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Operations Preserving Global Rigidity of Generic Direction-Length Frameworks

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Abstract

A two-dimensional direction-length framework is a pair (G, p), where $G = (V; D, L)$ is a graph whose edges are labeled as 'direction' or 'length' edges, and a map p from V to $\mathbb{R}^2$. The label of an edge uv represents a direction or length constraint between $p(u)$ and $p(v)$. The framework (G, p) is called globally rigid if every other framework (G, q) in which the direction or length between the endvertices of corresponding edges is the same, is 'congruent' to (G, p), i.e. it can be obtained from (G, p) by a translation and, possibly, a dilation by -1.

We show that labeled versions of the two Henneberg operations (0-extension and 1-extension) preserve global rigidity of generic direction-length frameworks. These results, together with appropriate inductive constructions, can be used to verify global rigidity of special families of generic direction-length frameworks.

1 Introduction

Consider a point configuration $p_1, p_2, ..., p_n$ in $\mathbb{R}^d$ together with a set of constraints which fix the direction or the length between some pairs p_i, p_j. A basic question is whether the configuration, with the given constraints, is locally or globally unique, up to 'congruence'. Results of this type have applications in localization problems of sensor networks, CAD, and molecular conformation [4, 14, 17]. The configuration and the constraints form a 'direction-length framework'.

A *mixed graph* $G = (V; D, L)$ is an undirected graph together with a labeling (or bipartition) $D \cup L$ of its edge set. We refer to edges in D as *direction edges* and edges in L as *length edges*. A *direction-length framework*, or more simply *mixed framework*, is a pair (G, p), where $G = (V; D, L)$ is a mixed graph and p is a map from V to $\mathbb{R}^d$, such that $p(u) \neq p(v)$ whenever $uv \in D \cup L$. When $L = \emptyset$, or $D = \emptyset$, we say that (G, p) is a *direction framework* or *length framework*, respectively, or simply that

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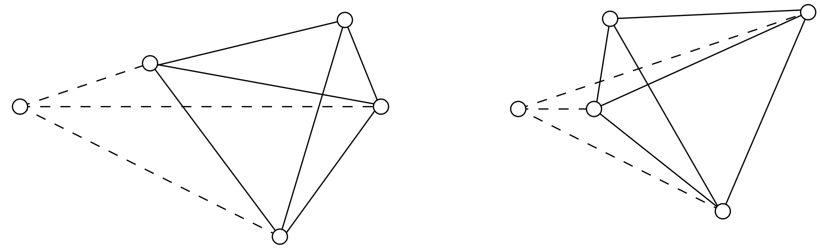

Figure 1: Two equivalent but non-congruent realizations of a mixed graph. We use solid or dashed lines to indicate edges with length or direction labels, respectively.

(G, p) is a *pure framework*. We also say that (G, p) is a *realization* of G in $\mathbb{R}^d$. Two frameworks (G, p) and (G, q) are *mixed-equivalent* if (i) $p(u) - p(v)$ is a scalar multiple of $q(u) - q(v)$ for all $uv \in D$ and (ii) $||p(u) - p(v)|| = ||q(u) - q(v)||$ for all $uv \in L$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$.

The mixed frameworks (G, p) and (G, q) are *direction-congruent* if $p(u) - p(v)$ is a scalar multiple of $q(u) - q(v)$ for all $u, v \in V$, *length-congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ for all $u, v \in V$, and *mixed-congruent* if they are both direction-congruent and length-congruent.

A mixed framework (direction framework, length framework) (G, p) is *globally mixed (direction, length) rigid* if every framework which is equivalent to (G, p) is mixed-congruent (direction-congruent, length-congruent, respectively) to (G, p). See Figure 1.

It is *mixed (direction, length) rigid* if there exists an $\epsilon > 0$ such that every framework (G, q) which is equivalent to (G, p) and satisfies $\|p(v) - q(v)\| < \epsilon$ for all $v \in V$, is mixed-congruent (direction-congruent, length-congruent, respectively) to (G, p). This is the same as saying that every continuous motion of the points $p(v)$, $v \in V$, which preserves the distances between pairs of points joined by length edges and directions between pairs joined by direction edges, results in a framework which is mixed (direction, length) congruent to (G, p).

It is a hard problem to decide if a given framework is rigid or globally rigid. Indeed Saxe [13] has shown that the global rigidity problem is NP-hard even for 1-dimensional length frameworks. The problem becomes more tractable, however, if we assume that there are no algebraic dependencies between the coordinates of the points of the framework. A framework (G, p) is said to be *generic* if the set containing the coordinates of all its points is algebraically independent over the rationals. We say that a mixed graph G is *globally mixed (direction, length) rigid* in $\mathbb{R}^d$ if all generic realizations of G in $\mathbb{R}^d$ as a mixed (direction, length) framework are globally rigid. Similarly, we say that a mixed graph G is *mixed (direction, length) rigid* in $\mathbb{R}^d$ if all generic realizations of G in $\mathbb{R}^d$ as a mixed (direction, length) framework are rigid. See Figure 2.

Whiteley [16] showed that a graph is direction rigid in $\mathbb{R}^d$ if and only if it is globally direction rigid in $\mathbb{R}^d$ and characterized the graphs which have this property. In contrast, length rigidity of graphs in $\mathbb{R}^d$ is a weaker property than global length rigidity for all $d \geq 1$, and graphs with these properties have been characterized only for

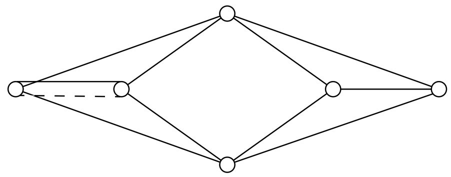

Figure 2: A mixed graph G for which no generic realization is globally mixed rigid. This follows from the fact that one side of a 2-separation of G contains length edges only and hence it can be flipped.

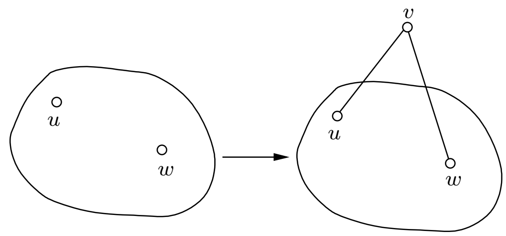

Figure 3: The 0-extension operation.

$d = 1, 2$. The cases when $d = 1$ are not difficult. The characterizations of length rigidity and global length rigidity when $d = 2$ are given in [11] and [8], respectively.

