Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2009-03. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

Maximum Number of Cycles and Hamiltonian Cycles in Sparse Graphs

Zoltán Király

April 2, 2009

Maximum Number of Cycles and Hamiltonian Cycles in Sparse Graphs

Zoltán Király*

Abstract

We mainly focus in this paper the maximum number of cycles in the union of two trees. In order to prove non-trivial bounds we also need some upper bounds on the number of Hamiltonian cycles in 3- and 4-regular graphs.

1 Introduction

We examine some important special cases of the following problem. Let $\mathcal{H}$ be a set of graphs. A graph is called $\mathcal{H}$-free, if it does not have any subgraph isomorphic to any member of $\mathcal{H}$. Let $f^k_{\mathcal{H}}(n)$ denote the maximum number of subgraphs isomorphic to a member of $\mathcal{H}$ in any graph that is a union of k $\mathcal{H}$-free graphs on the vertex set $V := \{1, 2, \dots, n\}$.

By "graph" we always mean a simple graph, otherwise, when parallel edges and loops are allowed, we are speaking about multigraphs. All graphs in question are on vertex set $V := \{1, 2, \dots, n\}$, and we always suppose $n \geq 2$. By "cycle" we mean a simple cycle (circuit), including cycles of length two in multigraphs. Δ will denote the maximum degree in G.

The union of graphs $G_i = (V, E_i)$ $1 \leq i \leq k$ is the graph $G = (V, \bigcup_{i=1}^k E_i)$. Sometimes we also need the sum (multi-union) as well, where if uv is an edge in $l \leq k$ graphs G_i then we put l parallel edges between u and v in the sum. We use $\hat{f}^k_{\mathcal{H}}(n)$ for the maximum number of subgraphs isomorphic to a member of $\mathcal{H}$ in any graph that is a sum of k $\mathcal{H}$-free graphs.

We mainly focus the problem, where $\mathcal{H} = \mathcal{C}$, here $\mathcal{C}$ denotes the set of all cycles. Thus in this problem every G_i is a tree (forest) and we are going to calculate (bound) the maximum number of cycles in their union. Our first goal is to prove upper and lower bounds for $f(n) := f^2_{\mathcal{C}}(n)$. Interestingly, giving a non-trivial upper bound for $f(n)$ needs non-trivial upper bound for the number of Hamiltonian cycles in a 4-regular graph. And, for proving this bound, we need an upper bound on the number of Hamiltonian cycles in a 3-regular graph.

*Department of Computer Science and Communication Networks Laboratory, Eötvös University, Pázmány Péter sétány 1/C Budapest, Hungary H-1117. Research is supported by EGRES group (MTA-ELTE) and OTKA grants NK 67867, K 60802. E-mail: `kiraly@cs.elte.hu`

In the next section we review the known as well as the simple bounds, and also some other special cases of the general problem.

In Section 3 we deal with the number of Hamiltonian cycles in a 3-regular graph. In Section 4 we examine the maximum number of Hamiltonian cycles in a 4-regular graph. In our main Section 5 we prove our upper bound on the number of cycles in a graph with $2n$ edges and derive an upper bound for $f(n)$ from this result. In Section 6 we give an upper bound for sum and union of k trees. Finally in Section 7 we give lower bounds, i.e., constructions.

2 Known results

In [3, 4] Balázs Patkós examined similar problems for hypergraphs and graphs. For graphs he studied two special cases, in both he had $k = 2$. The first was the case of $\mathcal{H} = \{K_r\}$ (the complete graph on r vertices), and the second was $\mathcal{H} = \mathcal{C}$. He gave sharp bounds for the complete graph case, and gave upper and lower bounds for the cycles case. His upper bound relies only on the next well known folklore theorem (we call a not necessarily connected subgraph Eulerian, if the degree of every vertex is even):

Theorem 2.1 (Folklore)**.** *If $G = (V, E)$ is a connected multigraph on n vertices having m edges, then the number of Eulerian subgraphs is exactly 2^{m-n+1}. Consequently $2^{m-n+1} - 1$ is an upper bound on the number of cycles.*

Proof. Let F denote a spanning tree, and $E' := E \setminus F$. For each subset $\hat{E} \subseteq E'$ clearly there exists a unique subset $\hat{F} \subseteq F$ such that $\hat{E} \cup \hat{F}$ is Eulerian. □

The theorem is not true for disconnected graphs, consider e.g., a triangle and many isolated vertices. But taking a spanning forest instead of a spanning tree, we easily get

Corollary 2.2. *If $G = (V, E)$ is a multigraph on n vertices having m edges and p components, then it has at most $2^{m+p-n} - 1$ cycles.*

We also need the following form

Corollary 2.3. *If $G = (V, E)$ is a loopless multigraph on n vertices having p components and maximum degree four, then it has at most $4 \cdot 2^{n-p}$ cycles.*

Proof. Let $H_1, \ldots, H_t$ be the components with at least one cycle, and $n_i = |V(H_i)| \geq 2$ and $n' = \sum n_i$. Clearly $n - p \geq n' - t$. As H_i has at most $2n_i$ edges, we have that the number of cycles is at most $\sum 2^{n_i+1} = 4 \sum 2^{n_i-1} \leq 4 \prod 2^{n_i-1} = 4 \cdot 2^{n'-t} \leq 4 \cdot 2^{n-p}$. □

The previous theorem gives an upper bound of $2^{n-1} - 1$ for $\hat{f}^2_{\mathcal{C}}(n)$. An almost matching lower bound, 2^{n-2}, can be easily achieved: we take the sum of paths $1, 2, \ldots, n$ and $2, 3, \ldots, n, 1$.

