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Rigid and Globally Rigid Graphs with Pinned Vertices

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Abstract

We consider rigid and globally rigid bar-and-joint frameworks (resp. graphs) in which some joints (resp. vertices) are pinned down and hence their positions are fixed. We give an overview of some old and new results of this branch of combinatorial rigidity with an emphasis on the related optimization problems.

In one of these problems the goal is to find a set P of vertices of minimum total cost for which the positions of all vertices become uniquely determined when P is pinned down. For this problem, which is motivated by the localization problem in wireless sensor networks, we give a constant factor approximation algorithm.

1 Introduction

A *bar-and-joint framework* (or simply *framework*) (G, p) in d-space is a graph $G = (V, E)$ and a map $p : V \to \mathbb{R}^d$. We also say that (G, p) is a d-dimensional *realization* of G. We can think of the edges and vertices of G in the framework as rigid (fixed length) bars and universal joints, respectively. An *infinitesimal motion* is a map $x : V \to \mathbb{R}^d$ satisfying

$$(p(v_i) - p(v_j))(x(v_i) - x(v_j)) = 0$$

for all edges $v_iv_j \in E$. The initial velocities obtained by differentiating a smooth motion of the (vertices of the) framework which preserves the edge lengths give rise to an infinitesimal motion of (G, p). The *rigidity matrix* of the framework (G, p) is the matrix $R(G, p)$ of size $|E| \times d|V|$, where, for each edge $e = v_iv_j \in E$, in the row corresponding to e, the entries in the two columns corresponding to vertices i and j contain the d coordinates of $(p(v_i) - p(v_j))$ and $(p(v_j) - p(v_i))$, respectively, and the remaining entries are zeros.

Example. The rigidity matrix of the framework of Figure 1 is as follows. The rows correspond to edges ab, bc, ca, cd, in this order, and consecutive pairs of columns correspond to vertices a, b, c, d.

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$$\begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \end{pmatrix}$$

Thus x (viewed as a vector in $\mathbb{R}^{d|V|}$) is an infinitesimal motion if and only if $R(G,p)x = 0$. Each translation and rotation of $\mathbb{R}^d$ gives rise to a smooth motion of (G,p) and hence to an infinitesimal motion of (G,p). These rigid motions of $\mathbb{R}^d$ give rise to a subspace of dimension $\binom{d+1}{2}$ in the null-space of $R(G,p)$. Hence

Lemma 1.1. *[31, Lemma 11.1.3] Let (G,p) be a framework in $\mathbb{R}^d$. Then*

$$\textit{rank}\, R(G,p) \leq S(n,d), \tag{1}$$

where $n = |V(G)|$ and

$$S(n,d) = \begin{cases} nd - \binom{d+1}{2} & \textit{if } n \geq d+2 \\ \binom{n}{2} & \textit{if } n \leq d+1. \end{cases}$$

We say that a framework (G,p) is *infinitesimally rigid* in $\mathbb{R}^d$ if the rank of its rigidity matrix $R(G,p)$ is maximum, i.e. if equality holds in (1). A framework is *rigid* if it has no non-trivial smooth motions. Thus infinitesimal rigidity is a sufficient condition for rigidity. It is known that for “generic” frameworks the two notions are the same. We refer the reader to [10, 31, 32] for more details on the theory of rigid frameworks.

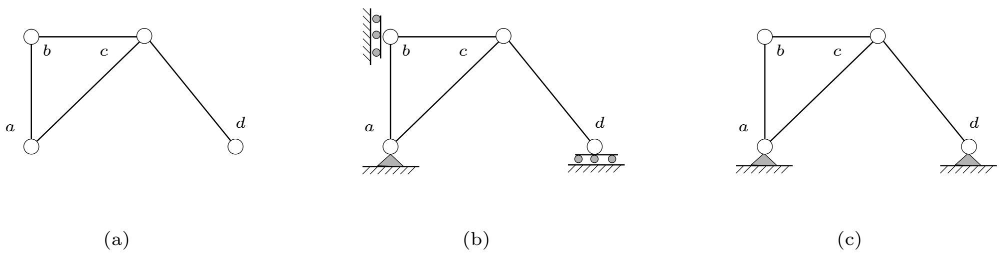

Figure 1: A framework in $\mathbb{R}^2$ on four vertices (left). The coordinates of the vertices are as follows: $p(a) = (0,0), p(b) = (0,1), p(c) = (1,1), p(d) = (2,0)$. Since $2|V| - \textit{rank}R(G,p) = 4$, to fix the framework one needs tracks of co-dimension four in total, which can be achieved by two one-dimensional tracks and a pin (middle) or two pins (right).

2 Rigid frameworks with pinned vertices

Let $G = (V,E)$ be graph and consider a d-dimensional realization (G,p) of G. We may fix (G,p) in $\mathbb{R}^d$ by restricting the infinitesimal motions of its vertices to given subspaces of $\mathbb{R}^d$. Suppose that for all vertices $v \in V$ we are given a subspace $U(v) \subseteq \mathbb{R}^d$, generated by a subset of the standard basis of $\mathbb{R}^d$. We call $U(v)$ the *track* of v and we say that (G, p) is *fixed* by the given set of tracks if the only infinitesimal motion x of (G, p) satisfying $x(v) \in U(v)$ for all $v \in V$ is the zero vector $x = 0$. In most cases we shall be interested in the special case when each track is either zero- or d-dimensional. We say that $P \subseteq V$ is a *pinning set* if (G, p) is fixed by the tracks $U(v) = \{0\}$ if $v \in P$, $U(v) = \mathbb{R}^d$ if $v \notin P$. We also say that the vertices in P are *pinned down*, or that each vertex of P is a *pin*.

The following lemma establishes the connection between tracks (pins) that fix a framework and its rigidity matrix (see also [26, Statement 8.2.1]). Note that each track $U(v)$ of dimension k, $0 \le k \le d$, corresponds naturally to a subset of size k of the d columns of the rigidity matrix which belong to v.