Henceforth, we assume that $d = 2$ unless specified otherwise. The characterizations of global direction and length rigidity in this case can both be formulated as inductive constructions using the following graph operations. The operation 0-*extension* on vertices u, w of a graph G adds a new vertex v and new edges vu, vw. The operation 1-*extension* on edge uw and vertex z of G deletes the edge uw and adds a new vertex v and new edges vu, vw, vz. See Figures 3, 4. These operations are known as *Henneberg operations* since they were first used in the study of rigidity by Henneberg [7]. Part (a) of the next theorem follows from results of Whiteley [16] and Tay and Whiteley [15]. Part (b) is from [8].

Theorem 1.1. *Let G be a graph. Then*
(a) G is globally direction rigid if and only if G can be obtained from K_2 by 0-extensions, 1-extensions, and edge-additions.
(b) G is globally length rigid if and only if $G = K_2$, $G = K_3$ or G can be obtained from K_4 by 1-extensions and edge-additions.

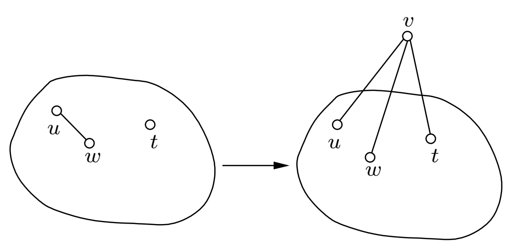

Figure 4: The 1-extension operation.

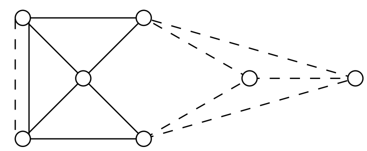

Figure 5: A globally mixed rigid mixed graph. The global rigidity of G can be verified by using the main results of this paper along with an appropriate inductive construction using labeled Henneberg operations, see also [10].

The problem of characterizing when a mixed graph G is globally mixed rigid is still an open problem. To solve this problem using a similar strategy to that for global length rigidity, we would require results which assert that the Henneberg operations preserve global mixed rigidity, as well as an inductive construction for the conjectured family of globally rigid mixed graphs which uses these operations. In this paper we present results of the first type by showing that mixed versions of the Henneberg operations preserve global rigidity. In [10] we use these results, together with new inductive constructions, to give a characterization for globally rigid mixed graphs G for which the edge set of G is a circuit in the direction-length rigidity matroid. (These concepts will be defined in Section 2.) This complements the results on globally length rigid graphs whose edge set is a circuit in the length-rigidity matroid [2] and may serve as a building block in a more complete characterization of mixed global rigidity.

Mixed versions of 0- and 1-extension are defined as follows. A 0-extension (on vertices u, w) of a mixed graph G, adds a new vertex v and new edges vu, vw in such a way that if vu, vw are of the same type (i.e. either they are both length edges or both direction edges) then u, w must be distinct. A 1-extension (on edge uw and vertex z) of G, deletes an edge uw and adds a new vertex v and new edges vu, vw, vz for some vertex $z \in V(G)$, with the provisos that at least one of the new edges has the same type as the deleted edge and, if $z = u$, then the two edges from z to u are of different type. (Servatius and Whiteley [14] used these operations in the proof of their characterization of mixed rigid mixed graphs.) In Sections 3, 4 we shall show that special cases of both these operations preserve global rigidity. Our main results are as follows.

Theorem 1.2. *Let G and H be mixed graphs with $|V(H)| \geq 2$. Suppose that G can be obtained from H by a 0-extension which adds a vertex v incident to two direction edges. Then G is globally mixed rigid if and only if H is globally mixed rigid.*

Theorem 1.3. *Let G and H be mixed graphs with $|V(H)| \geq 3$. Suppose that G can be obtained from H by a 1-extension on an edge uw. Suppose further that H is globally mixed rigid and $H - uw$ is mixed rigid. Then G is globally mixed rigid.*

A result on globally length rigid graphs analogous to Theorem 1.3 is given in [9], see also [3].

In Section 5 we will show that infinitesimal rigidity (defined in the next section) is a sufficient condition for rigidity, and that the two conditions are equivalent for generic mixed frameworks. Section 6 is devoted to concluding remarks.

2 Infinitesimal rigidity and the rigidity matrix

Servatius and Whiteley [14] developed a rigidity theory for mixed frameworks analogous to that given for pure frameworks. For $(x, y) \in \mathbb{R}^2$ let $(x, y)^\perp = (y, -x)$. The *rigidity matrix* of a mixed framework (G, p) is the matrix $R(G, p)$ of size $(|D| + |L|) \times 2|V|$, where, for each edge $uv \in D \cup L$, in the row corresponding to uv, the entries in the 2 columns corresponding to the vertex w are given by: $(p(u) - p(v))^\perp$ if $uv \in D$ and $w = u$; $-(p(u) - p(v))^\perp$ if $uv \in D$ and $w = v$; $(p(u) - p(v))$ if $uv \in L$ and $w = u$; $-(p(u) - p(v))$ if $uv \in L$ and $w = v$; $(0, 0)$ if $w \notin \{u, v\}$. The rigidity matrix of (G, p) defines the *rigidity matroid* of (G, p) on the ground set $D \cup L$ by linear independence of the rows of the rigidity matrix. The framework is said to be *independent* if the rows of $R(G, p)$ are linearly independent and *infinitesimally rigid* if the rank of $R(G, p)$ is $2|V| - 2$ (which is its maximum possible value). We will show in Section 5 that infinitesimal rigidity is a sufficient condition for rigidity, and that the two conditions are equivalent for generic mixed frameworks.

Any two generic frameworks (G, p) and (G, p') have the same rigidity matroid. We call this the (2-dimensional) *rigidity matroid* $\mathcal{R}(G) = (D \cup L, r)$ of the mixed graph G. We denote the rank of $\mathcal{R}(G)$ by $r(G)$. The mixed graph G is *independent*, or *rigid*, if $r(G) = |D| + |L|$, or $r(G) = 2|V| - 2$, respectively. Independent mixed graphs were characterized in [14]. This gives a characterization of the rigidity matroid of a mixed graph.

3 Generic points and quasi-generic frameworks

In this section we prove some preliminary results on generic frameworks which we will use in our proof that extensions preserve global rigidity. A point $\mathbf{x} \in \mathbb{R}^n$ is *generic* if its coordinates form an algebraically independent set over $\mathbb{Q}$.