Returning to the main question (about simple graphs), Patkós' lower bound (construction) gives $f(n) \geq c \cdot \kappa^n$, where c is a constant and κ is the unique real root of the equation $x^3 - x^2 - x - 1 = 0$, so $\kappa \approx 1.839$. We will describe this construction in Section 7. As an upper bound he proved $f(n) \leq 2^{n-1} - 1$, the same we showed for the sum case. Our main purpose is to revise this bound to $(2-\varepsilon)^n$ for some positive constant ε. We think that there should be some direct proof of this fact, but we could not find it. The problem is that this theorem is not true for the sum of the trees, so we must use the simplicity of the union graph.

Let $\mathcal{C}_o$ denote the set of odd cycles, and $T(n,p)$ denote the Turán graph on n vertices and with p classes (this is a complete p-partite graph with p almost equal classes).

We omit the simple proof of the next theorem, and we also leave out the explicit calculation of the value.

Theorem 2.4. *$f^k_{\mathcal{C}_o}(n)$ is the number of odd cycles in $T(n, 2^k)$.*

Many papers deal with the number of cycles or Hamiltonian cycles, but usually in another context. The proximate result for the number of Hamiltonian cycles is given by Teunter and Poort [8]. Rautenbach and Stella [7] gave an upper bound on the number of cycles slightly improving on 2^{m-n+1} of Theorem 2.1. Other results try to exactly determine the number of cycles in cubic graphs [1, 2, 6].

3 Hamiltonian cycles in a 3-regular graph

Let $h(G)$ denote the number of Hamiltonian cycles in graph G, and $h(n) := \max\{h(G) \mid G$ is a 3-regular graph on n vertices$\}$. Our main goal in this section is to prove the following theorem, which was independently proved by Dániel Gerbner [5]. Fix $\alpha = \sqrt[8]{8} \approx 1.2968$, $\beta = \sqrt[4]{2} \approx 1.1892$, $c_1 = 3/(2\beta) \approx 1.2613$.

Theorem 3.1. *$h(n) \leq c_1 \cdot \alpha^n$.*

We need an auxiliary theorem in order to use induction on the number of edges. Suppose $G = (V, E)$ is a loopless multigraph with $\Delta \leq 3$, and E is two-colored by red and blue. The coloring is called proper, if there are no two parallel blue edges. A two-coloring will be identified with $E_r \subseteq E$, the set of red edges. In this sense E_r is proper for G, if $G - E_r$ is simple. The number of edges is denoted by $m = |E|$, and the number of red edges is denoted by $r = |E_r|$. We are going to upper bound the number of Hamiltonian cycles that use all red edges (we call these cycles *nice*) in a proper coloring. Let $h(G, E_r)$ denote the number of nice cycles in G, where E_r is a proper coloring, and let $h(G, r)$ denote the maximum of $h(G, E_r)$ in any proper two-coloring E_r with $r = |E_r|$ red edges (defining $h(G, r) = 0$, if no such proper coloring exists). Finally let $h(n, r) := \max\{h(G, r) \mid$ loopless multigraph G has n vertices and $\Delta \leq 3\}$.

Theorem 3.2. *$h(n, r) \leq c_1 \cdot \alpha^n / \beta^r$.*

Proof. We are going to prove this statement by induction on m. Let G be a loopless multigraph with $\Delta \leq 3$, and E_r is a proper two-coloring. If G is not Hamiltonian, or has a vertex incident to 3 red edges then $h(G, r) = 0$. So we may suppose that G

Hamiltonian: it is connected, $n \geq 2$, G has minimum degree 2; and every vertex is incident to at most 2 red edges. If G is a cycle then $h(G,r) = 1$, so the statement holds again, thus we may suppose that G has a vertex of degree 3.

We prove, as a matter of fact, that

Lemma 3.3. *If $r > 0$ then $h(G,r) \leq \alpha^n/\beta^r$.*

Suppose we have already proved this statement. In order to prove Theorem 3.2 for the case $r = 0$ as well, take a vertex v with degree 3, and let e_1, e_2, e_3 denote the incident edges. We consider three new proper two-coloring: in each we color one e_i by red. Every Hamiltonian cycle of G is nice in two such coloring, so we have: $h(G,0) \leq (3/2) \cdot h(G,1) \leq (3/2) \cdot \alpha^n/\beta = c_1 \cdot \alpha^n$.

If $n = 2$ then, using that we have no two parallel blue edges, $h(G,r) \leq 1 \leq \alpha^2/\beta^r$ for all possible values of $r = 0, 1, 2$. We suppose $n \geq 3$ from now on.

If red edge e and blue edge f are parallel, then we safely delete f, neither n, nor r, nor the number of nice cycles will be changed. If there are two parallel red edges then (as we have $n \geq 3$), $h(G,r) = 0$. So we may assume, that G is simple.

Suppose u is vertex with degree 2, let its neighbors be a and b. Construct a new graph G' as follows. Delete u and put a new red edge between a and b. Clearly the number of nice cycles will not be changed. So $h(G,r) = h(G',r')$ where r' is between $r-1$ and $r+1$ depending on the colors of edges ua and ub. By induction $h(G',r') \leq \alpha^{n-1}/\beta^{r-1} < \alpha^n/\beta^r$. From now on we suppose G is a 3-regular graph. If G is a K_4, then $h(G,0) = 3 < c \cdot \alpha^4$ and $h(G,1), h(G,2) \leq 2 = \alpha^4/\beta^2$ and $h(G,3), h(G,4) \leq 1 < \alpha^4/\beta^4$.

If there is vertex u with two red neighbors then we delete u and put a new red edge between its two red neighbors, and use $h(G,r) \leq h(n-1, r-1) < \alpha^n/\beta^r$. From now on the red edges form a matching.

Call a vertex blue, if all 3 incident edges are blue, otherwise red. Let u be a red vertex.

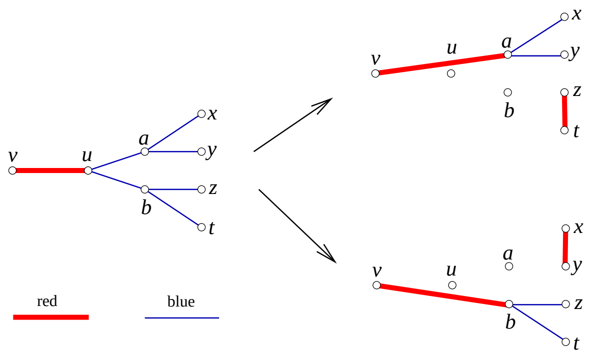

Figure 1: Case 1.