Lemma 2.1. *Let (G, p) be a framework in $\mathbb{R}^d$, let $U = (U(v) : v \in V)$ be a family of tracks, and let R_U be the matrix consisting of all columns of $R(G, p)$ which correspond to the tracks $U(v)$, $v \in V$. Then*
(i) U fixes (G, p) if and only if the columns of R_U are linearly independent,
(ii) P is a pinning set if and only if the $d|V - P|$ columns of $R(G, p)$ indexed by $V - P$ are linearly independent.

One may ask for an *optimal family of tracks* that fixes a given framework by using the least possible total restriction, i.e. an assignment $U = (U(v), v \in V)$ for which U fixes (G, p) and

$$\sum_{v \in V} (d - \dim U(v))$$

is minimum. By Lemma 2.1(i) an optimal family of tracks is easy to find by using a greedy algorithm to identify a maximum size independent set of columns in $R(G, p)$. Furthermore, the optimum is unchanged if we restrict the matrix to a maximum size set of independent rows (or if we consider the corresponding subgraph of G). It is also clear that

$$\min\{\sum_{v \in V} (d - \dim U(v)) : U \text{ fixes } (G, p)\} = d|V| - rank R(G, p).$$

We obtain a much more difficult problem if we impose restrictions on the dimension of the tracks. This is the case, for example, when we consider pinning sets. The *pinning number*, $pin_d(G, p)$, of (G, p) is defined to be the size of a smallest pinning set for (G, p). For $d = 2$ Lemma 2.1(ii) implies that the smallest pinning set problem can be formulated as a matroid matching problem in a linearly represented matroid and hence $pin_2(G, p)$ can be computed in polynomial time by using the algorithm of Lovász [21]. A combinatorial formula for $pin_2(G, p)$ was also given by Lovász [22]. Mansfield [25] proved that the problem of computing $pin_3(G, p)$ for a framework (G, p) is NP-hard.

On the other hand, a recent result of Szabó [30] shows that there exist tractable cases even in dimensions larger than two.

Theorem 2.2. *[30] The following problem is polynomial time solvable. Given a framework (G, p) in $\mathbb{R}^3$ and a partition $V = V_1 \cup V_2$, find a family $U = (U(v) : v \in V)$ of*

tracks minimizing $\sum_{v \in V}(d - \dim U(v))$ *such that* U *fixes* (G, p) *and*
(i) $\dim U(v) \in \{0, 1, 3\}$ *for* $v \in V_1$, *and*
(ii) $\dim U(v) \in \{0, 2, 3\}$ *for* $v \in V_2$.

The algorithm in [30] also uses Lovász' matroid matching algorithm as a subroutine.

If we are also given a cost function on the vertices, we may look for a pinning set of minimum total cost. Baudis et al [2] proposed an approximation algorithm for this more general problem. For d-dimensional frameworks the approximation guarantee is $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{d}$. Their algorithm is based on a general result about minimum cost spanning sets in d-polymatroids, see also Section 8.

3 Rigid graphs with pinned vertices

The rigidity matrix of a d-dimensional framework (G, p) defines the *rigidity matroid* of (G, p) on the ground set E where a set of edges $F \subseteq E$ is independent if and only if the rows of the rigidity matrix indexed by F are linearly independent. A framework (G, p) is *generic* if the set of coordinates of the points $p(v)$, $v \in V$, is algebraically independent over the rationals. Thus, since the entries of the rigidity matrix are polynomial functions with integer coefficients, any two generic d-dimensional frameworks (G, p) and (G, q) have the same rigidity matroid. We call this the *d-dimensional rigidity matroid* $\mathcal{R}_d(G)$ of the graph G. We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$. We say that a graph $G = (V, E)$ is *generically infinitesimally rigid*, or simply *rigid*, in $\mathbb{R}^d$ if $r_d(G) = S(n, d)$. We say that a graph $G = (V, E)$ is *independent* in $\mathbb{R}^d$ if E is independent in $\mathcal{R}_d(G)$. It is not difficult to see that $\mathcal{R}_1(G)$ is the cycle matroid of G. It remains an open problem to find good characterizations for independence or, more generally, the rank function in the d-dimensional rigidity matroid of a graph when $d \geq 3$.

Similarly, any two generic d-dimensional frameworks on G have the same pinning number. Thus we may define the *pinning number of* G, $pin_d(G)$, as the pinning number of (G, p) of any generic framework (G, p) in $\mathbb{R}^d$. It is easy to see that $pin_d(G) \leq pin_d(G, p)$ for all frameworks (G, p). The next lemma implies that computing the pinning number of G is the same as finding a smallest complete graph whose addition to G makes it rigid. For a set $P \subseteq V(G)$ let $G + K(P)$ denote the graph obtained from G by joining all pairs of non-adjacent vertices of P.

Lemma 3.1. *Let* $G = (V, E)$ *be a graph and* $P \subseteq V$ *with* $|P| \geq d$. *Let* (G, p) *be a generic realization of* G *in* $\mathbb{R}^d$. *Then* P *is a pinning set for* (G, p) *if and only if* $G + K(P)$ *is rigid in* $\mathbb{R}^d$.

Proof: Let $G' = G + K(P)$. First suppose that G' is rigid and consider the rigidity matrix $R(G', p)$. Since G' is rigid, the only solutions u to the equation $R(G', p)u = 0$ are from rigid congruences of $\mathbb{R}^d$. Thus, since (G', p) is generic, each non-zero solution leaves at most $(d-1)$ vertices fixed i.e. has at most $(d-1)$ zero entries. Suppose $R(G[V-P], p)$ has linearly dependent columns. Then we can find a non-zero solution u' to $R(G[V-P], p)u' = 0$. By extending u' to u by putting 0 in the components corresponding to P we obtain a non-zero solution to $R(G',p)u = 0$ with at least $|P| \geq d$ zeros, a contradiction. Thus P is a pinning set by Lemma 2.1(ii).

