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Local Edge-Connectivity Augmentation in Hypergraphs is NP-complete

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Abstract

We consider a local edge-connectivity hypergraph augmentation problem. Specifically, we are given a hypergraph $G = (V, E)$ and a subpartition of V. We are asked to find the smallest possible integer γ, for which there exists a set of size-two edges F, with $|F| = \gamma$, such that in $G' = (V, E \cup F)$, the local edge-connectivity between any pair of vertices lying in the same set in the subpartition is at least a given value k. Using a transformation from the bin-packing problem, we show that the associated decision problem is NP-complete, even when $k = 2$.
Keywords: Hypergraphs, edge-connectivity augmentation, NP-completeness.

1 Preliminaries

Connectivity augmentation is concerned with adding a set of new edges to a graph, digraph, or hypergraph G in order to satisfy a given requirement function that specifies the desired connectivity between every pair of vertices. We are usually concerned with minimising the number of new edges to be added. When the requirement function is constant over all pairs of vertices we refer to this problem as *global connectivity augmentation*, and otherwise as *local connectivity augmentation*.

The global edge-connectivity augmentation problems for graphs and digraphs (where one wants to make a graph or a digraph k-edge-connected) have been shown to be polynomially solvable. These results are due to Watanabe and Nakamura [18] for graphs, and Frank [6] for digraphs. The global vertex-connectivity augmentation

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problem was shown to be polynomially solvable for digraphs by Frank and Jordán [9], and, for a fixed value of k, for graphs by Jackson and Jordán [11]. Local edge-connectivity augmentation was also considered by Frank in [6], where he showed that the problem is polynomially solvable for graphs, but NP-hard for digraphs. Local vertex-connectivity augmentation is NP-hard for both graphs [15] and digraphs, see [6].

The complexity of the connectivity augmentation problem for hypergraphs depends on how we restrict the sizes of the edges to be added. Clearly, if there is no restriction, then the solution is simply to add edges of size $|V(G)|$ until the connectivity requirement function is satisfied. When we specify that the new edges should all have size two, global edge-connectivity augmentation is polynomially solvable by a result of Bang-Jensen and Jackson [1]. This result was extended to adding edges of any fixed size by T. Király [13]. If we allow the addition of edges of arbitrary size, but seek to minimise the sum of the sizes of the added edges, then the local edge-connectivity augmentation problem is polynomially solvable by a result of Szigeti [17]. In [2], Benczúr and Frank solved a special case of local augmentation problem by size-two edges, see later.

The result presented in this paper was announced at a workshop on graph connectivity augmentation held in Bonn in June 1999, and the first written version appeared in 2000 in [5]. As a consequence, researchers started to investigate special cases not covered by this hardness result and our result has been cited in several papers which have since appeared. Jordán and Szigeti [12] proved that if the starting hypergraph has only size-two and size-three edges then the local edge-connectivity augmentation problem is polynomially solvable in the following strong sense: given integers β and γ, it is decidable, whether a local edge-connectivity requirement can be satisfied by adding at most β size-two and at most γ size-three edges. The result of Szigeti [17] was subsequently extended independently by Cosh [5], Bernáth and T. Király [3] and Nutov [16] by showing that the minimum sum augmentation can be achieved using at most one edge of size bigger than two. This result has consequences for polynomial approximation, see the last section. For more results on connectivity augmentation and its algorithmic aspects, see the survey papers by Frank [7, 8] and Nagamochi [14], respectively.

The purpose of this note is to show that local edge-connectivity augmentation for hypergraphs, using edges of size two, is NP-hard. Before we can prove our complexity result, we need some notation.

A *hypergraph* is a pair of disjoint sets called vertices and edges (where the edge-set is allowed to be a multiset), together with an incidence relation that associates a subset of the vertices with each edge. We shall identify an edge with its corresponding subset of the vertex set. We denote an arbitrary hypergraph by $G = (V, E)$, where V is the vertex set and E is the edge set. The *size* of an edge e is $|e|$. If an edge has size two, say $\{x, y\}$, we use the usual graph notation xy to represent the edge. Let $e \in E$,

$x \in V$ and $X \subseteq V$. We say that e is *incident* with x when $x \in e$, and that e *intersects* X when $e \cap X$ is non-empty.