Lemma 3.1. *Let f_i and g_i be non-zero polynomials with integer coefficients in the indeterminates $x_1, x_2, \ldots, x_n$, and $r_i = f_i/g_i$ for $1 \leq i \leq m$. Let $T_i = \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \neq 0\}$ for $1 \leq i \leq m$ and put $T = \bigcap_{i=1}^m T_i$. Let $f : T \to \mathbb{R}^m$ by $f(\mathbf{x}) = (r_1(\mathbf{x}), r_2(\mathbf{x}), \ldots, r_m(\mathbf{x}))$. Suppose that $\max_{\mathbf{x} \in \mathbb{R}^n}\{\text{rank } df|_{\mathbf{x}}\} = m$. If $\mathbf{p}$ is a generic point in $\mathbb{R}^n$, then $\mathbf{p} \in T$ and $f(\mathbf{p})$ is a generic point in $\mathbb{R}^m$.*

Proof: Since $\mathbf{p}$ is generic, we have $\mathbf{p} \in T$ and rank $df|_{\mathbf{p}} = m$. Relabelling if necessary, we may suppose that the first m columns of $df|_{\mathbf{p}}$ are linearly independent. Let $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Define $f' : \mathbb{R}^m \to \mathbb{R}^m$ by $f'(x_1, x_2, \ldots, x_m) = f(x_1, x_2, \ldots, x_m, p_{m+1}, \ldots, p_n)$. Let $\mathbf{p}' = (p_1, p_2, \ldots, p_m)$. Then $f'(\mathbf{p}') = f(\mathbf{p})$ and rank $df'|_{\mathbf{p}'} = m$.

Let $f'(\mathbf{p}') = (\beta_1, \beta_2, \ldots, \beta_m)$. Suppose that $h(\beta_1, \beta_2, \ldots, \beta_m) = 0$ for some polynomial h with integer coefficients. Then $h(r_1(\mathbf{p}), r_2(\mathbf{p}), \ldots, r_m(\mathbf{p})) = 0$. Since $\mathbf{p}$ is generic, we must have $h(f'(\mathbf{x})) = 0$ for all $\mathbf{x} \in \mathbb{R}^m$. By the inverse function theorem f' maps a sufficiently small open neighbourhood U of $\mathbf{p}'$ diffeomorphically onto $f'(U)$. Thus $h(\mathbf{y}) = h(f'(\mathbf{x})) = 0$ for all $\mathbf{y} \in f'(U)$. Since h is a polynomial map and $f'(U)$ is an open subset of $\mathbb{R}^m$, we have $h = 0$. Hence $f'(\mathbf{p}') = f(\mathbf{p})$ is generic. •

Given a point $\mathbf{p} \in \mathbb{R}^n$ we use $\mathbb{Q}(\mathbf{p})$ to denote the field extension of $\mathbb{Q}$ by the coordinates of $\mathbf{p}$. Given fields K, L with $K \subseteq L$ the *transcendence degree* of L over K, $td[L : K]$, is the size of the largest subset of L which is algebraically independent over K, see [5]. We use $\tilde{K}$ to denote the algebraic closure of K. Note that $td[\tilde{K} : K] = 0$.

Lemma 3.2. *Let f_i and g_i be non-zero polynomials with integer coefficients in the indeterminates $x_1, x_2, \ldots, x_n$ and $r_i = f_i/g_i$ for $1 \leq i \leq n$. Let $T_i = \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \neq 0\}$ for $1 \leq i \leq n$ and put $T = \bigcap_{i=1}^n T_i$. Let $f : T \to \mathbb{R}^n$ by $f(\mathbf{x}) = (r_1(\mathbf{x}), r_2(\mathbf{x}), \ldots, r_n(\mathbf{x}))$. Suppose that $f(\mathbf{p})$ is a generic point in $\mathbb{R}^n$. Let $L = \mathbb{Q}(\mathbf{p})$ and $K = \mathbb{Q}(f(\mathbf{p}))$. Then $\tilde{K} = \tilde{L}$.*

Proof: Since $f_i(\mathbf{x})$ is a ratio of two polynomials with integer coefficients, we have $f_i(\mathbf{p}) \in L$ for all $1 \leq i \leq n$. Thus $K \subseteq L$. Since $f(\mathbf{p})$ is generic, by Lemma 3.1 we have $td[K : \mathbb{Q}] = n$. Since $K \subseteq L$ and $L = \mathbb{Q}(\mathbf{p})$ we have $td[L : \mathbb{Q}] = n$. Thus $\tilde{K} \subseteq \tilde{L}$ and $td[\tilde{K} : \mathbb{Q}] = n = td[\tilde{L} : \mathbb{Q}]$. Suppose $\tilde{K} \neq \tilde{L}$, and choose $\gamma \in \tilde{L} - \tilde{K}$. Then γ is not algebraic over K so $S = \{\gamma, f_1(\mathbf{p}), f_2(\mathbf{p}), \ldots, f_n(\mathbf{p})\}$ is algebraically independent over $\mathbb{Q}$. This contradicts the facts that $S \subseteq \tilde{L}$ and $td[\tilde{L} : \mathbb{Q}] = n$. •

A *configuration* C is a sequence $(P_1, P_2, \ldots, P_n)$ of points in $\mathbb{R}^2$. Let $P_i = (x_i, y_i)$. We say that C is *generic* if the point $p = (x_1, y_1, x_2, y_2, \ldots, y_n)$ is a generic point in $\mathbb{R}^{2n}$. A configuration $C' = (Q_1, Q_2, \ldots, Q_n)$ is *congruent* to C if there exists a $t \in \mathbb{R}^2$ and a $\lambda \in \{-1, 1\}$ such that $Q_i = \lambda P_i + t$ for all $1 \leq i \leq n$. This is equivalent to saying that C' can be obtained from C by either a translation or a rotation through $180°$. We say that C is *quasi-generic* if C is congruent to a generic configuration, and that C is in *standard position* if $P_1 = (0, 0)$.

Lemma 3.3. *Let $C = (P_1, P_2, \ldots, P_n)$ be a configuration. Then C is quasi-generic if and only if $(P_2 - P_1, P_3 - P_1, \ldots, P_n - P_1)$ is generic.*

Proof: Suppose C is quasi-generic. Then there exists a $t \in \mathbb{R}^2$ and a $\lambda \in \{-1, 1\}$ such that $(\lambda P_1 + t, \lambda P_2 + t, \ldots, \lambda P_n + t)$ is generic. This implies that $(\lambda(P_2 - P_1), \lambda(P_3 - P_1), \ldots, \lambda(P_n - P_1))$ is generic and hence $(P_2 - P_1, P_3 - P_1, \ldots, P_n - P_1)$ is generic.

