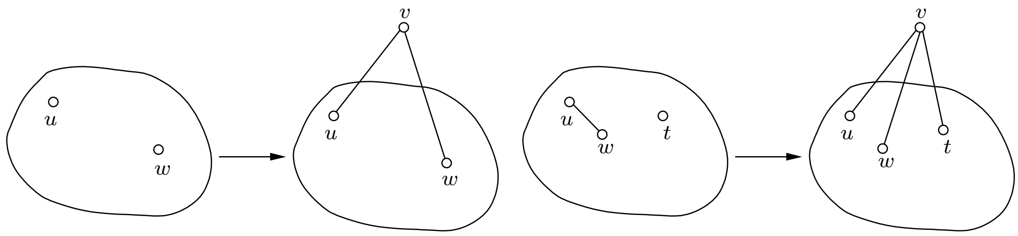

Figure 1: The extension operations.

We shall also consider other types of frameworks. In a *direction framework* each edge of G represents a direction constraint for the line containing its end-vertices. A *mixed framework* is defined on a graph with two types of edge labels and the edges may represent length as well as direction contraints, according to the labels. In a *body-bar framework* on a multigraph G each vertex corresponds to a rigid body and each edge to a rigid bar joining the bodies of its end-vertices. Rigidity and global rigidity in $\mathbb{R}^d$ can be defined analogously for each type of frameworks. It is known (with the exception of globally rigid mixed frameworks) that if the framework is *generic*, which means that there is no algebraic dependency between the coordinates of its points, then rigidity and global rigidity depends only on the graph of the framework. Thus we may speak about rigid or globally rigid graphs (when the framework type and the dimension are fixed). We refer the reader to [15] for the basic concepts and definitions of (globally) rigid graphs and frameworks.

2 Bar-and-joint frameworks

Let H be a graph. The operation 0-*extension* adds a new vertex v to G and two edges vu, vw with $u \neq w$. The operation 1-*extension* subdivides an edge uw of G by a new vertex v and adds a new edge vt for some $t \neq u, w$. An *extension* is either a 0-extension or a 1-extension, see Figure 1. Extensions are also called *Henneberg operations* in the literature.

Let $G = (V, E)$ be a graph. For a set $X \subseteq V$ let $i(X)$ denote the number of edges in $G[X]$, i.e. in the subgraph of G induced by X. We say that G is $(k, k+1)$-*sparse* if $i(X) \leq k|X| - (k+1)$ for all $X \subseteq V$ with $|X| \geq 2$. If, in addition, $|E| = k|V| - (k+1)$ holds then G is said to be $(k, k+1)$-*tight.* It is not difficult to see that if G is minimally rigid in $\mathbb{R}^2$ then G is $(2, 3)$-tight. The following result provides an inductive construction.

Theorem 2.1. *[10, 14] A graph is* $(2, 3)$*-tight if and only if it can be obtained from* K_2 *by a sequence of extensions.*

Since each of the two extension operations preserves (minimal) rigidity, Theorem 2.1 can be used to complete the characterization by showing that every $(2, 3)$-tight graph is minimally rigid.

Theorem 2.2. *[10] A graph* G *is minimally rigid in* $\mathbb{R}^2$ *if and only if* G *is* $(2, 3)$*-tight.*

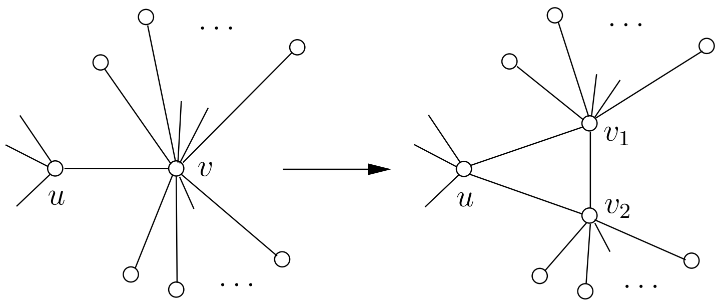

Figure 2: The plane vertex splitting operation on edge uv and vertex v.

As an immediate corollary we deduce that a graph is rigid if and only if it can be obtained from K_2 by extensions and edge additions.