Case 1. u is connected to two blue vertices: a and b. (See Figure 1). Denote u's red neighbor by v, a's other two (different from u) neighbors by x, y and b's other two neighbors by z, t. Note, that v, x, y, z, t are not necessarily distinct, but we have no parallel edges.

We partition the set of nice cycles of G into two classes: the first, when a cycle uses edge ua, and the second, when a cycle uses edge ub. We construct two new graphs, G_a and G_b. G_a is derived from G by deleting u and b, and adding new red edges va and zt. Similarly, G_b is derived from G by deleting u and a, and adding new red edges vb and xy. It is easy to observe, that each nice cycle of G corresponds to either a nice cycle of G_a or a nice cycle of G_b. Thus we have $h(G,r) = h(G_a, r+1) + h(G_b, r+1) \le 2 \cdot \alpha^{n-2}/\beta^{r+1} = 2/(\alpha^2\beta) \cdot \alpha^n/\beta^r = \alpha^n/\beta^r$.

Case 2. u is connected to a red vertex v by a blue edge, and to a blue vertex a, and v is connected to u and b by blue edges. We have 3 subcases:

i) $a = b$.

ii) $a \ne b$ and b is red.

iii) $a \ne b$ and b is blue.

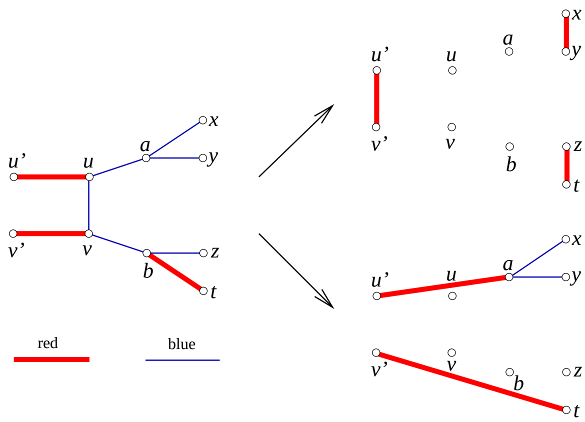

Figure 2: Case 2. Subcase ii)

Subcase i) is the easiest: if x is a's third neighbor, then no nice cycle may use edge ax, so we may delete it, and then use induction.

Denote u's red-edge neighbor by u', v's red-edge neighbor by v', a's other two neighbors by x, y and b's other two neighbors by z, t.

Subcase ii) (see Figure 2). We partition the set of nice cycles of G into two classes: the first, when a cycle uses edge uv, and the second, when a cycle uses edges ua and vb. We construct two new graphs, G_1 and G_2. G_1 is derived from G by deleting u, v, a, b and adding new red edges $u'v', xy, zt$. Graph G_2 is derived from G by deleting u, v, b and adding new red edges $u'a, v't$. It is easy to observe, that each nice cycle of G corresponds to either a nice cycle of G_1 or a nice cycle of G_2. Thus we have $h(G,r) = h(G_1, r) + h(G_2, r-1) \le \alpha^{n-4}/\beta^r + \alpha^{n-3}/\beta^{r-1} = (\frac{1}{\alpha^4} + \frac{\beta}{\alpha^3}) \cdot \alpha^n/\beta^r \approx 0.8988 \cdot \alpha^n/\beta^r < \alpha^n/\beta^r$.

Subcase iii) (b is blue, see Figure 3). We partition again the set of nice cycles of G into two classes in the same manner. We construct again two new graphs, G_1 and G_2, where G_1 is derived from G by deleting u, v, a, b and adding new red edges $u'v', xy, zt$, while G_2 is derived by deleting u, v and adding new red edges $u'a, v'b$. Each nice cycle

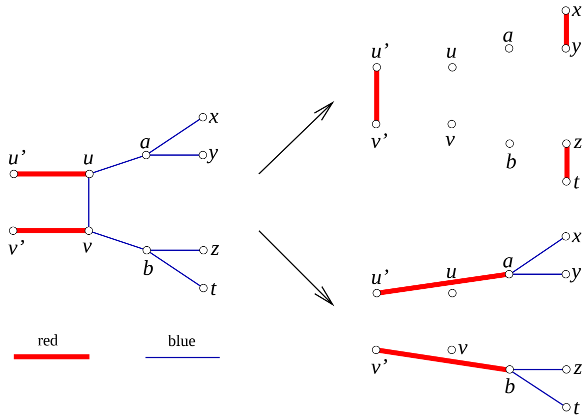

Figure 3: Case 2. Subcase iii)

of G corresponds to either a nice cycle of G_1 or a nice cycle of G_2. Thus we have $h(G,r) = h(G_1, r+1) + h(G_2, r) \leq \alpha^{n-4}/\beta^{r+1} + \alpha^{n-2}/\beta^r = (\frac{1}{\alpha^4\beta} + \frac{1}{\alpha^2}) \cdot \alpha^n/\beta^r \approx 0.8919 \cdot \alpha^n/\beta^r < \alpha^n/\beta^r$.

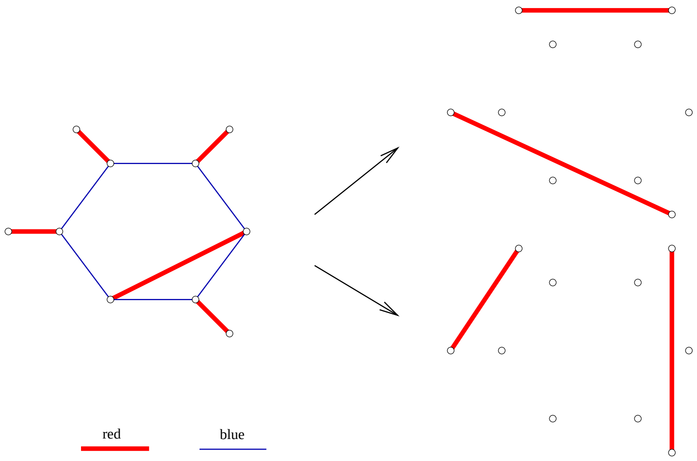

Figure 4: An example for Case 3.