Suppose $(P_2 - P_1, P_3 - P_1, \ldots, P_n - P_1)$ is generic. Choose $P_0 \in \mathbb{R}^2$ such that $(P_0, P_2 - P_1, P_3 - P_1, \ldots, P_n - P_1)$ is generic. Then $C' = (P_0, P_2 - P_1 + P_0, P_3 - P_1 + P_0, \ldots, P_n - P_1 + P_0)$ is generic. Since we can transform C' to C by translating by $t = P_1 - P_0$, C is quasi-generic. •

Let (G, p) be a mixed framework, where $G = (V; D, L)$. For $v_1, v_2 \in V$ with $p(v_i) = (x_i, y_i)$ let $l_p(v_1, v_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2$, and $s_p(u, v) = (y_1 - y_2)/(x_1 - x_2)$ whenever $x_1 \neq x_2$. Let $e = uv \in D \cup L$ and let $p(u) - p(v) = (x, y)$. We say that e is *vertical* in (G, p) if $x = 0$. The *length* of e in (G, p) is given by $l_p(e) = l_p(u, v)$, and the *slope* of e by $s_p(e) = s_p(u, v)$ whenever e is not vertical in (G, p). We say that (G, p) is *generic*, *quasi-generic* or *in standard position* if $p(V)$ is, respectively, generic, quasi-generic, or in standard position. The graph G is *independent* if $D \cup L$ is an independent set in the mixed rigidity matroid of G. We say that G is *minimally rigid* if G is rigid and independent. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $D \cup L = \{e_1, e_2, \ldots, e_m\}$.

We can view p as a point $(p(v_1), p(v_2), \ldots, p(v_n))$ in $\mathbb{R}^{2n}$. Let T be the set of all points $p \in \mathbb{R}^{2n}$ such that (G, p) has no vertical direction edges. Then the *rigidity map* $f_G : T \to \mathbb{R}^m$ is given by $f_G(p) = (h(e_1), h(e_2), \ldots, h(e_m))$, where $h(e_i) = l_p(e_i)$ if $e_i \in L$ and $h(e_i) = s_p(e_i)$ if $e_i \in D$. Note that each row in the evaluation of the Jacobian of the rigidity map at the point $p \in T$ is a non-zero scalar multiple of the corresponding row in the mixed rigidity matrix of the framework (G, p).

Lemma 3.4. *If (G, p) is a quasi-generic framework and G is independent then $f_G(p)$ is generic.*

Proof: Choose a generic framework (G, q) conguent to (G, p). Since G is independent, rank $df_G|_q = |E|$. Hence Lemma 3.1 implies that $f_G(q)$ is generic. The lemma now follows since $f_G(p) = f_G(q)$. •

Lemma 3.5. *Suppose that (G, p) is in standard position and has no vertical edges, G is minimally rigid and $f_G(p)$ is generic. Let $p = (0, 0, p_3, p_4, p_5, \ldots, p_{2n})$, $L = \mathbb{Q}(p)$ and $K = \mathbb{Q}(f_G(p))$. Then $(p_3, p_4, p_5, \ldots, p_{2n})$ is generic and $\tilde{K} = \tilde{L}$.*

Proof: Let T be the set of all points $q \in \mathbb{R}^{2n}$ such that (G, q) is in standard position and has no vertical direction edges. Let T' be the set of all points $(q_3, q_4, \ldots, q_{2n})$ such that $(0, 0, q_3, q_4, \ldots, q_{2n}) \in T$. Define $f : T' \to \mathbb{R}^{2n-2}$ by

$$f(q_3, q_4, \ldots, q_{2n}) = f_G(0, 0, q_3, q_4, \ldots, q_{2n}).$$

Let $p' = (p_3, p_4, \ldots, p_{2n})$. Then $f(p') = f_G(p)$ is generic. We have $L = \mathbb{Q}(p')$ and $K = \mathbb{Q}(f(p'))$. By Lemma 3.2, $\tilde{K} = \tilde{L}$. Furthermore, $2n - 2 = td[\tilde{K}, \mathbb{Q}] = td[\tilde{L}, \mathbb{Q}]$. Thus p' is a generic point in $\mathbb{R}^{2n-2}$. •

Corollary 3.6. *Suppose that (G, p) is a rigid generic mixed framework and that (G, q) is equivalent to (G, p). Then (G, q) is quasi-generic.*

Proof: Let H be a minimally rigid spanning subgraph of G. Choose translations of $\mathbb{R}^2$ which map (H, p) and (H, q) to two frameworks (H, p') and (H, q') in standard position. By Lemma 3.4, $f_H(p)$ is generic. Thus $f_H(q') = f_H(p') = f_H(p)$ is generic. By Lemmas 3.5 and 3.3, (H, q') is quasi-generic. Hence (H, q) and (G, q) are quasi-generic. •

4 Extensions and globally linked vertices

We first prove that 0-extension preserves global rigidity.

Theorem 4.1. *Let $G = (V, D \cup L)$ be a mixed graph, $v \in V$, with $d(v) = 2$ and $vu, vw \in D$, and $H = G - v$. Then G is globally rigid if and only if H is globally rigid.*

Proof: Suppose that H is globally rigid. Let (G,p) be a generic framework and (G,q) be equivalent to (G,p). Since H is globally rigid, we may assume (by applying a suitable translation and/or rotation by 180°) that $p|_H = q|_H$. In particular, $p(u) = q(u)$ and $p(w) = q(w)$. Since $vu, vw \in D$, this implies that $p(v) = q(v)$. Thus (G,p) and (G,q) are congruent.

Suppose that G is globally rigid. Let (H,p') be a generic framework and (H,q') be equivalent to (H,p'). Choose a point $P \in \mathbb{R}^2$ such that $p'(V-v) \cup \{P\}$ is generic. Let (G,p) be the generic realization of G with $p(x) = p'(x)$ for all $x \in V - v$ and $p(v) = P$. Let Q be the point of intersection of the lines through $q'(u)$ and $q'(w)$ with slopes $s_p(v,u)$ and $s_p(v,w)$, respectively. Let (G,q) be the realization of G with $q(x) = q'(x)$ for all $x \in V - v$ and $q(v) = Q$. Then (G,p) and (G,q) are equivalent. Since G is globally rigid, (G,p) and (G,q) are congruent. Hence (H,p') and (H,q') are congruent. •

To prove a similar result concerning 1-extensions we need a few more definitions. Let (G,p) be a generic mixed framework and $u,v \in V$. The pair $\{u,v\}$ is *globally distance linked*, respectively *globally direction linked*, in (G,p) if, in all equivalent frameworks (G,q), we have $l_p(u,v) = l_q(u,v)$, respectively $s_p(u,v) = s_q(u,v)$. It is *globally linked* in (G,p) if it is both globally distance linked and globally direction linked. The pair $\{u,v\}$ is *globally distance linked*, respectively *globally direction linked* or *globally linked*, in G if it is globally distance linked, respectively globally direction linked or globally linked, in all generic frameworks (G,p).