Now suppose that P is a pinning set and order the columns of $R = R(G',p)$ so that the columns of P come first and the rows of $E'' = E(G'[P])$ come first. (Then the upper right quarter is 0.) Hence $r(R) \geq r(R[P,E'']) + r(R[V-P, E-E'']) = d|P| - \binom{d+1}{2} + d|V-P| = d|V| - \binom{d+1}{2}$ (by using Lemma 2.1(ii) and that $G'[P]$ is rigid and $|P| \geq d$). Thus G' is rigid. •

Next we show that in the pinning problem we may assume that G is independent.

Lemma 3.2. *Let $F \subseteq E$ be a maximal edge set of $G = (V,E)$ for which $H = (V,F)$ is independent in $\mathcal{R}_d$. Then*
(i) each pinning set of G is a pinning set of H,
(ii) $pin_d(H) = pin_d(G)$.

Proof: To prove (i) suppose, for a contradiction, that there exists a pinning set P of G for which $H + K(P)$ is not rigid. Since $G + K(P)$ is rigid, we have $r_d(G+K(P)) > r_d(H+K(P))$, which implies that there is an edge $e \in E+E(K(P))-(F+E(K(P))) = E-F$ for which $F+e$ is independent, contradicting the maximality of F. This proves (i), from which (ii) follows immediately. •

It follows from the observations above that the pinning problem in graphs (or in generic frameworks) can be attacked by purely combinatorial methods provided good characterizations for independent and rigid graphs are available. This is the case when $d = 2$ and we shall discuss this approach in the 2-dimensional case in the forthcoming sections.

Mansfield [25] proved that the problem of computing $pin_3(G)$ for a graph G is NP-hard (see also [7] for a different proof), so the pinning problem in higher dimensions seems untractable. The following related result, however, might be useful in a different context as it points to a connection between high connectivity and rigidity in 3-space. It may be considered as a first step towards the Lovász-Yemini conjecture [23], which asserts that sufficiently highly connected graphs are rigid in 3-space (and hence their pinning number is three).

Theorem 3.3. *[14] Let $G = (V,E)$ be a 10-connected graph. Then $pin_3(G) \leq \frac{3|V|}{4} + 4$.*

4 The two-dimensional rigidity matroid

In the rest of the paper we will be concerned with the case when $d = 2$ and suppress the subscript d accordingly. In this section we first describe the characterization of independent and rigid graphs and prove some additional structural results which may also be useful in the solution of the pinning problem. For $X \subseteq V$ let $E_G(X)$ denote the set, and $i_G(X)$ denote the number of edges in $G[X]$, that is, in the subgraph induced by X in G. We say that a graph G is *sparse* if $i_G(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$. It is easy to show, by using the 2-dimensional case of Lemma 1.1, that independent graphs are sparse. Laman [18] proved that this necessary condition is also sufficient.

Theorem 4.1. *[18] A graph $G = (V, E)$ is independent if and only if G is sparse.*

A *cover* of $G = (V, E)$ is a collection $\mathcal{X} = \{X_1, X_2, ..., X_t\}$ of subsets of V, each of size at least two, such that $\cup_1^t E(X) = E$. The cover is said to be *thin* if $|X_i \cap X_j| \leq 1$ for all $i \neq j$. The *value* $val(\mathcal{X})$ of the cover is $\sum_{i=1}^{t}(2|X_i| - 3)$.

Let $\mathcal{X}$ be a thin cover of G and let $F \subseteq E$ be a set of edges for which $H = (V, F)$ is sparse. Then we have $|F \cap E_G(X_i)| \leq 2|X_i| - 3$ for all $1 \leq i \leq t$. Thus

$$|F| \leq val(\mathcal{X}). \tag{2}$$

We define a *rigid component* of a graph $G = (V, E)$ to be a maximal rigid subgraph of G. By the "plane gluing lemma" (see [31, Lemma 3.1.4]), which says that the union of two rigid graphs with at least two vertices in common is rigid, it follows that the vertex sets of the rigid components form a thin cover of G. In a sparse graph H we call a set $X \subseteq V(H)$ *critical* if $i_H(X) = 2|X| - 3$ holds. It follows from the gluing lemma that if $X, Y \subset V(H)$ are critical sets in H with $|X \cap Y| \geq 2$ then $X \cup Y$ is also critical (see also [13, Lemma 2.3]).

Lemma 4.2. *Let $G = (V, E)$ be a graph, let $F \subseteq E$ be a maximal edge set in G for which $H = (V, F)$ is sparse. Then the family $\mathcal{X} = \{X_1, X_2, ..., X_t\}$ of maximal critical sets in H satisfies that*
(a) $\mathcal{X}$ is a thin cover of G with $|F| = val(\mathcal{X})$,
(b) $\mathcal{X}$ is equal to the family of vertex sets of the rigid components of G.

Proof: (a) The maximality of the critical sets implies that $|X_i \cap X_j| \leq 1$ for all $1 \leq i < j \leq t$. Since every single edge of F induces a critical set, it follows that $\mathcal{X} = \{X_1, X_2, ..., X_t\}$ is a thin cover of H. Thus

$$|F| = \sum_{1}^{t} |E_H(X_i)| = \sum_{1}^{t}(2|X_i| - 3).$$

To complete the proof we show that $\mathcal{X}$ is a cover of G as well. Choose $uv \in E - F$. Since F is a maximal sparse subset of E, $F + uv$ is not sparse. Thus there exists a set $X \subseteq V$ such that $u, v \in X$ and $i_H(X) = 2|X| - 3$. Hence X is a critical set in H. This implies that $X \subseteq X_i$ and hence $uv \in E_G(X_i)$ for some $1 \leq i \leq t$.