Case 3. Every vertex is red. (Otherwise we can apply either Case 1, or Case 2, because if there is a blue vertex then there are connected blue and red vertices as well.) Consequently the blue edges form some disjoint cycles. It is an easy observation, that if C is a blue cycle then every nice cycle uses every second edge of it. Thus, if there is a blue odd cycle then $h(G,r) = 0$, and we are done. Suppose C is a blue cycle of length $2l$ with edges $e_1, e_2, \ldots, e_{2l}$. Nice cycles are of two types: either they use $e_1, e_3, \ldots, e_{2l-1}$ and miss $e_2, e_4, \ldots, e_{2l}$, or they use $e_2, e_4, \ldots, e_{2l}$ and miss $e_1, e_3, \ldots, e_{2l-1}$ (see Figure 4). For both cases we construct smaller graphs. We delete every second edge of C and color the remaining edges of C by red.

Suppose, that at least one of the two reduced graphs has a red cycle. Then, in this graph, either there is no nice cycle, or the red edges form a Hamiltonian cycle. It is easy to conclude that in this case the statement follows by induction.

Otherwise in both reduced graphs we *replace* each red path by one red edge (connecting its end-nodes), so all vertices of C will be deleted, and the number of red edges will be decreased by exactly l. Here we have $h(G,r) \leq 2 \cdot h(n-2l, r-l) = 2(\beta/\alpha^2)^l \cdot \alpha^n/\beta^r = 2^{1-l/2} \cdot \alpha^n/\beta^r \leq \alpha^n/\beta^r$, because $l \geq 2$. □

Corollary 3.4. *Let G be a graph with m edges and $\Delta \leq 3$. Then $h(G) \leq c_1 \cdot \gamma^{m-n}$, where $\gamma = \alpha^2 \approx 1.6818$.*

Proof. If G is disconnected or has a pendant vertex then $h(G) = 0$, or if G is a cycle then $h(G) = 1$, in these cases we are done. Otherwise let $D_2 \subseteq V$ denote the set of degree-2 vertices. Originally color all edges of G by blue. Take the vertices of D_2 one by one. If u has two neighbors a and b, then delete u and add ab as a new edge (if $a = b$ then either G is a cycle, or G is not Hamiltonian). The number of nice cycles will not be changed. After this procedure the resulting graph G' will have $n - |D_2|$ vertices and $m - |D_2|$ edges, it is 3-regular with a proper two-coloring, so $m - |D_2| = (3/2) \cdot (n - |D_2|)$, i.e., $m - n = (n - |D_2|)/2$. Using Theorem 3.1 we get: $h(G) = h(G') \leq c_1 \cdot \alpha^{n-|D_2|} = c_1 \cdot (\alpha^2)^{m-n}$. □

4 Hamiltonian cycles in sparse graphs

Theorem 4.1. *If G is a graph with m edges then $h(G) \leq c_1 \cdot (2 - \varepsilon_1)^{m-n}$, where $\varepsilon_1 = 2 - \left(2^{\frac{11}{12}} \cdot \gamma^{\frac{1}{12}}\right) \approx 0.0287$.*

Proof.

We use again induction on m. If G has $\Delta \leq 3$ then we are done by Corollary 3.4. If G is not Hamiltonian then we are also done.

Let u be a vertex of degree d, and for each edge incident to u, we calculate, how many Hamiltonian cycles use it. We call edge uv a "*least used edge at u*" if this number reaches the minimum. Obviously at most $2/d$ fraction of the Hamiltonian cycles use this edge.

If G has a vertex u of degree ≥ 5 then let uv denote the least used edge at u. At least 3/5 of the Hamiltonian cycles avoids uv, so $h(G) \leq (5/3) \cdot h(G - uv) \leq (5/3) \cdot c_1 \cdot (2 - \varepsilon_1)^{m-1-n} < c_1 \cdot (2 - \varepsilon_1)^{m-n}$, by induction. Thus we may suppose that G has $\Delta = 4$.

The outline of the proof is the following. We take the degree four vertices, one by one. When considering vertex u, we delete uv, the least used edge at u. We have $h(G) \leq 2 \cdot h(G - uv)$. Suppose, that this procedure stops after z steps, i.e., in the remaining graph $\Delta \leq 3$. We are going to apply Corollary 3.4 for this graph. This will give altogether $2^z \cdot c_1 \cdot \gamma^{m-z-n} \leq c_1 \cdot (2 - \varepsilon_1)^{m-n}$, unless at the end of the procedure we have $m' := m - z = n + o(m - n)$ edges. We must do some efforts to avoid this case.

If $G = K_5$ then $h(G) = 12 < 1.65^{10-5}$, so we may suppose that G differs from K_5. Let $D_4 \subseteq V$ denote the set of vertices of degree 4, and $D_3 \subseteq V$ denote the set of vertices of degree 3. As G is Hamiltonian and $\Delta = 4$, we have $m - n = |D_4| + \frac{1}{2}|D_3|$.

By Brooks' theorem G can be colored by 4 colors, let I denote the largest intersection of a color class and D_4. Clearly $|I| \geq \frac{1}{4}|D_4|$ and I is an independent set.

First we make the following procedure, starting with $G = G_0$. Until there exists a vertex u in $D_4 \setminus I$ that has degree four in G_i, take the least used edge uv at u, and let $G_{i+1} = G_i - uv$. This procedure ends after t steps. In the resulting graph G_t, every vertex of degree 4 is a member of I. And we also have $h(G) \leq 2^t \cdot h(G_t)$.