There is a different inductive construction for $(2,3)$-tight graphs embedded in the plane. The construction uses (the topological version of) vertex splitting. Given a graph $G=(V,E)$, an edge $uv \in E$, and a bipartition F_1, F_2 of the edges incident to v in $G-uv$, the *vertex splitting* operation *on edge uv at vertex v* replaces the vertex v by two new vertices v_1 and v_2, replaces the edge uv by three new edges uv_1, uv_2, v_1v_2, and replaces each edge $wv \in F_i$ by an edge wv_i for $i \in \{1,2\}$, see Figure 2. When G is a plane graph and each of the edge sets F_1, F_2 form an interval in the natural cyclic ordering given by the embedding then we call the operation *plane vertex splitting*. It is easy to see that vertex splitting, when applied to a $(2,3)$-tight graph, yields a $(2,3)$-tight graph. On the other hand, the general version of vertex splitting may destroy planarity.

Theorem 2.3. *[3] A graph is a plane $(2,3)$-tight graph if and only if it can be obtained from an edge by plane vertex splitting operations.*

Theorem 2.3 can be used to give a different algorithmic proof for the fact that every planar minimally rigid graph can be embedded as a so-called pointed pseudo-triangulation. The original proof, due to Haas et al., used (a topological version of) Theorem 2.1.

We say that a graph $H=(V,E)$ is a *circuit* if it is minimally non-$(2,3)$-tight (or equivalently, if E is a circuit in the "rigidity matroid"). We say that G is *redundant* if it has at least one edge and each edge of G is in a circuit. A graph G is *redundantly rigid* if $G-e$ is rigid for all edges e of G. It follows that a graph G is redundantly rigid if and only if G is rigid and redundant. It is known that circuits are redundantly rigid graphs. See [6] for more details on the properties of circuits (which are also called 2-circuits, M-circuits, or generic cycles in the literature).

Hendrickson [5] proved that if G is a globally rigid graph in $\mathbb{R}^2$ on at least four vertices then G is 3-connected and redundantly rigid in $\mathbb{R}^2$. The following result gives an inductive construction for this family of graphs.

Theorem 2.4. *[6] Let G be a 3-connected graph which is redundantly rigid in $\mathbb{R}^2$. Then G can be obtained from K_4 by a sequence of 1-extensions and edge additions.*

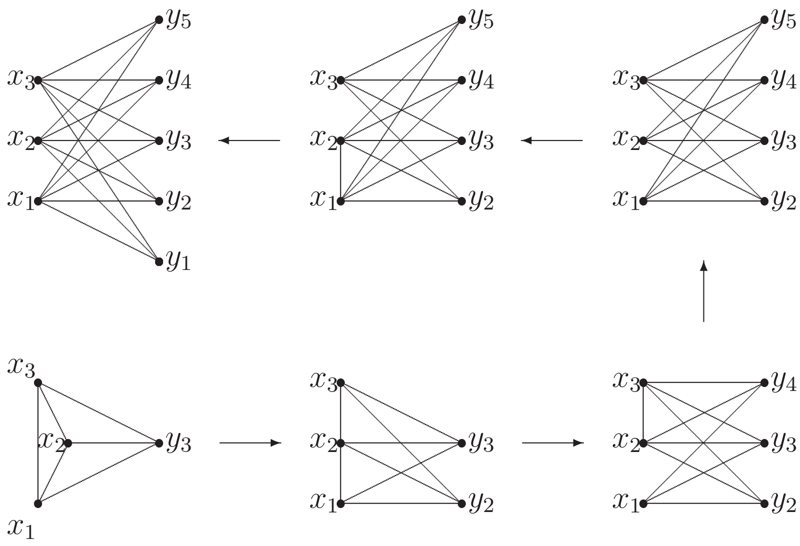

Figure 3: A construction of $K_{3,5}$ from K_4 using 1-extensions and edge additions.

We illustrate Theorem 2.4 by constructing $K_{3,5}$ from K_4, see Figure 3. Connelly [1] proved that each of the two operations in Theorem 2.4 preserves global rigidity in $\mathbb{R}^2$. Thus, by combining these results, we obtain a complete characterization.