For $x \in V$ we use $d(x)$ to denote the number of edges which are incident with x. For $X \subseteq V$, the *degree* of X, denoted by $d(X)$, is the number of edges which intersect both X and $V - X$. As for graphs, we define a *path* in a hypergraph as an alternating sequence of distinct vertices and edges $v_1, e_1, v_2, e_2, \ldots e_{p-1}, v_p$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $i = 1, \ldots, p-1$. A (u,v)*-path* has end vertices u and v. A hypergraph is *connected* if there is a (u,v)-path for every pair of vertices u, v. We say a set $X \subseteq V$ *separates* vertices u and v, if $|X \cap \{u,v\}| = 1$. We use $\lambda_G(x,y)$ to denote the maximum number of edge-disjoint (x,y)-paths in G. When there is no confusion about which hypergraph we are referring to, we drop the subscript. By a version of Menger's Theorem $\lambda_G(x,y)$ is equal to the minimum value of $d(X)$ over all sets $X \subseteq V$ which separate x, y.

For a set $T \subset V$, we say that G is k*-edge-connected in* T when, for all pairs of vertices $x, y \in T$, we have $\lambda(x,y) \geq k$. We say a set X *separates* T when both $X \cap T$ and $T - X$ are non-empty, and a subpartition $\mathcal{P}$ *separates* T when every member of the subpartition separates T. By the above mentioned version of Menger's theorem, G is k-edge-connected in T if and only if $d(X) \geq k$ for every set $X \subset V$ separating T.

The general local edge-connectivity augmentation problem for hypergraphs can be stated as follows.

PROBLEM: Local Edge-Connectivity Augmentation in Hypergraphs **(LECA)**
Instance: A hypergraph $G = (V, E)$, a function $r : V^2 \to \mathbb{N}$ defined for each pair of vertices and an integer γ.

Question: Can we add γ size-two edges to G so that in the resulting hypergraph $\lambda(x,y) \geq r(x,y)$ for all pairs $x, y \in V$?

As noted above, Bang-Jensen and Jackson [1] determined the minimum number of size-two edges required to make any given hypergraph k-edge-connected and provided a polynomial algorithm for performing a minimum augmentation. In [2], Benczúr and Frank considered a slightly more local version: making a given hypergraph $G = (V, E)$ k-edge-connected in a specified set T, with $\emptyset \subset T \subset V$. Their paper contains a minimax result and a polynomial algorithm that determines the minimum number of size-two edges that must be added to G to satisfy this connectivity requirement.

In this paper, we consider the natural extension of their result in which we have pairwise disjoint sets $T_1, T_2, \ldots T_t$ to be made k-edge-connected. That is, we try to satisfy a local demand function, $r : V^2 \to \mathbb{N}$ where

$$r(x,y) := \begin{cases} k & \text{if } x, y \in T_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

The $k = 1$ case of this problem is fairly straightforward and is dealt with in [5]. Here, we consider the case when $k = 2$ and look at the associated decision problem.

PROBLEM: Subpartition Edge-Connectivity Augmentation **(SPCA)**
Instance I_{SPCA}**:** A $(k-1)$-edge-connected hypergraph $G = (V, E)$, a subpartition $T_1, T_2, \ldots T_t$ of V, and an integer γ.

Question: Can we add γ size-two edges to G so that in the resulting hypergraph, whenever x and y are both members of the same set T_i, $\lambda(x, y) \geq k$?

Using a transformation from the bin-packing problem, we show that SPCA is NP-complete for any $k \geq 2$, and hence that the general local edge-connectivity augmentation problem LECA is also NP-complete for hypergraphs.