(b) Clearly, $G[X_i]$ is rigid for all $1 \leq i \leq t$ by Theorem 4.1. Suppose that $H[C]$ is not critical, where C is the set of vertices of some rigid component of G. Thus $|J| \leq 2|C| - 4$, where $J = E(H[C])$. Since $G[X_i]$ is rigid, it follows from the gluing lemma that $\mathcal{X}' = \{X_i \in \mathcal{X} : |X_i \cap C| \geq 2\}$ is a thin cover of $G[C]$ with $|J| = val(\mathcal{X}')$. Thus we can use (2) to deduce that for any subset $F' \subseteq E(G[C])$ which induces a sparse subgraph on vertex set C we have $|F'| \leq val(\mathcal{X}') = |J| \leq 2|C| - 4$, contradicting the fact that $G[C]$ is rigid. This completes the proof of (b). •

Lemma 4.2(a) shows that for the edge set of a maximal sparse subgraph and for its maximal critical sets we have equality in (2). This implies the following rank formula of the rigidity matroid, due to Lovász and Yemini, and shows that a maximum size sparse edge set can be found greedily.

Theorem 4.3. *[23] Let $G = (V, E)$ be a graph. Then*

$$r(G) = \min\{val(\mathcal{X}) : \mathcal{X} \text{ is a thin cover of } G\}.$$

A maximum size sparse edge set (and the rigid components of a graph) can be found in time $O(n^2)$, see e.g. [4].

We may simplify the min-max formula of Theorem 4.3 when the graph is obtained from a sparse graph by 'pinning' a set of vertices. For a set $X \subseteq V$ let $e(X)$ denote the number of edges with at least one end-vertex in X.

Lemma 4.4. *Suppose that $G = (V, E)$ is a sparse graph and let $P \subseteq V$ with $|P| \geq 2$. Let $G' = G + K(P)$. Then*

$$r(G') = \min_{P \subseteq Z} 2|Z| - 3 + e(V - Z).$$

Proof: Let $Z \subseteq V$ with $P \subseteq Z$ and consider the thin cover $\mathcal{Z} = \{Z \cup \{\{u, v\} : uv \in E - E(Z)\}\}$ of G'. Then $r(G') \leq val(\mathcal{Z}) = 2|Z| - 3 + e(V - Z)$.

To see that equality holds for some $Z \subseteq V$ choose a maximal edge set F in G' for which $H = (V, F)$ is sparse and P is a critical set in H. Such an F can be constructed by extending the edge set F' of a minimally rigid subgraph of the complete graph $G'[P]$. Let $\mathcal{X}$ be the family of maximal critical sets of H. By Lemma 4.2(a) and Theorem 4.3 we have $r(G') = val(\mathcal{X})$. Since P is critical in H, there is a set $Z \in \mathcal{X}$ with $P \subseteq Z$. Thus, since G is sparse and all edges of $K(P)$ are covered by Z, we have $i_G(X) = i_H(X) = 2|X| - 3$ for all $X \in \mathcal{X} - Z$. Hence $r(G') = val(\mathcal{X}) = 2|Z| - 3 + \sum_{X \in \mathcal{X} - Z} i_G(X) = 2|Z| - 3 + e(V - Z)$, which completes the proof. •

5 Optimal families of tracks and smallest pinning sets

Let $G = (V, E)$ be a graph. First consider the problem of finding an optimal family of tracks, $U = (U(v) : v \in V)$, which fixes (G, p) for a generic realization of G in $\mathbb{R}^2$. As we have observed earlier, we may assume that G is independent (or equivalently, that G is sparse). Thus $|E| = 2|V| - k$ for some integer $k \geq 3$. It is also clear that $\sum_{v \in V}(2 - \dim U(v)) = k$ for an optimal family of tracks. The following algorithm, due to Lee et al. [19], determines an optimal family of tracks in $O(n^2)$ time. It uses $k - 2$ one-dimensional tracks (also called *sliders*) and one pin to fix (G, p). (For the remaining vertices the tracks are two-dimensional.)

The algorithm works as follows. First identify the rigid components of G. Mark one of the components, say C, as the *base*. For some edge uv in C assign a pin to

u and a slider to v. This fixes the base. Then repeat the following until one rigid component remains: pick an edge ij which leaves the base and assign a slider to j. Update G by adding a new edge jk, where ik is an edge in the base. Replace the base C by the rigid component of the updated graph containing uv.

The correctness of this algorithm follows from the fact that if C' is a rigid component that shares vertex i with C then the only motion of C' with respect to C is a rotation about i. Since the framework is generic, assigning a slider to j eliminates this motion and hence the distance between j and k becomes fixed. Thus every iteration increases the rank by one and therefore the algorithm will terminate with a rigid graph after adding at most $k-3$ sliders (not counting the pin and the slider added to fix the original base). The algorithm, when applied to (a generic realization of) the graph of Figure 1(a), may give the family of tracks shown by Figure 1(b).

We remark that combinatorial characterizations for the generic rigidity of bar-and-slider frameworks (which are bar-and-joint frameworks equipped with sliders at given joints) have been given in [19], and also in [17], where the authors consider the version in which the directions of the slider lines are also given.

Next consider the pinning problem.

Lemma 5.1. *[7] Let* $G=(V,E)$ *be a sparse graph and let* $P \subseteq V$ *with* $|P| \geq 2$. *Then* P *is a pinning set for* G *if and only if* $2|X| \leq e(X)$ *for all* $X \subseteq V-P$.

Proof: Suppose, for a contradiction, that P is a pinning set and $2|X| > e(X)$ for some $X \subseteq V-P$ and let $Z = V-X$. Then $\mathcal{X} = \{Z \cup \{\{u,v\} : uv \in E - E(Z)\}\}$ is a thin cover of $G+K(P)$ with $val(\mathcal{X}) \leq 2|Z|-3+e(X) < 2|V|-3$. Thus, by Theorem 4.3, $G+K(P)$ is not rigid. Hence P is not a pinning set by Lemma 3.1, a contradiction.