Theorem 2.5. *[6] A graph G on at least four vertices is globally rigid in $\mathbb{R}^2$ if and only if G is 3-connected and redundantly rigid in $\mathbb{R}^2$.*

No similar inductive construction is known for redundantly rigid graphs. However, redundant graphs do have a constructive characterization, which was used in [9] to give an efficient combinatorial algorithm for finding a cable-strut labeling for a redundantly rigid graph for which the resulting tensegrity graph is rigid. Note that every 3-connected redundant graph is redundantly rigid, see [6].

Suppose that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs on disjoint vertex sets and $u_i v_i \in E_i$, $i = 1, 2$, are designated edges. Then the 2-*sum of* G_1 *and* G_2 is the graph obtained from $G_1 - u_1 v_1$ and $G_2 - u_2 v_2$ by identifying the pairs u_1, u_2 and v_1, v_2. The *merging* operation (along k vertices) is also applied to two graphs G_1, G_2 on disjoint vertex sets. It identifies k pairs of designated vertices from the two graphs.

Theorem 2.6. *[9] A graph G is redundant if and only if G can be obtained from disjoint K_4's by recursively applying 1-extensions or edge-additions within some connected component, 2-sums to two connected components, and merging components along at most two vertices.*

A recursive construction of a redundant graph can be obtained in polynomial time. The labeling algorithm is based on the fact that a rigid cable-strut labeling of a graph can easily be extended to a rigid labeling of a bigger graph obtained by some of the operations in Theorem 2.6.

3 Direction frameworks

Recall that in a direction framework each edge represents a direction constraint for the line containing its end-vertices. Whiteley [15] characterized the direction rigid graphs in $\mathbb{R}^d$ and pointed out that rigidity and global rigidity are the same for generic direction frameworks.

Theorem 3.1. *[15] A graph G is direction rigid in $\mathbb{R}^d$ if and only if $(d-1)G$, the graph obtained from G by replacing each edge e by $d-1$ copies of e, contains a $(d, d+1)$-tight spanning subgraph.*

Note that the characterization of 2-dimensional minimally direction rigid frameworks given by Theorem 3.1 is identical to the characterization of 2-dimensional minimally rigid bar-and-joint frameworks given in Theorem 2.2. In fact, in the 2-dimensional case, there is a simple transformation which shows that a graph G is minimally direction rigid if and only if it is minimally 'length-rigid'.

An inductive construction for the family of minimally direction rigid graphs is due to Frank and Szegő. The operation *pinching k edges (with z)* subdivides k edges by k new vertices, adds a new vertex z, and identifies the k subdividing vertices with z.

Theorem 3.2. *[4] A graph G is $(k, k+1)$-tight if and only if G can be constructed from an initial graph, consisting of two vertices and $k-1$ parallel edges connecting them, by the following operation: pinch $j \leq k-1$ edges with a new vertex z and then add $k-j$ new edges connecting z with existing vertices without creating k parallel edges.*

4 Mixed frameworks

A *mixed graph* is a graph together with a bipartition $D \cup L$ of its edge set. We refer to edges in D as *direction edges* and edges in L as *length edges*.

Minimally rigid mixed graphs in $\mathbb{R}^2$ were characterized by Servatius and Whiteley. For $G = (V; D, L)$ a mixed graph and $X \subseteq V$, let $E_D(X)$ and $E_L(X)$ denote the sets, and $i_D(X)$ and $i_L(X)$ the numbers, of direction and length edges, respectively, in $G[X]$.

Theorem 4.1. *[12] A mixed graph $G = (V; D, L)$ is minimally rigid if and only if $|D \cup L| = 2|V| - 2$ and, for all $X \subseteq V$ with $|X| \geq 2$,*

$$i(X) \leq 2|X| - 2, \tag{1}$$

and

$$i_D(X) \leq 2|X| - 3 \text{ and } i_L(X) \leq 2|X| - 3. \tag{2}$$

It is straightforward to use this result to obtain a characterization of rigid mixed graphs. The proof of Theorem 4.1 is based on a Henneberg type inductive construction for minimally rigid mixed graphs (and some lemmas which show that each of the operations preserves rigidity). We shall give a different proof for a slight extension of their construction result. The number of operations in our construction is smaller, which might be useful in possible applications (for example, some lemmas used in the original proof of Theorem 4.1 can be omitted).