Now we make the following procedure, starting with $G_i = G_t$. Until there exists a vertex u that has degree four in G_i, take the least used edge uv at u, and let $G_{i+1} = G_i - uv$. This procedure ends after $z - t$ steps. In the resulting graph G_z every vertex has degree ≤ 3. And we also have $h(G) \leq 2^z \cdot h(G_z)$.

As we always deleted the least used edge, thus G_z is also Hamiltonian, so every degree is at least two. Let m_z denote the number of edges in G_z, such that $m_z = m - z$. If D_3^* denotes the set of vertices with degree 3 in G_z then $m_z - n = \frac{1}{2}|D_3^*|$.

Lemma 4.2.
$$\frac{m_z - n}{m - n} \geq \frac{1}{12}, \text{ or equivalently } z \leq \frac{11}{12}(m - n).$$

Proof. (of the lemma).

Case 1. In G_t we have less than $|D_4|/6 + |D_3|/12$ vertices of degree four. Then $z - t \leq |D_4|/6 + |D_3|/12$, and by the definition of the first procedure, $t \leq |D_4 \setminus I| \leq (3/4) \cdot |D_4|$. Thus we have $z \leq (11/12) \cdot |D_4| + |D_3|/12$. This results

$$z \leq (11/12) \cdot |D_4| + (11/24) \cdot |D_3| = \frac{11}{12}(m - n).$$

Case 2. In G_t we have at least $|D_4|/6 + |D_3|/12$ vertices of degree four. Since these vertices are independent, after the second procedure these vertices will have degree three in G_z, so $|D_3^*| \geq |D_4|/6 + |D_3|/12$. We have

$$\frac{m_z - n}{m - n} = \frac{|D_3^*|/2}{D_4 + |D_3|/2} \geq \frac{|D_4|/12 + |D_3|/24}{D_4 + |D_3|/2} = \frac{1}{12}. \qquad \square$$

It is easy to finish now the proof of the Theorem. We apply the two procedures above and at the end we use Corollary 3.4 for G_z. We get:

$$h(G) \leq 2^z \cdot h(G_z) \leq 2^z \cdot c_1 \cdot \gamma^{m_z - n} = c_1 \cdot \big(2/\gamma\big)^z \cdot \gamma^{m-n} \leq c_1 \cdot \big(2/\gamma\big)^{\frac{11}{12}(m-n)} \cdot \gamma^{m-n} \leq$$
$$\leq c_1 \cdot \Big(\big(2/\gamma\big)^{\frac{11}{12}} \cdot \gamma\Big)^{m-n} = c_1 \cdot \Big(2^{\frac{11}{12}} \cdot \gamma^{\frac{1}{12}}\Big)^{m-n} = c_1 \cdot (2 - \varepsilon_1)^{m-n}. \qquad \square$$

Corollary 4.3. *If G is a 4-regular graph, then $h(G) \leq c_1 \cdot (2 - \varepsilon_1)^n$.*

Let $hp(G)$ denote the number of Hamiltonian paths contained in G.

Corollary 4.4. *If G is a graph with m edges then $hp(G) \leq n^2 \cdot c_1 \cdot (2 - \varepsilon_1)^{m-n}$.*

Corollary 4.5. *If G is a graph with $\Delta \leq 4$ then $hp(G) \leq n^2 \cdot c_1 \cdot (2 - \varepsilon_1)^n$.*

5 Cycles in a graph with $2n$ edges

Let $c(G)$ denote the number of cycles in graph G. First we give an upper bound when G has exactly $m = 2n$ edges. Then, as a corollary, we show that the same bound holds for the union of two trees.

Theorem 5.1. *If G is a connected graph with $m = 2n$ edges, then $c(G) \le c \cdot n^2 \cdot (2-\varepsilon)^n$ for some positive constants c and ε.*

Proof. Here we will not calculate ε and c, and actually we will prove an asymptotic result, i.e. which holds for large enough n values. With an appropriate increase of value c it will also hold for every n. This allows us, for example, to omit integer parts.

We need four constants $0 < \varepsilon_i < 0.1$. Remember that ε_1 has been already defined, and $\varepsilon_1 \approx 0.0287$. Define ε_2 by the identity $(1-\varepsilon_2)\cdot(2-\varepsilon_2) = 2-\varepsilon_1$. Then define ε_3, such that for every large enough n

$$\binom{n}{\varepsilon_3 \cdot n} \le \Big(\frac{1}{1-\varepsilon_2}\Big)^n.$$

By using simple tools from calculus it is easy to see that such a positive $\varepsilon_3 < 0.1$ exists. Define ε_4 by the identity $(1-4\varepsilon_4)\cdot(1-\varepsilon_3/2) = 1-\varepsilon_3$. Let L denote $\varepsilon_3 \cdot n$.

If there exists an edge e, such that at most $(1/2-\varepsilon_4)$ fraction of the cycles use e, then delete e. We repeat this procedure until such an edge exists. We make l_1 deletion steps, resulting graph G' with n vertices and $m' = m - l_1$ edges. We separate the cases when we deleted a cut edge of the current graph. Suppose this happened l_2 times (in these cases the number of cycles did not changed), and let $l = l_1 - l_2$. Thus G' has exactly $l_2 + 1$ components. We have $c(G) \le 1/(1/2+\varepsilon_4)^l \cdot c(G')$. It is easy to see that the maximum degree in G' is at most four, because at a vertex v with degree at least five, there must exists an edge vw that is used only by forty percent of the cycles. And, as the procedure finished, (*) every edge of G' is used by $> (1/2-\varepsilon_4)$ fraction of the cycles.