Lemma 4.2. *Let $G = (V, D \cup L)$ be a mixed graph and $v \in V(G)$ with $d(v) = 3$. Let vu, vw, vt be the edges incident to v and suppose that $G - v$ is rigid.*
(a) If $\{vu, vw, vt\} \subseteq D$, then $\{u,w\}$, $\{u,t\}$ and $\{w,t\}$ are globally direction linked in G.
(b) If $\{vu, vw, vt\} \subseteq L$, then $\{u,w\}$, $\{u,t\}$ and $\{w,t\}$ are globally distance linked in G.
(c) If $\{vu, vw, vt\} \cap D \neq \emptyset \neq \{vu, vw, vt\} \cap L$, then $\{u,w\}$, $\{u,t\}$ and $\{w,t\}$ are globally linked in G.

Proof: Note that v must have three distinct neighbours in cases (a) and (b), but may only have two distinct neighbours in case (c). Relabelling if necessary, we may suppose that $u \neq w \neq t$ in all cases. Let $V = \{v_1, v_2, \ldots, v_n\}$, where $v_1 = u$, $v_2 = w$, $v_3 = t$ if $u \neq t$, and $v_n = v$. Let (G,p^*) and (G,q^*) be equivalent quasi-generic frameworks. By Lemma 3.3, (G,p^*) is congruent to a framework (G,p), where $p(v_1) = (0,0)$, $p(v_i) = (p_{2i-1}, p_{2i})$ for $2 \leq i \leq n$, and $\{p_3, p_4, \ldots, p_{2n}\}$ is algebraically independent over $\mathbb{Q}$. Similarly (G,q^*) is congruent to a framework (G,q), where $q(v_1) = (0,0)$, and $q(v_i) = (q_{2i-1}, q_{2i})$ for $2 \leq i \leq n$.

Let $p' = p|_{V-v}$ and $q' = q|_{V-v}$. Consider the equivalent quasi-generic frameworks $(G-v,p')$ and $(G-v,q')$. Applying Lemmas 3.4 and 3.5 to a minimally rigid spanning subgraph of $G - v$, we have $\tilde{K} = \tilde{L}$ where $K = \mathbb{Q}(p')$ and $L = \mathbb{Q}(q')$. Thus $q_3, q_4, q_5, q_6 \in \tilde{K}$ and q_3, q_4, q_5, q_6 are algebraically independent over $\mathbb{Q}$.

(a) Since (G, q) is equivalent to (G, p), we have the following equations:

$$\begin{aligned} q_{2n}/q_{2n-1} &= p_{2n}/p_{2n-1} && (1)\\ (q_{2n}-q_4)/(q_{2n-1}-q_3) &= (p_{2n}-p_4)/(p_{2n-1}-p_3) && (2)\\ (q_{2n}-q_6)/(q_{2n-1}-q_5) &= (p_{2n}-p_6)/(p_{2n-1}-p_5). && (3) \end{aligned}$$

We may rewrite each of the above equations as:

$$\begin{aligned} q_{2n}p_{2n-1} &= p_{2n}q_{2n-1} && (4)\\ q_{2n}(p_{2n-1}-p_3)-q_{2n-1}(p_{2n}-p_4) &= q_4(p_{2n-1}-p_3)-q_3(p_{2n}-p_4) && (5)\\ q_{2n}(p_{2n-1}-p_5)-q_{2n-1}(p_{2n}-p_6) &= q_6(p_{2n-1}-p_5)-q_5(p_{2n}-p_6) && (6) \end{aligned}$$

Using equations (4) and (5) we obtain

$$q_{2n-1}p_4 - q_{2n}p_3 = q_3(p_4 - p_{2n}) - q_4(p_3 - p_{2n-1}). \tag{7}$$

Similarly, using equations (4) and (6), we obtain

$$q_{2n-1}p_6 - q_{2n}p_5 = q_5(p_6 - p_{2n}) - q_6(p_5 - p_{2n-1}). \tag{8}$$

We may solve (7) and (8) for q_{2n-1} and q_{2n} and then substitute into (4) to obtain

$$a_{2,0}p_{2n-1}^2 + a_{0,2}p_{2n}^2 + a_{1,1}p_{2n-1}p_{2n} + a_{1,0}p_{2n-1} + a_{0,1}p_{2n} + a_{0,0} = 0,$$

where $a_{i,j} \in \tilde{K}$. This means, that there is a polynomial

$$f = a_{2,0}z_1^2 + a_{0,2}z_2^2 + a_{1,1}z_1z_2 + a_{1,0}z_1 + a_{0,1}z_2 + a_{0,0} \in \tilde{K}[z_1, z_2]$$

such that $f(p_{2n-1}, p_{2n}) = 0$. Since $\{p_3, p_4, \ldots, p_{2n}\}$ is algebraically independent over $\mathbb{Q}$, $\{p_{2n-1}, p_{2n}\}$ is algebraically independent over $\tilde{K}$. Thus $f \equiv 0$. We have, $a_{2,0} = q_4p_6 - q_6p_4$, $a_{0,2} = q_3p_5 - q_5p_3$ and $a_{1,1} = q_6p_3 - q_4p_5 + q_5p_4 - q_3p_6$. Putting $a_{2,0} = 0$ we obtain $p_6 = q_6p_4/q_4$. Similarly, $a_{0,2} = 0$ implies that $p_5 = q_5p_3/q_3$. We may substitute these values for p_5, p_6 into the equation $a_{1,1} = 0$ to obtain $p_4/p_3 = q_4/q_3$. Thus the pair (u, w) is globally direction linked. Symmetry now implies that (u, w) and (w, t) are also globally direction linked.

(b) Since (G, q) is equivalent to (G, p), we have the following equations:

$$\begin{aligned} q_{2n-1}^2 + q_{2n}^2 &= p_{2n-1}^2 + p_{2n}^2 && (9)\\ (q_{2n-1}-q_3)^2 + (q_{2n}-q_4)^2 &= (p_{2n-1}-p_3)^2 + (p_{2n}-p_4)^2 && (10)\\ (q_{2n-1}-q_5)^2 + (q_{2n}-q_6)^2 &= (p_{2n-1}-p_5)^2 + (p_{2n}-p_6)^2. && (11) \end{aligned}$$

Using equations (9) and (10) we obtain

$$q_3(2q_{2n-1} - q_3) + q_4(2q_{2n} - q_4) = p_3(2p_{2n-1} - p_3) + p_4(2p_{2n} - p_4). \tag{12}$$

Similarly, using equations (9) and (11), we obtain

$$q_5(2q_{2n-1} - q_5) + q_6(2q_{2n} - q_6) = p_5(2p_{2n-1} - p_5) + p_6(2p_{2n} - p_6). \tag{13}$$