Let G be a mixed graph. The operation 0-*extension* adds a new vertex v and two edges vu, vw such that no parallel edges of the same type may be created. The operation 1-*extension* (on edge uw and vertex z) for a mixed graph G deletes an edge uw and adds a new vertex v and new edges vu, vw, vz for some vertex $z \in V(G)$, such that at least one of the new edges has the same type as the deleted edge and, if $z = u$, then the two edges from z to u are of different type. These operations are sufficient to construct all minimally mixed rigid graphs G starting with a single vertex.

Theorem 4.2. *[12] Let G be minimally mixed rigid. Then G can be obtained from a vertex by* 0*-extensions and* 1*-extensions.*

A mixed graph is *independent* if it satisfies (1) and (2). Thus an independent mixed graph $G = (V; D, L)$ with $|D \cup L| = 2|V| - 2$ is *minimally mixed rigid*. Let $G = (V; D, L)$ be an independent mixed graph and $X \subseteq V$ with $|X| \geq 2$. Then X is *mixed critical* if $i_{D \cup L}(X) = 2|X| - 2$, *direction critical* if $i_D(X) = 2|X| - 3$ and $E_L(X) = \emptyset$, and *length critical* if $i_L(X) = 2|X| - 3$ and $E_D(X) = \emptyset$. We say that X is *pure critical* if X is either direction critical or length critical, and X is *critical* if X is either mixed critical or pure critical. We use $d(X, Y)$ to denote the number of edges of G joining $X - Y$ and $Y - X$ and put $d(X) = d(X, V - X)$. We use $d(X, Y, Z)$ to denote the number of edges of G which belong to $G[X \cup Y \cup Z]$ but not to $G[X] \cup G[Y] \cup G[Z]$. We shall need the following two lemmas on critical sets.

Lemma 4.3. *[7] Let $G = (V; D, L)$ be an independent mixed graph.*
(a) If X, Y are mixed critical sets with $X \cap Y \neq \emptyset$ then $X \cap Y$ and $X \cup Y$ are both mixed-critical and $d(X, Y) = 0$,
(b) If X, Y are direction (resp. length) critical sets with $|X \cap Y| \geq 2$ then either
(i) $d(X, Y) = 0$ and $X \cap Y$ and $X \cup Y$ are both direction (resp. length) critical, or
(ii) $d(X, Y) = 1$, $X \cup Y$ is mixed critical, and $i_D(X \cup Y) = 2|X \cup Y| - 3$ (resp. $i_L(X \cup Y) = 2|X \cup Y| - 3$) holds.
(c) If X is mixed critical and Y is pure critical with $|X \cap Y| \geq 2$ then $X \cup Y$ is mixed critical, $X \cap Y$ is pure critical and $d(X, Y) = 0$.
(d) If X is length critical and Y is direction critical with $|X \cap Y| \geq 2$ then $X \cup Y$ is mixed critical, $d(X, Y) = 0$, and $|X \cap Y| = 2$.

Lemma 4.4. *[7] Let $G = (V; D, L)$ be an independent mixed graph and let X, Y, Z be critical sets satisfying $|X \cap Y| = |Y \cap Z| = |Z \cap X| = 1$ and $X \cap Y \cap Z = \emptyset$.*
(a) If X is mixed critical then Y, Z are both pure critical, $X \cup Y \cup Z$ is mixed critical, and $d(X, Y, Z) = 0$.
(b) If X, Y, Z are direction (resp. length) critical then either
(i) $d(X, Y, Z) = 0$ and $X \cup Y \cup Z$ is direction (resp. length) critical, or
(ii) $d(X, Y, Z) = 1$, $X \cup Y \cup Z$ is mixed critical, and $i_D(X \cup Y \cup Z) = 2|X \cup Y \cup Z| - 3$ (resp. $i_L(X \cup Y \cup Z) = 2|X \cup Y \cup Z| - 3$) holds.