Now suppose $2|X| \leq e(X)$ for all $X \subseteq V-P$. It follows from Lemma 4.4 that there is a thin cover $\mathcal{X}$ of $G+K(P)$ with $P \subseteq Z$ for some $Z \in \mathcal{X}$ and $r(G+K(P)) = val(\mathcal{X}) = 2|Z|-3+e(V-Z)$. Since $e(V-Z) \geq 2|V-Z|$ this gives $val(\mathcal{X}) = 2|V|-3$. Hence $G+K(P)$ is rigid and, by Lemma 3.1, P is a pinning set. •

Thus finding a smallest pinning set is equivalent to finding a largest set $Y \subseteq V$ for which $e(X) \geq 2|X|$ for all $X \subseteq Y$. This can be formulated as a matching problem in an auxiliary graph (see Fekete [7]) and can be solved in $O(n^2)$ time. Fekete [7] also provides a min-max formula for $pin_2(G)$. Makai and Szabó [24] deduce this formula by using polymatroidal methods.

We note that Servatius, Shai, and Whiteley [28] consider a different version of the pinning problem and provide a characterization and a decomposition result for the so-called pinned isostatic graphs.

6 The network localization problem

In the network localization problem the locations of some nodes (called anchors) of a network as well as the distances between some pairs of nodes are known, and the goal is to determine the location of all nodes. This is one of the fundamental algorithmic problems in the theory of wireless sensor networks and has been the focus of a number of recent research articles and survey papers, see for example [1].

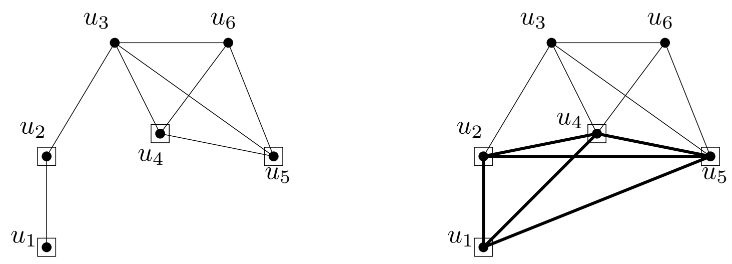

Figure 2: The distance graph and the grounded graph of a network on six nodes, including four anchor nodes. The anchor nodes are in boxes. The network is uniquely localizable since it has at least three anchors and its grounded graph is globally rigid. This is a smallest anchor set which can guarantee unique localizability for the given set of distances.

A natural additional question is whether a solution to the localization problem is unique. The network, with the given locations and distances, is said to be *uniquely localizable* if there is a unique set of locations consistent with the given data. As we shall see, the unique localizability of a two-dimensional network, whose nodes are in generic position, can be characterized by using results from graph rigidity theory. In this case unique localizability depends only on the combinatorial properties of the network: it is determined completely by the *distance graph* of the network and the set of anchors, or equivalently, by the *grounded graph* of the network and the number of anchors. The vertices of the distance and grounded graph correspond to the nodes of the network. In both graphs two vertices are connected by an edge if the corresponding distance is explicitly known. In the grounded graph we have additional edges: all pairs of vertices corresponding to anchor nodes are adjacent. See Figure 2. The grounded graph represents all known distances, since the distance between two anchors is determined by their locations. Before stating the basic observation about unique localizability we need some additional terminology. It is convenient to investigate localization problems with distance information by using frameworks, the central objects of rigidity theory.

Two frameworks (G,p) and (G,q) are *equivalent* if corresponding edges have the same lengths, that is, if $||p(u)-p(v)|| = ||q(u)-q(v)||$ holds for all pairs u,v with $uv \in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G,p), (G,q) are *congruent* if $||p(u)-p(v)|| = ||q(u)-q(v)||$ holds for all pairs u,v with $u,v \in V$. This is the same as saying that (G,q) can be obtained from (G,p) by an isometry of $\mathbb{R}^d$. We shall say that (G,p) is *globally rigid*, or that (G,p) is a *unique realization* of G in $\mathbb{R}^d$, if every framework which is equivalent to (G,p) is congruent to (G,p). We say that a graph G is *globally rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is globally rigid.

The next observation shows that unique localizability and global rigidity are, in some sense, the same.

Lemma 6.1. *[1, 29] Let N be a network in $\mathbb{R}^d$ consisting of m anchors located at positions $p_1, ..., p_m$ and $n - m$ ordinary nodes located at $p_{m+1}, ..., p_n$. Suppose that there are at least $d + 1$ anchors in general position. Let G be the grounded graph of N and let $p = (p_1, ..., p_n)$. Then the network is uniquely localizable if and only if (G, p) is globally rigid.*

Globally rigid graphs in $\mathbb{R}^2$ have been characterized by Jackson and Jordán [13], relying on earlier results of Hendrickson [11] and Connelly [6]. We say that a graph G is *redundantly rigid* in $\mathbb{R}^2$ if $G - e$ is rigid in $\mathbb{R}^2$ for all $e \in E(G)$.

Theorem 6.2. *[13] Let (G, p) be a generic framework in $\mathbb{R}^2$. Then (G, p) is globally rigid if and only if G is a complete graph on at most three vertices or G is 3-connected and redundantly rigid.*

Theorem 6.2 implies that global rigidity is indeed a generic property. It also implies that global rigidity can be tested in $O(n^2)$ time.

We shall consider the *minimum cost anchor set problem* in which the goal is, given the set of known distances in a network and a cost function on the nodes, to designate a minimum cost set of anchor nodes which makes the network uniquely localizable. Lemma 6.1 and Theorem 6.2 imply that for generic networks we may reformulate the above problem in the following purely combinatorial form:

Given a graph $G = (V, E)$ and a function $c : V \to \mathbb{R}_+$, find a set $P \subseteq V$, $|P| \geq 3$, for which $G + K(P)$ is 3-connected and redundantly rigid, and $c(P) = \sum_{v \in P} c(v)$ is minimum.

In the next sections first we shall show that a relaxed version (in which the requirement is that the rigidity matroid of $G + K(P)$ must be connected) can be formulated as a matroid optimization problem. Then, based on this formulation, we shall develop a polynomial time approximation algorithm for the minimum cost anchor set problem. Note that the complexity status of each of the above problems is still open.