Case 1. $l \ge L/4$. In this case we have

$$c(G) \le \Big(\frac{1}{\frac{1}{2}+\varepsilon_4}\Big)^l \cdot c(G') \le \Big(\frac{1}{\frac{1}{2}+\varepsilon_4}\Big)^l \cdot 2^{(2n-l_1)+(l_2+1)-n} = \Big(\frac{1}{1+2\varepsilon_4}\Big)^l \cdot 2^{n+1} \le$$

$$\le 2 \cdot \Big[2 \cdot \Big(\frac{1}{1+2\varepsilon_4}\Big)^{\varepsilon_3/4}\Big]^n < 2\cdot(2-\varepsilon)^n$$

for an appropriate positive ε. For the second inequality we used Corollary 2.2.

Case 2. $l < L/4$. First we claim, that if G' has at least two components with some edges, H_1 and H_2, then G' consists of (at most) two disjoint cycles and some isolated vertices. First suppose that H_1 has a vertex v with degree three. As at least $> (1/2 - \varepsilon_4)\cdot c(G')$ cycles use every edge vw at v, we have that at least $(3/4-(3/2)\varepsilon_4)\cdot c(G')$ cycles use v. Next suppose that H_1 has a vertex v with degree four. Similarly we have that at least $(1-2\varepsilon_4)\cdot c(G')$ cycles use v. In both cases any edge of H_2 can be used by less than $(1/4+(3/2)\varepsilon_4)\cdot c(G') \le 0.4 \cdot c(G')$ cycles, this contradicts to property

(*). Thus we have that the maximum degree in H_1 is two, and clearly the same is true for H_2. If we have an edge outside $H_1 \cup H_2$ then we also get a contradiction. If G' is the union of two disjoint cycles then the theorem holds, so from now on we may suppose that G' has one huge component H with size $|H| = n_0$ and $n - n_0$ isolated vertices.

Let D'_4 denote the set of vertices having degree four in G'. We claim that $|D'_4| \geq n - 2l$. If the degree of a vertex v in G' is denoted by $d'(v)$ then it is enough to prove that $\sum_{v \in V(H)} d'(v) \geq 3n_0 + n - 2l$, because there are no vertices of degree larger than four, so $\sum_{v \in V(H)} d'(v) \leq 3n_0 + |D'_4|$. We have $l_2 + 1$ components, that is $n - n_0 = l_2$. Thus we have $\sum_{v \in V(H)} d'(v) = \sum_{v \in V} d'(v) = 4n - 2l_1 = (3n_0 + n - 2l) + 3(n - n_0) - 2l_2 = (3n_0 + n - 2l) + (n - n_0) \geq 3n_0 + n - 2l$.

We proved that at least $(1 - 2\varepsilon_4) \cdot c(G')$ cycles use v for any $v \in D'_4$. Next we claim that at least the half of the cycles use $> (1 - 4\varepsilon_4)$ fraction of D'_4 (we call them long cycles). This claim can be proved by simple double-counting.

Therefore the long cycles have length $> (1 - 4\varepsilon_4)|D'_4| \geq (1 - 4\varepsilon_4)(n - 2l) \geq (1 - 4\varepsilon_4)(n - L/2) = (1 - 4\varepsilon_4)(1 - \varepsilon_3/2) \cdot n = (1 - \varepsilon_3) \cdot n$, by the definition of ε_4.

We are going to upper bound the number of long cycles (i.e., the cycles of length $> (1 - \varepsilon_3) \cdot n$), by the observation above $c(G') \leq$ twice the number of long cycles. We take each subset $V'' \subseteq V$ with size $(1 - \varepsilon_3) \cdot n$, and first upper bound the number of Hamiltonian paths in the induced graph $G'[V'']$; and finally we add up these upper bounds. We get

$$c(G) \leq \left(\frac{1}{\frac{1}{2} + \varepsilon_4}\right)^l \cdot c(G') \leq \left(\frac{1}{\frac{1}{2} + \varepsilon_4}\right)^l \cdot 2 \cdot \binom{n}{\varepsilon_3 \cdot n} \cdot hp(n - \varepsilon_3 \cdot n) \leq$$
$$\leq 2c_1 n^2 \cdot \left(\frac{1}{\frac{1}{2} + \varepsilon_4}\right)^l \cdot \left(\frac{1}{1 - \varepsilon_2}\right)^n \cdot (2 - \varepsilon_1)^{(1 - \varepsilon_3)n} \leq$$
$$\leq 2c_1 n^2 \cdot 2^l \cdot \left(\frac{1}{1 - \varepsilon_2}\right)^n \cdot (1 - \varepsilon_2)^n \cdot (2 - \varepsilon_2)^n \cdot \left(\frac{1}{2 - \varepsilon_1}\right)^{\varepsilon_3 n} \leq$$
$$\leq 2c_1 n^2 \cdot (2 - \varepsilon_2)^n \cdot 2^{(\varepsilon_3/4)n} \cdot \left(\frac{1}{2 - \varepsilon_1}\right)^{\varepsilon_3 n} \leq 2c_1 n^2 \cdot (2 - \varepsilon_2)^n \cdot \left(\frac{\sqrt[4]{2}}{2 - \varepsilon_1}\right)^{\varepsilon_3 n} \leq$$
$$\leq 2c_1 n^2 \cdot (2 - \varepsilon_2)^n,$$

because

$$\left(\frac{\sqrt[4]{2}}{2 - \varepsilon_1}\right) \approx 0.6033 < 1.$$

□

Corollary 5.2. *If G is a connected 4-regular graph then $c(G) \leq c \cdot n^2 \cdot (2 - \varepsilon)^n$ for some positive constants c and ε.*

Corollary 5.3. *If G is a connected graph with $m \leq 2n$ edges, then $c(G) \leq c \cdot n^2 \cdot (2 - \varepsilon)^n$ for some positive constants c and ε.*

Proof. We can add $2n - m$ new edges (that are not parallel to any existing one), and this will only increase the number of cycles. □

Corollary 5.4. *If G is a union of two trees then $c(G) \leq c \cdot n^2 \cdot (2 - \varepsilon)^n$ for some positive constants c and ε. Consequently $f(n) \leq c \cdot n^2 \cdot (2 - \varepsilon)^n$.*

6 Union and sum of k trees

In this section we have a much less precise bound.