Let $G = (V; D, L)$ be a mixed graph. We call a vertex of degree three a *node* of G. A node is said to be *pure* if the three edges it is incident with are of the same type. Let $v \in V$ be a node. The *1-reduction* operation *at* v, *on edges* vu, vw deletes v and all edges incident with v, and adds a new edge uw. The type of the new edge is arbitrary, unless v is a pure node, in which case the type of uw must be the same as the type of v. The graph obtained by the operation is denoted by G_v^{uw}. Note that 1-reduction is the inverse operation of 1-extension. The 1-reduction is *inconsistent* if both vu and vw have the same type and uw has the opposite type, and otherwise it is *consistent*.

Let G be a minimally rigid mixed graph and let v be a node in G. A pair of edges uv, wv incident to v (and the corresponding 1-reduction) is said to be *suitable* if G_v^{uw} is a minimally mixed rigid graph. For $X \subseteq V$ let $N(X)$ denote the set of *neighbours* of X (that is, $N(X) := \{v \in V - X : uv \in D \cup L \text{ for some } u \in X\}$).

Lemma 4.5. *Let v be a vertex in a minimally rigid mixed graph $G = (V; D, L)$.*
(a) If $d(v) = 2$ then $G - v$ is a minimally rigid mixed graph.
(b) If $d(v) = 3$ then there is a consistent suitable 1-reduction at v.

Proof. Part (a) follows easily from the counts. To prove (b) consider a node v in G. Suppose that v is not length pure. Then a direction 1-reduction at v on a pair vu, vw is suitable if and only if G_v^{uw} is independent. It follows that the 1-reduction is not suitable if and only if there is a mixed critical set X with $u, w \in X$ and $v \notin X$ or there is a direction critical set Y with $u, w \in Y$ and $v \notin Y$.

Claim 4.6. *Suppose that v is a node, which is not length (direction) pure, and let X be a mixed critical set with $v \notin X$ and $|X \cap N(v)| = 2$. Then there is a suitable direction (resp. length) 1-reduction at v.*

Proof. Let $X \cap N(v) = \{u, w\}$. First observe that no mixed critical set X' can contain all neighbours of v, since $N(v) \subseteq X'$ would imply $i(X' + v) \geq 2|X'| - 2 + 3 = 2|X' + v| - 1$, contradicting the independence of G. Thus v has a neighbour t with $t \notin X$. For a contradiction suppose that the direction 1-reductions on edges vt, vu and vt, vw are both non-suitable. By using Lemma 4.3(a,b,c) and the fact that there is no mixed critical set X' with $N(v) \subseteq X'$ we can deduce that there exist direction critical sets Y, Z with $u, t \in Y$, $w, t \in Y$, and $|X \cap Y| = |X \cap Z| = |Y \cap Z| = 1$. Now Lemma 4.4(a) implies that $X \cup Y \cup Z$ is mixed critical, a contradiction (since $N(v) \subseteq X \cup Y \cup Z$). □

Claim 4.7. *Suppose that v is a node, which is not length pure, and that there is no mixed critical set X with $v \notin X$ and $|X \cap N(v)| = 2$. Then either there is a suitable direction 1-reduction at v or v is a mixed node and every length 1-reduction is suitable.*

Proof. Suppose that there is no suitable direction 1-reduction at v. Since there is no mixed critical set containing two neighbours of v, it follows that each pair of neighbours of v belongs to a direction critical set in $G - v$. Furthermore, Lemma 4.3(d) implies that no pair of neighbours of v belongs to a length critical set in $G - v$. This proves that every length 1-reduction at v gives rise to an independent graph and completes the proof when v is a mixed node.