7 Graphs with a connected rigidity matroid

Given a matroid $\mathcal{M} = (E, \mathcal{I})$, we define a relation on E by saying that $e, f \in E$ are related if $e = f$ or if there is a circuit C in $\mathcal{M}$ with $e, f \in C$. It is well-known that this is an equivalence relation. The equivalence classes are called the *components* of $\mathcal{M}$. If $\mathcal{M}$ has at least two elements and only one component then $\mathcal{M}$ is said to be *connected*.

We say that a graph $G = (V, E)$ is *M-connected* if $\mathcal{M}(G)$ is connected. For example, $K_{3,m}$ is M-connected for all $m \geq 4$. The *M-components* of G are the subgraphs of G induced by the components of $\mathcal{M}(G)$. It is easy to see that the M-components are pairwise edge-disjoint induced subgraphs. Theorem 6.2 and the following result show that M-connectivity is in between redundant rigidity and global rigidity.

Theorem 7.1. *[13] Let G be a graph. Then*
(a) if G is M-connected then G is redundantly rigid, and
(b) if G is 3-connected and redundantly rigid then G is M-connected.

Since the M-components of G are redundantly rigid by Theorem 7.1, the partition of $E(G)$ given by the M-components is a refinement of the partition given by the rigid components, see Figure 3. The rigidity matroid of a graph G is the direct sum of the rigidity matroids of either the rigid components of G or the M-components of G. Furthermore, the vertex sets of the components in each of the above decompositions form a thin cover of G with minimum value. This minimum value is equal to the rank of $\mathcal{R}_2(G)$ by Theorem 4.3.

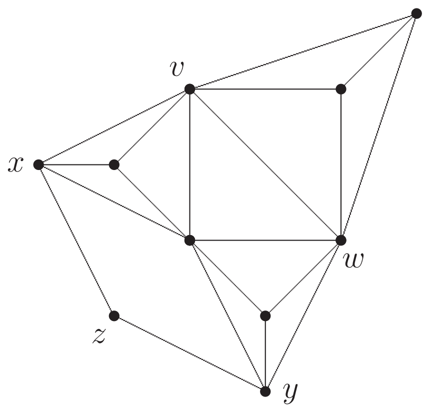

Figure 3: This graph is rigid so has exactly one rigid component. It has five M-connected components: each of the three copies of K_4, and the remaining two copies of K_2.

The following lemma is easy to prove by standard matroid techniques.

Lemma 7.2. *Let $\mathcal{M} = (E, r)$ be a matroid on ground set E with rank function r and let $E_1, E_2, \ldots, E_t$ be the components of $\mathcal{M}$. Then*
(i) $r(E) = \sum_1^t r(E_i)$, and
(ii) if $r(E) = \sum_1^q r(F_i)$ for some partition $F_1, F_2, \ldots, F_q$ of E and E_i is a component of $\mathcal{M}$ for some $1 \leq i \leq t$, then $E_i \subseteq F_j$ for some $1 \leq j \leq q$.

The next lemma shows how this general result can be formulated in terms of subgraphs and covers in the special case when the matroid is the rigidity matroid of a graph. We say that a cover is *non-trivial* if it contains at least two sets.

Lemma 7.3. *[9] $G = (V, E)$ is M-connected if and only if $val(\mathcal{X}) \geq 2|V| - 2$ for all non-trivial covers $\mathcal{X}$ of G.*

Proof: First suppose that G is M-connected. Then G is rigid, and hence $val(\mathcal{X}) \geq 2|V| - 3$ for all covers $\mathcal{X}$ of G by (the easy direction of) Theorem 4.3. Suppose that $val(\mathcal{X}) = 2|V| - 3$ for some non-trivial cover $\mathcal{X} = \{X_1, X_2, ..., X_q\}$ of G. Let $F_i = E(G[X_i])$, $1 \leq i \leq q$. We have $r(F_i) = 2|X_i| - 3$ for all $1 \leq i \leq q$, as $\mathcal{X}$ is a cover of G which minimizes $val(\mathcal{X})$. Thus $r(E) = val(\mathcal{X}) = \sum_1^q r(F_i)$, which contradicts Lemma 7.2(ii).

To prove the other direction suppose that $val(\mathcal{X}) \geq 2|V| - 2$ for all non-trivial covers $\mathcal{X}$ of G, but G is not M-connected. Let $H_1, H_2, ..., H_t$ be the M-components of G. Lemma 7.2(i) now implies that $2|V| - 3 \geq r(E) = \sum_1^t r(E(H_i)) = \sum_1^t (2|V(H_i)| - 3)$. Thus, since each edge of G belongs to some M-component and $t \geq 2$, $\mathcal{X} = \{V(H_1), V(H_2), ..., V(H_t)\}$ is a non-trivial cover of G with $val(\mathcal{X}) \leq 2|V| - 3$. This contradicts our assumption. •

7.1 M-connected graphs with pinned vertices

In the *M-connected pinning problem* the goal is to find a (smallest) set $P \subseteq V$ for which $G + K(P)$ is M-connected. The following lemma establishes the connection between the feasible solutions of the M-connected pinning problem and the M-components of G.

Lemma 7.4. *[9] Let $G = (V, E)$ be a graph, let $\mathcal{H} = \{H_1, H_2, ..., H_t\}$ be the M-components of G, and let $P \subseteq V$ with $|P| \geq 4$. Then $G + K(P)$ is M-connected if and only if*

$$2|V| - 2 \leq 2|Z| - 3 + \sum_{H_i \in \mathcal{H}_Z} (2|V(H_i)| - 3) \quad (3)$$

holds for all $Z \subset V$ with $P \subseteq Z$, $Z \neq V$, where $\mathcal{H}_Z = \{H_i \in \mathcal{H} : V(H_i) \cap (V - Z) \neq \emptyset\}$.