We may solve (12) and (13) for q_{2n-1} and q_{2n} and then substitute into (9) to obtain

$$a_{2,0}p_{2n-1}^2 + a_{0,2}p_{2n}^2 + a_{1,1}p_{2n-1}p_{2n} + a_{1,0}p_{2n-1} + a_{0,1}p_{2n} + a_{0,0} = 0,$$

where $a_{i,j} \in \tilde{K}$. We may deduce, as in (a), that $a_{i,j} = 0$ for all $0 \leq i + j \leq 2$. In particular,

$$\begin{aligned}
a_{2,0} &= (p_3q_6 - p_5q_4)^2 + (p_3q_5 - p_5q_3)^2 - (q_3q_6 - q_5q_4)^2 = 0 \\
a_{0,2} &= (p_4q_6 - p_6q_4)^2 + (p_4q_5 - p_6q_3)^2 - (q_3q_6 - q_5q_4)^2 = 0 \\
a_{1,1} &= p_3(p_4q_6^2 + p_4q_5^2 - p_6q_3q_5 - p_6q_4q_6) + \\
&\quad p_5p_6q_3^2 + p_5p_6q_4^2 - p_4p_5q_4q_5 - p_4p_5q_3q_5 = 0.
\end{aligned}$$

We may solve the $a_{1,1}$-equation for p_3 and substitute into the $a_{2,0}$-equation to obtain $(q_3q_6 - q_5q_4)^2 b_{2,0} = 0$, where

$$b_{2,0} = p_5^2[(p_4q_6 - p_6q_4)^2 + (p_4q_5 - p_6q_3)^2] - [q_6(p_4q_6 - p_6q_4) + q_5(p_4q_5 - p_6q_3)]^2.$$

Since q_3, q_4, q_5, q_6 are algebraically independent over $\mathbb{Q}$, we have $(q_3q_6 - q_5q_4)^2 \neq 0$ and hence $b_{2,0} = 0$. If we now use the $a_{0,2}$-equation to replace $p_5^2(p_4q_6 - p_6q_4)^2 + (p_4q_5 - p_6q_3)^2$, $q_6^2(p_4q_6 - p_6q_4)^2$, and $q_5^2(p_4q_5 - p_6q_3)^2$ by $p_5^2(q_3q_6 - q_5q_4)^2$, $q_6^2[(q_3q_6 - q_5q_4)^2 - (p_4q_5 - p_6q_3)^2]$, and $q_5^2[(q_3q_6 - q_5q_4)^2 - (p_4q_6 - p_6q_4)^2]$, respectively, in the $b_{2,0}$-equation, we obtain

$$(q_3q_6 - q_5q_4)^2(p_5^2 + p_6^2 - q_5^2 - q_6^2) = 0.$$

Since $(q_3q_6 - q_5q_4)^2 \neq 0$, this gives $(p_5^2 + p_6^2) - (q_5^2 + q_6^2) = 0$. Thus the pair (u, t) is globally distance linked. Symmetry now implies that (u, w) and (w, t) are also globally distance linked.

(c) We need to consider two cases, depending on whether v is incident to two direction edges or two length edges.

Case 1: $vu \in L$ and $vw, vt \in D$.

We first consider the subcase when $u \neq t$. Since (G, q) is equivalent to (G, p), we have the following equations:

$$\begin{aligned}
q_{2n-1}^2 + q_{2n}^2 &= p_{2n-1}^2 + p_{2n}^2 && (14)\\
(q_{2n} - q_4)/(q_{2n-1} - q_3) &= (p_{2n} - p_4)/(p_{2n-1} - p_3) && (15)\\
(q_{2n} - q_6)/(q_{2n-1} - q_5) &= (p_{2n} - p_6)/(p_{2n-1} - p_5). && (16)
\end{aligned}$$

Using equation (15) we obtain

$$q_{2n-1}(p_4 - p_{2n}) - q_{2n}(p_3 - p_{2n-1}) = q_3(p_4 - p_{2n}) - q_4(p_3 - p_{2n-1}). \quad (17)$$

Similarly, using equation (16), we obtain

$$q_{2n-1}(p_6 - p_{2n}) - q_{2n}(p_5 - p_{2n-1}) = q_5(p_6 - p_{2n}) - q_6(p_5 - p_{2n-1}). \quad (18)$$

We may solve (17) and (18) for q_{2n-1} and q_{2n} and then substitute into (14) to obtain $\sum_{0\le i+j\le 4} a_{i,j}p_{2n-1}^i p_{2n}^j = 0$, where $a_{i,j} \in \tilde{K}$ for all $0 \le i+j \le 6$. Again we have $a_{i,j} = 0$ for all $0 \le i+j \le 4$. In particular,

$$\begin{aligned}
a_{4,0} &= (p_6 - p_4)^2 - (q_6 - q_4)^2 = 0 \\
a_{3,1} &= 2((p_5 - p_3)(p_6 - p_4) - (q_5 - q_3)(q_6 - q_4)) = 0 \\
a_{3,0} &= 2[(p_6 - p_4)(p_4p_5 - p_6p_3) + (q_6 - q_4)(p_4q_3 - p_6q_5) + \\
&\quad (q_6 - q_4)^2(p_5 + p_3)] = 0 \\
a_{0,3} &= 2[(p_5 - p_3)(p_3p_6 - p_5p_4) + (q_5 - q_3)(p_3q_4 - p_5q_6) + \\
&\quad (q_5 - q_3)^2(p_6 + p_4)] = 0
\end{aligned}$$

The $a_{4,0}$-equation tells us that $q_6 - q_4 = \alpha(p_6 - p_4)$ for some $\alpha \in \{1, -1\}$. Substituting into the $a_{3,1}$- and $a_{3,0}$-equations, we obtain $q_5 - q_3 = \alpha(p_5 - p_3)$, and $(p_6 - p_4)(p_6[p_5 - \alpha q_5] - p_4[p_3 - \alpha q_3]) = 0$. Since $p_6 \ne p_4$, we may use both equations to deduce that $p_3 = \alpha q_3$ and $p_5 = \alpha q_5$. A similar argument using the $a_{4,0}$-, $a_{3,1}$- and $a_{0,3}$-equations gives $p_4 = \alpha q_4$ and $p_6 = \alpha q_6$. Hence either $p_i = q_i$ for all $3 \le i \le 6$ or $p_i = -q_i$ for all $3 \le i \le 6$. Thus $\{u, w\}$, $\{u, t\}$ and $\{w, t\}$ are globally linked in G.

The subcase when $u = t$ can be handled similarly. We replace each of p_5, p_6, q_5, q_6 by zero in the above analysis. The resulting $a_{4,0}$- and $a_{3,1}$-equations then imply that $(p_3, p_4) = \pm(q_3, q_4)$ and hence (u, w) is globally linked in G.