2 Bin-Packing

Suppose we are given a multiset of "weights" and are asked to fit them into bins, each of which can only carry a certain total weight. It is sensible to ask, "What is the smallest number of bins required?" The decision problem associated with this (optimization) question is known to be strongly NP-complete. (See Garey and Johnson, [10].)

PROBLEM: Bin-Packing **(BP)**
Instance I_{BP}**:** A multiset of weights $W = \{w_1, w_2, \ldots, w_n\}$ with each $w_i \in \mathbb{N}$, a bin size b^* and an integer m such that $\sum_{w_i \in W} w_i \leq mb^*$.

Question: Is there a partition $W_1, \ldots, W_m$ of W such that the sum of the weights in each W_i is at most b^*?

The *strong* NP-completeness of BP means that the problem is NP-complete when the weights and bin size are stored as a string of 1's (that is, an integer n is represented by a list of n 1's) instead of the more usual binary string. This is called a *unary* encoding. For a more complete discussion of strong NP-completeness see [10].

For our purposes we require a slight refinement of the bin-packing problem. Namely, one in which we can vary the sizes of the bins, but only in such a way that the sum of the weights equals the sum of the bin sizes. We also include the condition that each weight and bin must have size at least three. We use this condition in our reduction for the hypergraph problem.

PROBLEM: Special Bin-Packing **(SBP)**
Instance I_{SBP}**:** A multiset of weights $W = \{w_1, w_2, \ldots, w_n\}$ with each $w_i \in \mathbb{N}$, a

multiset of bin sizes $B = \{b_1, b_2, \ldots, b_m\}$, such that every w_i and b_i is at least 3, $\sum_{w_i \in W} w_i = \sum_{b_i \in B} b_i$, and every number is given by a unary encoding.

Question: Is there a partition $W_1, \ldots, W_m$ of W such that $\sum_{w_i \in W_j} w_i = b_j$, for each $j = 1, \ldots, m$?

It is not difficult to show that SBP remains strongly NP-complete. The details are given in [5].

3 SPCA is NP-complete

We are now in a position to provide our main result. We show that SPCA is NP-complete using a transformation from SBP.

Given a decision problem PROB, let Y_{PROB} denote the set of instances of PROB for which the answer is "yes". Also, given a hypergraph G and a connectivity requirement, we say a set of edges F is an *augmenting set* when $G + F$ satisfies the requirement.

Theorem 3.1. *SPCA is NP-complete.*

Proof: SPCA is in NP since we can verify that an augmentation with γ edges satisfies the connectivity requirement by exhibiting edge-disjoint paths. To establish NP-completeness we start by describing how to construct an instance of SPCA from one for SBP. First we prove the case $k = 2$. Let I_{SBP} consists of a weight set $\{w_1, w_2, \ldots, w_n\}$ and a bin set $\{b_1, b_2, \ldots, b_m\}$ such that $\sum w_i = \sum b_i$ and all $w_i, b_i \geq 3$.

For each w_i, create a vertex set X_i with w_i vertices and for each b_i create a vertex set Y_i with $b_i + 1$ vertices (all these sets are pairwise disjoint). We form a hypergraph $G = (V, E)$ as follows. Let $V = X_1 \cup \ldots \cup X_n \cup Y_1 \cup \ldots \cup Y_m$. Let $E = \{e_0, e_1, e_2, \ldots, e_m\}$ where e_0 is incident with every vertex in $X_1 \cup X_2 \cup \ldots \cup X_n$ and exactly one vertex from each Y_i, calling this vertex y_i in each case, (note that y_i is not an additional vertex, rather it is a free choice from the existing $b_i + 1$ vertices in Y_i) and, for $i = 1, \ldots, m$, the edge e_i is incident with every vertex in Y_i and no others.