Proof: First suppose that $G + K(P)$ is M-connected. Since every edge of G belongs to an M-component of G and $P \subseteq Z$, it follows that $\{Z\} \cup \{V(H_i) : H_i \in \mathcal{H}, V(H_i) \cap (V - Z) \neq \emptyset\}$ is a cover of $G + K(P)$. This cover is non-trivial, since $Z \neq V$. Thus (3) follows from Lemma 7.3.

To prove the other direction suppose, for a contradiction, that (3) holds but $G' = G + K(P)$ is not M-connected. Let $\mathcal{H}' = \{H'_1, H'_2, ..., H'_q\}$ denote the M-components of G'. Since complete graphs on at least four vertices are M-connected, and $|P| \geq 4$, it follows that $G'[P]$ is M-connected. Thus there is an M-component of G', say H'_1, for which $P \subseteq V(H'_1)$. Let $Z' = V(H'_1)$ and $\mathcal{H}_{Z'} = \{H_i \in \mathcal{H} : V(H_i) \cap (V - Z') \neq \emptyset\}$. Note that $Z' \neq V$.

Claim 7.5. *Let $X \subseteq V$ be a set of vertices. Then $X = V(H'_j)$ for some M-component H'_j of G' with $2 \leq j \leq q$ if and only if $X = V(H)$ for some $H \in \mathcal{H}_{Z'}$.*

Proof: First consider an M-component $H'_j \in \mathcal{H}'$ with $j \geq 2$ and let $X = V(H'_j)$. Since $P \subseteq Z'$ and H'_1 is an induced subgraph of G' which has no edge in common with H'_j, it follows that $G[X]$ is M-connected and $X \cap (V - Z') \neq \emptyset$. Thus $X = V(H)$ for some $H \in \mathcal{H}_{Z'}$.

Next consider an M-component $H_i \in \mathcal{H}_{Z'}$ of G and put $X = V(H_i)$. $G'[X]$ is clearly M-connected. For a contradiction suppose that there is an M-component H'_j of G' with $V(H'_j) = Y \subseteq V$ for which X is a proper subset of Y. Then $|Y \cap Z'| \geq |Y \cap P| \geq 2$ must hold. Since $X \cap (V - Z') \neq \emptyset$, we have $j \geq 2$. This contradicts the fact that the M-components of G' are pairwise edge-disjoint. Thus $G'[X]$ is an M-component of G', which completes the proof. •

By using Claim 7.5 and Lemma 7.2(i), and by applying (3) with $Z = Z'$, we obtain

$$2|V| - 3 \geq r(G') = 2|V(H'_1)| - 3 + \sum_{H_i \in \mathcal{H}_{Z'}} (2|V(H_i)| - 3) \geq 2|V| - 2,$$

a contradiction. •

Let $G = (V, E)$ be a graph and let $\mathcal{H} = \{H_1, H_2, ..., H_t\}$ be the M-components of G. Let $H(G) = (V, \mathcal{E})$ be the hypergraph which contains $2|V(H_i)| - 3$ copies of the hyperedge $V(H_i)$ for each $H_i \in \mathcal{H}$, $1 \leq i \leq t$. Note that since the M-components are rigid it follows from Lemma 7.2(i) that $|\mathcal{E}| = r(G) \leq 2|V| - 3$. By letting $Y = V - Z$ in Lemma 7.4 and using the above definitions we obtain:

Lemma 7.6. *Let $G = (V, E)$ be a graph, let $\mathcal{H} = \{H_1, H_2, ..., H_t\}$ be the M-components of G, and let $P \subseteq V$ with $|P| \geq 4$. Then $G + K(P)$ is M-connected if and only if*

$$2|Y| + 1 \leq e_{H(G)}(Y) \tag{4}$$

holds for all non-empty subsets $Y \subseteq V - P$, where $e_{H(G)}(Y)$ denotes the number of hyperedges $e \in \mathcal{E}$ with $e \cap Y \neq \emptyset$.

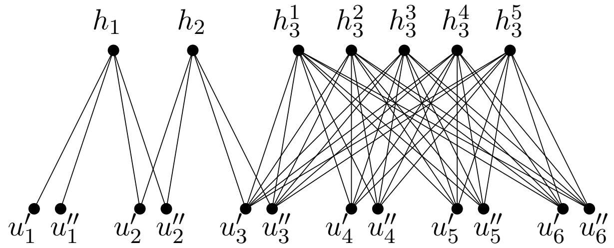

Figure 4: The bipartite incidence graph of $L(G)$, where G is the graph of Figure 2.

A hypergraph $F = (V, \mathcal{F})$ satisfying $|\cup \mathcal{F}'| \geq |\mathcal{F}'| + 1$ for all $\emptyset \neq \mathcal{F}' \subseteq \mathcal{F}$ is called a *hyperforest*. Inequality (4) can be reformulated in terms of hyperforests as follows. Let $L(G) = (W, \mathcal{U})$ be the hypergraph obtained from the dual hypergraph of $H(G)$ by duplicating every hyperedge[1]. For a set $X \subseteq V$ let $\mathcal{U}(X)$ denote the set of hyperedges corresponding to X in $L(G)$. Thus $|\mathcal{U}(X)| = 2|X|$.

[1]Consider the bipartite incidence graph G^* of $H(G)$ and split each vertex $u \in V$ into two vertices u', u''. See Figure 4. Then we obtain the bipartite incidence graph of $L(G)$ by interchanging the color classes.

Lemma 7.7. *Let $G = (V, E)$ be a graph and let $P \subseteq V$ with $|P| \geq 4$. Then P satisfies (4) if and only if $\mathcal{U}(V - P)$ is a hyperforest.*

Lorea [20] proved that the edge sets of the subhypergraphs of a hypergraph H' which are hyperforests form the family of independent sets of a matroid. A matroid arising this way is called the *circuit matroid* of the hypergraph H' and will be denoted by $\mathcal{M}_{H'}$. We call a matroid which is the circuit matroid of a hypergraph a *hypergraphic matroid.* Let $\mathcal{M}$ be a matroid on ground-set S and suppose that S is partitioned into a set A of pairs. A subset $M \subseteq A$ is a *matroid matching* if the union of the pairs in M is independent in $\mathcal{M}$. In the *matroid matching problem* the goal is to find a largest matroid matching, see [27, Chapter 43]. Lovász [21] has shown that this problem may require exponential time in general but can be solved polynomially if the matroid is represented by a set of vectors in some linear space.