Case 2: $vu \in D$ and $vw, vt \in L$.
We first consider the subcase when $u \ne t$. Since (G, q) is equivalent to (G, p), we have the following equations:

$$\begin{aligned}
q_{2n}/q_{2n-1} &= p_{2n}/p_{2n-1} && (19)\\
(q_{2n-1} - q_3)^2 + (q_{2n} - q_4)^2 &= (p_{2n-1} - p_3)^2 + (p_{2n} - p_4)^2 && (20)\\
(q_{2n-1} - q_5)^2 + (q_{2n} - q_6)^2 &= (p_{2n-1} - p_5)^2 + (p_{2n} - p_6)^2. && (21)
\end{aligned}$$

Using equations (20) and (21) we obtain

$$\begin{aligned}
&(2q_{2n-1} - q_3 - q_5)(q_5 - q_3) + (2q_{2n} - q_4 - q_6)(q_6 - q_4) = \\
&\qquad (2p_{2n-1} - p_3 - p_5)(p_5 - p_3) + (2p_{2n} - p_4 - p_6)(p_6 - p_4). \qquad (22)
\end{aligned}$$

We may solve (19) and (22) for q_{2n-1} and q_{2n} and then substitute into (20) to obtain $\sum_{0\le i+j\le 4} a_{i,j}p_{2n-1}^i p_{2n}^j = 0$, where $a_{ij} \in \tilde{K}$ for all $0 \le i+j \le 4$. Again we have $a_{i,j} = 0$ for all $0 \le i+j \le 4$. In particular,

$$\begin{aligned}
a_{4,0} &= (q_5 - q_3)^2 - (p_5 - p_3)^2 = 0 \\
a_{0,4} &= (q_6 - q_4)^2 - (p_6 - p_4)^2 = 0 \\
a_{3,1} &= 2(-p_5p_6 - p_3p_4 + p_3p_6 + p_4p_5 + q_5q_6 - q_3q_6 - q_5q_4 + q_3q_4) = 0 \\
a_{3,0} &= (p_5 - p_3)(p_5^2 - p_3^2 + p_6^2 - p_4^2 + q_4^2 - q_6^2) - (p_5 + p_3)(q_5 - q_3)^2 = 0 \\
a_{0,3} &= (p_6 - p_4)(p_6^2 - p_4^2 + p_5^2 - p_3^2 + q_3^2 - q_5^2) - (p_6 + p_4)(q_6 - q_4)^2 = 0
\end{aligned}$$

The $a_{4,0}$-equation tells us that $(q_5 - q_3)^2 = (p_5 - p_3)^2$. We may use this to replace $(q_5 - q_3)^2$ by $(p_5 - p_3)^2$ in the last term of the $a_{3,0}$ equation to obtain $(p_5 - p_3)(p_6^2 - p_4^2 + q_4^2 - q_6^2) = 0$. Since $\{p_3, p_4, \ldots, p_{2n}\}$ is algebraically independent over $\mathbb{Q}$, $(p_5 - p_3) \neq 0$. Thus $p_6^2 - p_4^2 = q_6^2 - q_4^2$. We may combine this with the $a_{4,0}$-equation to deduce that $p_4^2 = q_4^2$ and $p_6^2 = q_6^2$. A similar analysis using the $a_{0,4}$- and $a_{0,3}$-equations yeilds $p_3^2 = q_3^2$ and $p_5^2 = q_5^2$. Thus $p_i = \alpha_i q_i$ for some $\alpha_i \in \{1, -1\}$ and all $3 \le i \le 6$. Substituting into the $a_{3,1}$-equation we obtain

$$(1 - \alpha_5\alpha_6)q_5q_6 + (1 - \alpha_3\alpha_4)q_3q_4 + (\alpha_3\alpha_6 - 1)q_3q_6 + (\alpha_4\alpha_5 - 1)q_4q_5 = 0.$$

Since $\{q_3, q_4, q_5, q_6\}$ is algebraically independent over $\mathbb{Q}$, all coefficients must be zero. This implies that all the α_i are equal and hence either $p_i = q_i$ for all $3 \le i \le 6$ or $p_i = -q_i$ for all $3 \le i \le 6$. Thus $\{u, w\}$, $\{u, t\}$ and $\{w, t\}$ are globally linked in G.

The subcase when $u = t$ can be handled similarly. We replace each of p_5, p_6, q_5, q_6 by zero in the above analysis. The resulting $a_{4,0}$-, $a_{0,4}$- and $a_{3,1}$-equations then imply that $(p_3, p_4) = \pm(q_3, q_4)$ and hence (u, w) is globally linked in G. •

Theorem 4.3. *Let G be a 1-extension of a mixed graph H, so $H = G - v + uw$ for some vertex v of G where u, w are neighbours of v. Suppose that $H - uw$ is rigid and that x, y are vertices of H. If (x, y) is globally linked in H then (x, y) is globally linked in G.*

Proof: Suppose (G, p) is a generic mixed framework and that (G, q) is equivalent to (G, p). Let $p' = p|_{V-v}$ and $q' = q|_{V-v}$. Since $G - v = H - uw$ is rigid, Lemma 4.2 implies that $\{u, w\}$ is globally direction linked in G if G is a direction 1-extension of H, that $\{u, w\}$ is globally distance linked in G if G is a distance 1-extension of H, and that $\{u, w\}$ is globally linked in G if G is a mixed 1-extension of H. Thus (H, p') and (H, q') are equivalent. Since $\{x, y\}$ is globally linked in H, we have

$$l_p(x, y) = l_{p'}(x, y) = l_{q'}(x, y) = l_q(x, y)$$

and

$$s_p(x, y) = s_{p'}(x, y) = s_{q'}(x, y) = s_q(x, y).$$

Thus $\{x, y\}$ is globally linked in G. •

Corollary 4.4. *Let H be a globally rigid mixed graph with $|V(H)| \ge 3$ and G be obtained from H by a 1-extension, which deletes an edge uw and adds a vertex v joined to vertices u, w, t. Suppose that $H - uw$ is rigid. Then G is globally rigid.*

Proof: Theorem 4.3 and the fact that H is globally rigid imply that all pairs $\{x, y\} \subseteq V - v$ are globally linked in G. Suppose (G, p) is a generic framework and that (G, q) is equivalent to (G, p). Let $p' = p|_{V-v}$ and $q' = q|_{V-v}$. Since all pairs $\{x, y\} \subseteq V - v$ are globally linked in G, we may assume (by applying a suitable translation and/or rotation by 180° to (G, p)) that $p' = q'$. Since (G, p) is generic and $d_G(v) = 3$, this implies that we must also have $p(v) = q(v)$. Thus (G, p) and (G, q) are congruent. •

5 Rigidity and infinitesimal rigidity

In this section we show that infinitesimal rigidity and rigidity are equivalent conditions for generic mixed frameworks. It follows that infinitesimal rigidity is a necessary condition for global mixed rigidity in mixed graphs. (Other necessary conditions for mixed global rigidity are given in [10].) Our proofs use the rigidity map and some elementary differential topology. They are similar to those of the analogous results for length frameworks given by Asimow and Roth [1].