So we may assume that v is a direction pure node. Then we must have $|N(v)| = 3$. Observe that no direction critical set Y' can contain all neighbours of v, since

$N(v) \subseteq Y'$ would imply $i_D(Y'+v) \geq 2|Y'| - 3 + 3 = 2|Y'+v| - 2$, contradicting the independence of G. Hence we can use Lemma 4.3(b) to deduce that there exist direction critical sets X, Y, Z in $G - v$, containing distinct pairs of neighbours of v, with $|X \cap Y| = |X \cap Z| = |Y \cap Z| = 1$ and $X \cap Y \cap Z = \emptyset$. Lemma 4.4(b) now implies that $X \cup Y \cup Z$ is direction critical, a contradiction. This proves that there is a suitable direction 1-reduction at v. □

We can now complete the proof of the lemma. First suppose v is pure. Without loss of generality v is direction pure. Then Claims 4.6 and 4.7 imply that there is a suitable direction 1-reduction at v. By definition, this 1-reduction is also consistent. Next suppose that v is a mixed node. By symmetry we may assume that v is incident with two direction edges vu, vw and one length edge vt. It follows from Claims 4.6 and 4.7 that either there is a suitable direction 1-reduction at v (which is also consistent, since there is only one length edge incident with v) or every length 1-reduction at v is suitable. Thus the 1-reduction at v on the pair vu, vt is consistent and suitable. □

We can now deduce the following strengthening of Theorem 4.2. The 1-extension is *consistent* if no direction (resp. length) edge uw is replaced by two length (resp. direction) edges vu, vw. Note that consistent 1-extension is the inverse of consistent 1-reduction.

Theorem 4.8. *Let G be minimally mixed rigid. Then G can be obtained from a vertex by 0-extensions and consistent 1-extensions.*

Proof. The result follows by observing that G must have a vertex v of degree two or three and then using the fact that there is a consistent suitable 1-reduction at v by Lemma 4.5, and applying induction on the number of vertices. □

The problem of characterizing when a generic mixed framework (G, p) is globally rigid is still an open problem and it is not known whether global rigidity of mixed frameworks is a generic property.

The following is a necessary condition for global rigidity, which is analogous to the '3-connectedness condition' of Theorem 2.5. It uses the following concept. Let G be a 2-connected mixed graph. A 2-*separation* of G is a pair of subgraphs G_1, G_2 such that $G = G_1 \cup G_2$, $|V(G_1) \cap V(G_2)| = 2$ and $V(G_1) - V(G_2) \neq \emptyset \neq V(G_2) - V(G_1)$. The 2-separation is *direction-balanced* if both G_1 and G_2 contain a direction edge. We say that G is *direction balanced* if all 2-separations of G are direction balanced. It was shown in [7] that every globally rigid mixed graph is direction balanced. Rigidity is also a necessary condition for global rigidity. Redundant rigidity, however, is no longer necessary.

We say that a mixed graph $G = (V; D, L)$ is a *mixed circuit* if $D \neq \emptyset \neq L$ and it is minimal with respect to violating (1) or (2) (or equivalently, if $D \cup L$ is a circuit in the "mixed rigidity matroid"). The smallest mixed circuits on three vertices are denoted by K_3^+ and K_3^-. They can be obtained from each other by interchanging the direction edges and the length edges. The first graph on Figure 4 is K_3^+. Theorem 4.1 can be used to characterize mixed circuits and, in particular, show that mixed