By the above discussion and Lemma 7.7 it follows that the problem of finding a smallest set P for which $G + K(P)$ is M-connected can be formulated as finding a largest matroid matching in the hypergraphic matroid $\mathcal{M}_{L(G)}$, in which the doubled hyperedges form the pairs. Hypergraphic matroids are known to be linear, but it is not known how to find a suitable linear representation. The complexity status of the matroid matching problem in hypergraphic matroids is still open. Nevertheless, this formulation can be used to design a randomized algorithm, see [8], or a constant factor approximation algorithm which works for the more difficult minimum cost version as well.

To describe the approximation algorithm we need the following concepts. A *2-polymatroid* is a pair (S, f), where S is a finite ground set and f is a non-negative, monotone increasing, integer-valued, and submodular function on the subsets of S, for which $f(s) \leq 2$ for all $s \in S$. A set $X \subseteq S$ is *spanning* if $f(X) = f(S)$.

Let $G = (V, E)$ be a graph and $X \subseteq V$. Let us define $b : 2^V \rightarrow \mathbb{Z}_+$ by letting

$$b(X) = r^*(\mathcal{U}(X)), \tag{5}$$

where r^* is the rank function of the matroid dual of the hypergraphic matroid $\mathcal{M}_{L(G)}$. Then (V, b) is a 2-polymatroid.

For a spanning set $X \subseteq V$ we have $r^*(\mathcal{U}(X)) = b(X) = b(V) = r^*(\mathcal{U}(V))$. Thus X is spanning if and only if the set corresponding to $\mathcal{U}(V - X)$ is independent in $\mathcal{M}_{L(G)}$. Together with Lemmas 7.6 and 7.7 this implies:

Lemma 7.8. *Let $G = (V, E)$ be a graph and $P \subseteq V$ with $|P| \geq 4$. Then $G + K(P)$ is M-connected if and only if P is a spanning set of the 2-polymatroid (V, b).*

8 Low cost anchor sets in uniquely localizable networks

Given a 2-polymatroid (S, f) and a cost function $c : S \rightarrow \mathbb{R}$, the *minimum cost spanning set problem* is to find a spanning set X of the 2-polymatroid that minimizes $c(X) = \sum_{s \in X} c(s)$. Baudis et al. [2] verified that the GSS (Greedy Spanning Set) algorithm is a constant factor approximation algorithm for this problem. Algorithm GSS starts with $X = \emptyset$ and, as long as $f(X) < f(S)$ holds, adds a new element s to X for which $\frac{c(s)}{f(X+s)-f(X)}$ is minimum.

Theorem 8.1. *[2] Let (S, f) be a 2-polymatroid, let $c : S \to \mathbb{R}$ be a cost function, and let X_{opt} be a spanning set of minimum cost. Then*

$$c(X) \leq \frac{3}{2}c(X_{opt}),$$

where X is the spanning set output by algorithm GSS.

Lemma 7.8 and Theorem 8.1 give rise to a $\frac{3}{2}$-approximation algorithm for the *minimum cost M-connected pinning problem.* To see this it remains to note that by using bipartite matching algorithms it is easy to test independence in $\mathcal{M}_{L(G)}$ and evaluate $b(X)$ for some $X \subseteq V$ in polynomial time.

To obtain an approximation algorithm for the minimum cost anchor set problem (defined in Section 6) we also need a subroutine for the *minimum cost 3-connected pinning problem.* Let $H = (V, E)$ be a 2-connected graph. For some $X \subseteq V$ let $N(X)$ denote the set of neighbours of X. We say that $X \subset V$ is *tight* if $|N(X)| = 2$ and $X \cup N(X) \neq V$. The following lemma shows that a minimum cost set P' for which $H + K(P')$ is 3-connected can be found, in a greedy manner, in linear time.

Lemma 8.2. *Let $H = (V, E)$ be 2-connected and let $P' \subseteq V$. Then $H + K(P')$ is 3-connected if and only if $P' \cap X \neq \emptyset$ for all minimal tight sets X of H. Furthermore, the minimal tight sets of H are pairwise disjoint and can be found in linear time.*

Recall that redundant rigidity and M-connectivity are the same for 3-connected graphs by Theorem 7.1. Thus, by combining the approximation algorithm for the minimum cost M-connected pinning problem and the algorithm for the minimum cost 3-connected pinning problem we obtain a constant factor approximation algorithm for the minimum cost anchor set problem.

Theorem 8.3. *There is a polynomial time $\frac{5}{2}$-approximation algorithm for the minimum cost anchor set problem.*

Proof: (sketch) Let c^* denote the optimum value. By checking all feasible solutions $P \subseteq V$ with $|P| = 3$ we may suppose that the optimal solution has at least four vertices. First we compute a close-to-optimal solution P for the minimum cost M-connected pinning problem with $c(P) \leq \frac{3}{2}c^*$. Since $G' = G + K(P)$ is M-connected, it is 2-connected. Then we compute an optimal solution P' for the minimum cost 3-connected pinning problem on G'. Clearly, $c(P') \leq c^*$. It is also clear that $G + K(P \cup P')$ is 3-connected and M-connected. Furthermore, $c(P \cup P') \leq c(P) + c(P') \leq \frac{5}{2}c^*$ holds. •

We remark that the above methods can be used to design a constant factor approximation algorithm for the corresponding *augmentation problem* as well, in which the goal is to add a smallest set F of new edges to G such that $G + F$ is globally rigid. We omit the details.

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