Let U be an open subset of $\mathbb{R}^m$, $f : U \to \mathbb{R}^n$ be a smooth map. For $X \subset \mathbb{R}^n$ let $f^{-1}(X) = \{u \in U \ : \ f(u) \in X\}$. Let k be the maximum rank of the Jacobian $df|_y$ over all $y \in U$. A point $x \in U$, is a *regular point* of f if rank $df|_x = k$, and $f(x)$ is a *regular value* of f if $f^{-1}(f(x))$ contains only regular points. Note that if $G = (V; D, L)$ is a rigid mixed graph and $f_G : T \to \mathbb{R}^{|D \cup L|}$ is its rigidity map, then a point $p \in T$ is a regular point of f_G if and only if (G, p) is infinitesimally rigid, and $f_G(p)$ is a regular value of f_G if and only if all frameworks (G, q) which are equivalent to (G, p) are infinitesimally rigid.

Lemma 5.1. *Suppose that (G, p) is an infinitesimally rigid mixed framework. Then (G, p) is rigid.*

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Since (G, p) is infinitesimally rigid, we may choose a spanning subgraph H of G such that H has $2n - 2$ edges and (H, p) is infinitesimally rigid. Relabelling the vertices of G if necessary, we may suppose that the last $2n - 2$ rows of $df_H|_p$ are linearly independent. Suppose that (H, p) is not rigid. Then for all $k \geq 1$, there exists a realization $(H, p(k))$ of H such that (H, p) and $(H, p(k))$ are equivalent non-congruent frameworks and $\|p - p(k)\| < 1/k$. Applying a suitable translation to (H, p) and to each $(H, p(k))$ if necessary, we may suppose that (H, p) and each $(H, p(k))$ are in standard position with v_1 mapped onto the origin. We may choose our coordinate system such that (H, p), and hence also each $(H, p(k))$, has no vertical direction edges. Let T be the set of all points $q \in \mathbb{R}^{2n}$ such that (H, q) is in standard position and has no vertical direction edges. For each $q = (0, 0, q_3, q_4, \ldots, q_{2n}) \in T$ let $\hat{q} = (q_3, q_4, \ldots, q_{2n})$ and put $\hat{T} = \{\hat{q} \ : \ q \in T\}$. Define $f : \hat{T} \to \mathbb{R}^{2n-2}$ by $f(\hat{q}) = f_H(q)$. Since the last $2n-2$ rows of $df_H|_p$ are linearly independent, $df|_{\hat{p}}$ is non-singular. By the inverse function theorem, f maps some open neighborhood U of $\hat{p}$ in $\hat{T}$ diffeomorphically onto $f(U)$. In particular, for all points $\hat{q} \in U - \hat{p}$, we have $f(\hat{q}) \neq f(\hat{p})$. This contradicts the fact that the points $p(\hat{k})$ converge to $\hat{p}$ and satisfy $f(p(\hat{k})) = f(\hat{p})$ for all $k \geq 1$. Thus (H, p) is rigid. Hence (G, p) is also rigid. •

We next show that the converse of Lemma 5.1 holds when (G, p) is quasi-generic. We will need the following basic result from differential topology, see for example [12, Lemma 1, page 11].

Lemma 5.2. *Let U be an open subset of $\mathbb{R}^m$ and $f : U \to \mathbb{R}^n$ be a smooth map. Suppose that $x \in U$ and that $f(x)$ is a regular value of f with rank $df|_x = k$. Then $f^{-1}(f(x))$ is an $(m - k)$-dimensional manifold.*

Lemma 5.3. *Suppose that (G, p) is a quasi-generic mixed framework. Then (G, p) is rigid if and only if (G, p) is infinitesimally rigid.*

Proof: Sufficiency follows from Lemma 5.1. Suppose that that (G, p) is not infinitesimally rigid. We may choose our coordinate system such that (G, p) is in standard position and has no vertical direction edges. Let T be the set of all points $q \in \mathbb{R}^{2n}$ such that (G, q) is in standard position and has no vertical direction edges, where $n = |V(G)|$. For each $q = (0, 0, q_3, q_4, \ldots, q_{2n}) \in T$ let $\hat{q} = (q_3, q_4, \ldots, q_{2n})$ and put $\hat{T} = \{\hat{q} \ : \ q \in T\}$. Define $f : \hat{T} \to \mathbb{R}^m$ by $f(\hat{q}) = f_G(q)$, where $m = |E(G)|$. Since $\hat{p}$ is generic, $\hat{p}$ is a regular point of f. Let $k = \text{rank } df|_{\hat{p}}$. By continuity, there exists an open neighbourhood $W \subset \hat{T}$ of $\hat{p}$ such that rank $df|_{\hat{w}} = k$ for all $w \in W$. Let $g = f|_W$. Then $f(\hat{p})$ is a regular value of g. By Lemma 5.2, $M = g^{-1}(g(\hat{p})) = f^{-1}(f(\hat{p})) \cap W$ is a $(2n-2-k)$-dimensional manifold. Since (G, p) is not infinitesimally rigid, $k < 2n-2$ and hence M has dimension at least one. Thus we may choose a sequence of points $\hat{p}_i \in M - \{\hat{p}\}$, converging to $\hat{p}$. Since $\hat{p}_i \in M$, $f(\hat{p}_i) = f(p)$ and hence (G, p_i) is equivalent to (G, p). Since (G, p_i) and (G, p) are in standard position and $\hat{p}_i \neq \hat{p}$, (G, p_i) is not a translation of (G, p). Furthermore, when $\hat{p}_i$ is close enough to $\hat{p}$, (G, p_i) is not a dilation of (G, p) by -1. Hence (G, p_i) is not congruent to (G, p) whenever p_i is close enough to p. Thus (G, p) is not rigid. •

6 Concluding remarks

In this paper we proved, among others, that mixed versions of the Henneberg operations preserve global mixed rigidity. In [10] we use these results, together with new inductive constructions, to give a characterization for globally rigid mixed graphs G for which the edge set of G is a circuit in the direction-length rigidity matroid. Although this characterization may lead to a more complete characterization of global mixed rigidity, the following example indicates that we may need additional operations (other than 0- and 1-extensions and edge-additions) in an inductive construction which could be used to characterize global mixed rigidity in general mixed graphs. Consider the mixed graph G obtained by connecting two disjoint copies of pure length K_4's by four disjoint direction edges. This graph, which appears to be globally mixed rigid, has no vertices of degree less than four and for all edges e of G the mixed graph $G - e$ is no longer globally mixed rigid. Thus it cannot be obtained from a smaller globally mixed rigid graph by any of the above operations.

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