Theorem 6.1. *Let G be a sum of k trees. Then*

$$c(G) \leq \frac{10}{3} \cdot \left(\sqrt{\frac{100}{27}} \cdot k\right)^n \approx \frac{10}{3} \cdot (1.925 \cdot k)^n.$$

Consequently $f_{\mathcal{C}}^k(n) \leq \hat{f}_{\mathcal{C}}^k(n) \leq \frac{10}{3} \cdot \left(\sqrt{\frac{100}{27}} \cdot k\right)^n \approx \frac{10}{3} \cdot (1.925 \cdot k)^n$.

Proof. Let G be a connected loopless multigraph with $m = kn$ edges. It is enough to prove that $c(G) \leq \frac{10}{3} \cdot \left(\sqrt{\frac{100}{27}} \cdot k\right)^n$ for $k \geq 3$.

We run the following process. Choose a vertex u with maximum degree and take the least used edge uv at u. Delete this edge. The process stops if the maximum degree reaches four.

We divide this process into phases. Phase d contains the steps, where the maximum degree is d, except the first phase – Phase $2k$ – which contains the steps, where the maximum degree is at least $2k$. For Phase d, let l'_d denote the number of steps, where we delete a non-cut edge, and let l''_d denote the number of steps, where we delete a cut edge. Let $p = 1 + \sum_{d=5}^{2k} l''_d$, the number of components at the end of the process. Using the previous arguments and Corollary 2.3, we have

$$c(G) \leq \left(\frac{2k}{2k-2}\right)^{l'_{2k}} \left(\frac{2k-1}{2k-3}\right)^{l'_{2k-1}} \left(\frac{2k-2}{2k-4}\right)^{l'_{2k-2}} \cdots \left(\frac{6}{4}\right)^{l'_6} \left(\frac{5}{3}\right)^{l'_5} \cdot 4 \cdot 2^{n-p}$$

We must calculate the maximum value of the right-hand side under the following conditions: For all $5 \leq j \leq 2k$ we have $\sum_{i=j}^{2k} (l'_i + l''_i) \geq (2k+1-j) \cdot n/2$ (we have to delete at least this many edges in order to reach maximum degree less than j). And $\sum_{i=5}^{2k} (l'_i + l''_i) \leq kn - (n-p)$, so $\sum_{i=5}^{2k} l'_i \leq (k-1)n + 1$ (at least $n-p$ edges remain at the end).

For $k \geq 4$, the maximum value of the right-hand side under these conditions is

$$\left(\frac{2k}{2k-2}\right)^{n/2-p+1} \left(\frac{2k-1}{2k-3}\right)^{n/2} \left(\frac{2k-2}{2k-4}\right)^{n/2} \cdots \left(\frac{6}{4}\right)^{n/2} \left(\frac{5}{3}\right)^{n/2+n+p} \cdot 4 \cdot 2^{n-p} =$$

$$= 4 \cdot \left(\frac{(2k)(2k-1) \cdot 5^2 \cdot 2^2}{4 \cdot 3 \cdot 3^2}\right)^{n/2} \cdot \left(\frac{(2k-2) \cdot 5}{2k \cdot 3 \cdot 2}\right)^{p-1} \cdot \frac{5}{6} =$$

$$= \frac{10}{3} \left(\frac{400k^2 - 200k}{108}\right)^{n/2} \cdot \left(\frac{10k-10}{12k}\right)^{p-1} \leq \frac{10}{3} \left(\frac{100k^2}{27}\right)^{n/2} = \frac{10}{3} \left(\sqrt{\frac{100}{27}} \cdot k\right)^n$$

if $p - 1 \leq n/2$, and

$$\left(\frac{2k-1}{2k-3}\right)^{n-p+1} \left(\frac{2k-2}{2k-4}\right)^{n/2} \cdots \left(\frac{6}{4}\right)^{n/2} \left(\frac{5}{3}\right)^{n/2+n+p} \cdot 4 \cdot 2^{n-p} =$$

$$= 4 \cdot \left(\frac{(2k-1)(2k-1)(2k-2) \cdot 5^2 \cdot 2^2}{(2k-3) \cdot 4 \cdot 3 \cdot 3^2}\right)^{n/2} \cdot \left(\frac{(2k-3) \cdot 5}{(2k-1) \cdot 3 \cdot 2}\right)^{p-1} \cdot \frac{5}{6} =$$

$$= \frac{10}{3}\Big(\frac{100\cdot(8k^3-16k^2+10k-2)}{108\cdot(2k-3)}\Big)^{n/2}\cdot\Big(\frac{10k-15}{12k-6}\Big)^{p-1}\le$$
$$\le\frac{10}{3}\Big(\frac{100k^2}{27}\Big)^{n/2}=\frac{10}{3}\Big(\sqrt{\frac{100}{27}}\cdot k\Big)^n$$

otherwise, as $8k^3-16k^2+10k-2\le 4k^2(2k-3)$, if $k\ge 3$.

For $k=3$, if $p-1\le n/2$ then

$$\Big(\frac{6}{4}\Big)^{n/2-p+1}\Big(\frac{5}{3}\Big)^{3n/2+p}\cdot 4\cdot 2^{n-p}=\frac{10}{3}\Big(\frac{6\cdot 5^3\cdot 2^2}{4\cdot 3^3}\Big)^{n/2}\cdot\Big(\frac{5}{9}\Big)^{p-1}\le\frac{10}{3}\Big(1.76\cdot k\Big)^n$$

and if $p-1>n/2$ then

$$\Big(\frac{5}{3}\Big)^{2n+1}\cdot 4\cdot 2^{n-p}=\frac{10}{3}\Big(\frac{50}{9}\Big)^n\cdot 2^{-p+1}\le\frac{10}{3}\Big(\frac{50}{9}\Big)^n\cdot 2^{-n/2}=$$
$$\frac{10}{3}\Big(\frac{50}{9\sqrt{2}}\Big)^n<\frac{10}{3}\Big(1.31\cdot k\Big)^n$$

□

7 Lower bounds – constructions

First we describe Patkós' construction [3, 4] for 4-regular graphs.