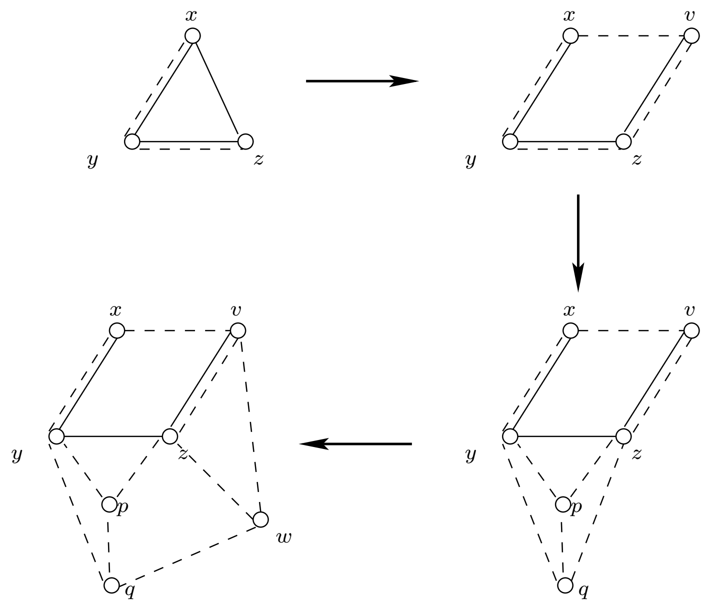

Figure 4: The inductive construction of a direction balanced (and hence globally rigid) mixed circuit. Solid (dashed) edges indicate length (resp. direction) constraints. The graph is obtained from K_3^+ by a 1-extension (adding vertex v), followed by a 2-sum with a direction-pure K_4 (on vertex set $\{y, z, p, q\}$) and then another 1-extension which adds w.

circuits are 2-connected and redundantly rigid. We recently showed that the direction balancedness condition is also sufficient to imply that mixed circuits are globally rigid.

We will need one further operation on mixed graphs. When $G_i = (V_i; D_i, L_i)$ is a mixed graph for each $i \in \{1, 2\}$ and uv has the same type in both G_1 and G_2, their *2-sum* is the mixed graph $(V_1 \cup V_2; (D_1 \cup D_2) - \{uv\}, (L_1 \cup L_2) - \{uv\})$.

Theorem 4.9. *[7] A mixed graph is a mixed circuit if and only if it can be obtained from K_3^+ or K_3^- by a sequence of 1-extensions and 2-sums with pure K_4's.*

When the mixed circuit is direction balanced, the 2-sum operation is more restricted.

Theorem 4.10. *[7] A mixed graph is a direction balanced mixed circuit if and only if it can be obtained from K_3^+ or K_3^- by a sequence of 1-extensions and 2-sums with direction-pure K_4's.*

The construction is illustrated by Figure 4. We also proved that each operation in Theorem 4.10, when applied to a mixed circuit, preserves global rigidity. More precisely, we showed in [8] that (i) if H is a globally rigid mixed graph with at least three vertices and G is obtained from H by a 1-extension on an edge uw, and $H - uw$ is rigid, then G is globally rigid, and (ii) if H is a globally rigid mixed graph and G is obtained from H by a 0-extension which adds a vertex v incident to two direction edges, then G is globally rigid. Note that a 2-sum with a direction-pure K_4 can be obtained by a 0-extension followed by a 1-extension. By combining these results we were able to characterize globally rigid mixed circuits.

Theorem 4.11. *[7] Let G be a mixed circuit. Then G is globally rigid if and only if G is direction balanced.*

5 Body-bar frameworks

A graph is *k-tree-connected* if it contains k edge-disjoint spanning trees. For this family of graphs Nash-Williams verified the following construction.

Theorem 5.1. *[11] A graph G is k-tree-connected if and only if G can be obtained from a vertex by edge additions and by pinching $i \leq k-1$ edges with a new vertex z and then adding $k-i$ new edges from z to existing vertices.*

Theorem 5.1 can be used to verify the following result of Tay.

Theorem 5.2. *[13] A multigraph G is body-bar rigid in $\mathbb{R}^d$ if and only if G is $\binom{d+1}{2}$-tree-connected.*

A graph $G = (V, E)$ is *highly k-tree-connected* if $G - e$ is k-tree-connected for all $e \in E$. The inductive construction of highly k-tree-connected graphs is due to Frank and Szegő.

Theorem 5.3. *[4] A graph G is highly k-tree-connected if and only if G can be obtained from a vertex by edge additions (which may be loops) and by pinching i edges ($1 \leq i \leq k-1$) with a new vertex z and adding $k-i$ new edges connecting z with existing vertices.*

It follows from Theorem 5.2 that a multigraph G is redundantly body-bar rigid if and only if it is highly k-tree-connected. In [2] we proved that the multigraph of a globally rigid generic body-bar framework is redundantly body-bar rigid. Furthermore, most of the pinching operations of Theorem 5.3 preserves body-bar global rigidity. Then we used Theorem 5.3, and some additional arguments, to obtain the following result.

Theorem 5.4. *[2] A multigraph G is globally body-bar rigid in $\mathbb{R}^d$ if and only if G is redundantly body-bar rigid in $\mathbb{R}^d$.*

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