Now, let I_{SPCA} be the hypergraph $G = (V, E)$ above, let $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m$ be the subpartition, (actually this is a partition of V, but this makes no difference), and let $\gamma = \sum w_i$. Recall that, thanks to the strong NP-completeness of the bin-packing problem, the instance considered was given by unary encoding. Hence, the size of I_{SPCA} is bounded by a polynomial of the size of I_{SBP} and thus we have indeed performed a polynomial transformation.

We now show that $I_{SPCA} \in Y_{SPCA}$ if and only if $I_{SBP} \in Y_{SBP}$.

Claim 3.2. *If $I_{SBP} \in Y_{SBP}$, then $I_{SPCA} \in Y_{SPCA}$.*

Proof: Suppose that $I_{SBP} \in Y_{SBP}$ and let $W_1, W_2, \ldots, W_m$ be the solution partition. We can assume, by renumbering if necessary, that $W_1 = \{w_1, w_2, \ldots, w_s\}$. Then $\sum_{w_i \in W_1} = b_1$ and so $X_1 \cup X_2 \cup \ldots \cup X_s$ has $b_1 = |Y_1| - 1$ vertices. Thus we can form a bijection, f, between $X_1 \cup X_2 \cup \ldots \cup X_s$ and $Y_1 - y_1$ (where y_1 is the vertex incident with both e_0 and e_1), and let F_1 be the set of edges $\{xy : x \in (X_1 \cup X_2 \cup \ldots \cup X_s)$ and $y = f(x)\}$. Note that $|F_1| = b_1$.

Then we can quickly see that in $G+F_1$, if $Z \subseteq V$ separates any of $X_1, X_2, \ldots, X_s, Y_1$, it has degree at least 2, and hence G is 2-edge-connected "inside" these sets. Repeating this process for each of the "bins" leads to an augmenting set $F = F_1 \cup \ldots \cup F_m$ with $\sum b_i = \sum w_i = \gamma$ size-two edges. Thus $I_{SPCA} \in Y_{SPCA}$. □

Before dealing with the other direction, we introduce some notation. Let $X = X_1 \cup \ldots \cup X_n$, and $Y = (Y_1 - y_1) \cup \ldots \cup (Y_m - y_m)$. Then $|X \cup Y| = 2\gamma$.

Claim 3.3. *If $I_{SPCA} \in Y_{SPCA}$, then $I_{SBP} \in Y_{SBP}$.*

Proof: Suppose that $I_{SPCA} \in Y_{SPCA}$ and let F be an augmenting set for G with γ size-two edges. Note that every singleton set containing a vertex from $X \cup Y$ separates some X_i or Y_i, and has degree one in G. Thus, in $G + F$ each vertex in $X \cup Y$ is incident with at least one edge from F. Since $|X \cup Y| = 2|F|$, F is a perfect matching between the vertices of $X \cup Y$.

We now show that if $uv \in F$ and $u \in X$, then $v \in Y$. To see this we suppose $uv \in F$ and that both u and v are in X. Then u is a member of some X_i, and because $|X_i| \geq 3$ the set $Z = \{u, v\}$ separates X_i. But then in $G+F$, $d(Z) = 1$, because the only edges incident with u and v are uv and e_0. That is X_i is not 2-edge-connected in $G + F$, which contradicts the fact that F is an augmenting set for G.

We now wish to show that if $uv \in F$ with $u \in X_i$ and $v \in Y_j$, then every $xy \in F$ with $x \in X_i$ has $y \in Y_j$. So we suppose that this is false and let $Z = \{z \in X :$ there exists $zy \in F$ with $y \in Y_j\}$. Then our supposition implies that X_i is not contained in Z and moreover that $Z \cup Y_j$ separates X_i. But in $G + F$, $d(Z \cup Y_j) = 1$ and so $G + F$ is not 2-edge-connected in X_i. This is a contradiction.