Construction 7.1. Suppose $n=2k+1$, and construct G as follows. Connect v_i and v_j if $i-j \mod n$ is one of the values $\{1,-1,k,-k\}$.

Theorem 7.2 (Patkós). *For graph G described in Construction 7.1, we have $c(G)\ge c\cdot\kappa^n$, where κ is the unique real root of the equation $x^3-x^2-x-1=0$, so $\kappa\approx 1.839$.*

For a construction for the union of two trees, take the same graph, and delete any two edges. The bound remains essentially the same (only the constant c changes).

Construction 7.3. Construct G on $n=3n'$ vertices: $V=a_1,b_1,c_1,a_2,\ldots\ldots,a_{n'},b_{n'},c_{n'}$ as follows. Connect b_i to c_i and a_i to b_{i-1},c_{i-1},b_i,c_i (where we identify b_0 with $b_{n'}$ and c_0 with $c_{n'}$). This graph has $m=5n'=\frac{5}{3}n$ edges.

Theorem 7.4. *For graph G described in Construction 7.3, we have $c(G)\ge 2^{m-n}$.*

Remark 7.5. If n is an odd number divisible by three, then Construction 7.3 can be derived from Construction 7.1 by deleting every third edge of the outer cycle. Consequently Construction 7.1 is derived from Construction 7.3 by adding the following edges: $b_1b_2, c_2c_3, b_3b_4, c_4c_5,\ldots$.

Next we construct a $2k$-regular multigraph, then a simple graph, and derive a construction for the union of k trees.

Construction 7.6. Construct G as follows. Take k parallel edges between v_i and v_i+1 as well as between v_n and v_1.

Theorem 7.7. *For graph G described in Construction 7.6, we have $c(G) = k^n$.*

Construction 7.8. Construct G as follows. We take $l = n/(2k+1)$ copies of complete graph K_{2k}. For each copy call the half of the vertices left, the other half right. Connect them cyclically as follows, adding l new vertices $a_1, a_2, \ldots, a_l$. Vertex a_i is connected to the left vertices of the ith copy and the right vertices of the $(i-1)$th copy (if $i = 1$ then the lth copy).

Theorem 7.9. *For graph G described in Construction 7.8, we have $c(G) \geq \left(\left(e \cdot k^2 \cdot (2k-2)!\right)^{1/(2k+1)}\right)^n \approx (2k/e)^n > (0.735 \cdot k)^n$, if k is large enough, and $c(G) \geq 2.48^n$, if $k = 3$.*

These two constructions have the property, that deleting any k edges, the resulting graph is the sum (union) of k trees, and if we delete a matching of size k then they have essentially the same number of cycles. Next we show some constructions, where the graph has many Hamiltonian cycles.

Construction 7.10 (Gerbner)**.** Construct a 3-regular graph G as follows. Let H denote $K_{3,3}$ with one edge deleted. Call one vertex of degree two in H left, the other vertex of degree two in H right. Take $n/6$ copies of H, and connect them cyclically: we put an edge between the right vertex of the ith copy and the left vertex of the $(i+1)$th copy (and also between the right vertex of the nth copy and the left vertex of the first copy).

Theorem 7.11 (Gerbner)**.** *For graph G described in Construction 7.10, we have $h(G) \geq 4^{n/6} \approx 1.2599^n$.*

Construction 7.12. Construct a 4-regular graph G as follows. Let H denote $K_{4,4}$ with one edge deleted. Call one vertex of degree three in H left, the other vertex of degree three in H right. Take $n/8$ copies of H, and connect them cyclically: we put an edge between the right vertex of the ith copy and the left vertex of the $(i+1)$th copy (and also between the right vertex of the nth copy and the left vertex of the first copy).

Theorem 7.13. *For graph G described in Construction 7.12, we have $h(G) \geq 36^{n/8} \approx 1.5651^n$.*

We conclude with three conjectures.

Conjecture 7.14. *If G is 4-regular then $c(G) \leq 1.99^n$.*

Conjecture 7.15. *If G is a graph with m edges then $c(G) \leq c \cdot 1.4^m$.*

Conjecture 7.16. *If G is 4-regular then $h(G) \leq 1.8^n$.*

References

[1] R. E: L. Aldred, C. Thomassen On the maximum number of cycles in a planar graph, *Journal of Graph Theory* **57** issue 3, (2008), pp. 255-264.

[2] C. A: Barefoot, L. Clark, R. Entringer Cubic graphs with the minimum number of cycles, *Proceedings of the seventeenth Southeastern international conference on combinatorics, graph theory, and computing (Boca Raton, Fla., 1986)* **Congr. Numer. 53** (1986), pp. 49-62.

[3] B. Patkós, Problems in extremal finite set theory, *PhD Thesis*, Central European University, Budapest, (2007)
`http://goya.ceu.hu/search/Xpatkos&SORT=A/Xpatkos&SORT=A`
`&SUBKEY=patkos/1,2,2,B/l856~b1127812&FF=Xpatkos&SORT=A&2,2,,1,0`

[4] B. Patkós, The distance of $\mathcal{F}$-free families, *to appear in Studia Scientiarum Mathematicarum Hungarica.*

[5] D. Gerbner, Personal communication.

[6] K. Markstrm Extremal graphs for some problems on cycles in graphs, *Congressus numeratium*, **171** (2004).

[7] D. Rautenbach, I. Stella, On the maximum number of cycles in a Hamiltonian graph, *Discrete Mathematics* **304** (2005), pp. 101-107.

[8] R. H. Teunter, E. S. van der Poort, The maximum number of Hamiltonian cycles in graphs with a fixed number of vertices and edges, *Operations Research Letters* **26** (2000), pp. 91-98.