Thus we have shown that every edge in F is from X into Y, and that for each X_i, there is a Y_j such that every edge $uv \in F$ with one end in X_i has the other end in Y_j. Hence there is partition $U_1, \ldots, U_m$ of the set $\{X_1, X_2, \ldots, X_n\}$ such that for every $1 \leq j \leq m$

$$\sum_{X_i \in U_j} |X_i| = |Y_j| - 1.$$

Therefore, if we let $W_j = \{w_i : X_i \in U_j\}$ for all $1 \leq j \leq m$, we have a solution partition of the weight set from I_{SBP}. That is $I_{SBP} \in Y_{SBP}$. □

Since SBP is NP-complete and since the above transformation from an instance of SBP to an instance of SPCA is polynomial, Claims 3.1.1 and 3.1.2 imply that SPCA is also NP-complete. In order to prove NP-completeness for general k, we use the same construction, but take $k-1$ parallel copies of each edge e_i.

□□□

We would like to point out that the above proof implies SPCA is NP-complete even when the starting hypergraph is connected.

Since SPCA is a special case of the general local edge-connectivity augmentation problem, LECA, we have the following Corollary.

Corollary 3.4. *LECA is NP-complete.* □

Our proof technique also gives the following result.

Theorem 3.5 (Nagamochi and Ishii[15]). *The following problem is NP-complete. Given a connected graph G, some pairs of vertices, and a number γ, decide whether γ edges can be added to G obtaining 2-vertex-connectivity between all prescribed pairs of vertices.*

Proof: We use almost the same construction, but instead of defining the sets $e_0, e_1, \dots, e_m$ as edges, we add new vertices $w_0, w_1, \dots, w_m$, and for each $0 \leq i \leq m$ we define size-two edges connecting w_i to each vertex in set e_i. The prescribed pairs $\{x, y\}$ are vertex-pairs lying in the same X_j or Y_j. □

4 Closing Remarks

(a) We may obtain an approximate solution to LECA using the result of Cosh, Bernáth and T. Király and Z. Nutov mentioned in the Preliminaries Section. Let $G = (V, E)$ be a connected hypergraph, $\alpha \geq 3$ be the maximum size of an edge of G, and $r : V^2 \to \mathbb{N}$ be an edge-connectivity requirement function. Szigeti [17] gave a minimax formula which determines $\min \sum_{e \in F} |e|$ over all augmenting edge-sets F for G. Cosh [5], Bernáth and Király [3] and Nutov [16] independently showed that this minimum may be attained by an augmenting edge-set F containing one edge e_{big} of size at most α and all other edges of size two. In addition, Bernáth and Király, and Nutov gave an explicit polynomial time algorithm for constructing such a set F. It is easy to see that if we replace the edge e_{big} of F by a tree of size-two edges on the same vertex set as e_{big}, then we obtain a solution to LECA which contains at most $\lfloor \frac{|e_{big}|}{2} \rfloor - 1 \leq \lfloor \frac{\alpha}{2} \rfloor - 1$ edges more than in the optimal augmenting set consisting of size-two edges. Nutov also observed that the algorithm is 7/4-approximating.

(b) András Frank has asked whether there is a polynomial algorithm for solving the following special case of the hypergraph local edge-connectivity augmentation problem.

Given hypergraph $H = (V, E)$, and a collection of sets, $T = \{T_1, \dots, T_k\}$ such that $T_k \subseteq T_{k-1} \subseteq \dots \subseteq T_1 \subseteq V$, what is the smallest number of size two edges we must add to satisfy the requirement function $r : V^2 \to \mathbb{N}$ where $r(x, y)$ is the largest i, such that $x, y \in T_i$?

Like SPCA, this problem is another special case of LECA which would extend the above mentioned result of Benczúr and Frank, but the nesting of the sets may make progress possible. Note, however, that the problem cannot be solved by simply using the algorithm of [2] to first find an optimal augmentation $G + F_1$ for T_1, and then extending it to an optimal augmentation $G + F_1 + F_2$ for T_2, and so on. This follows from an example of Cheng and Jordán [4] which shows that such a sequential augmentation need not yield the optimal solution even when $k = 2$ and $T_1 = T_2 = V$.

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