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A sufficient connectivity condition for generic rigidity in the plane

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Abstract

A graph $G = (V, E)$ is said to be 6-mixed-connected if $G-U-D$ is connected for all sets $U \subseteq V$ and $D \subseteq E$ which satisfy $2|U| + |D| \leq 5$. In this note we prove that 6-mixed-connected graphs are (redundantly globally) rigid in the plane. This improves on a previous result of Lovász and Yemini.

1 Introduction

All graph considered are without loops and multiple edges. In this note we consider sufficient connectivity conditions which imply the rigidity or global rigidity of a graph in two dimensions. For definitions and basic results on rigid and globally rigid bar-and-joint frameworks and graphs see e.g. [2, 3, 4, 9]. It is well-known, by a result of Lovász and Yemini [8, Theorem 2] from 1982, that 6-vertex-connected graphs are rigid in the plane. This implies, by using the more recent characterization of globally rigid graphs [4, Theorem 7.1], that 6-vertex-connectivity is also sufficient to ensure global rigidity [4, Theorem 7.2]. An infinite family of 5-vertex-connected non-rigid graphs given in [8] shows that hypothesis on the vertex connectivity in the Lovasz-Yemini theorem cannot be reduced from six to five. On the other hand, it was shown in [5] that the connectivity hypothesis can be replaced by a slightly weaker hypothesis of 'essential-6-vertex-connectivity' which allows vertex cuts of size four or five as long as they only separate one or at most three vertices, respectively, from the graph.

The purpose of this note is to show that the connectivity hypothesis of the Lovasz-Yemini theorem can be weakened in a more substantial way and still guarantee the rigidity and global rigidity of the graph. To this end we define the following form of 'mixed connectivity', which was introduced, in a more general form, by Kaneko and Ota [6] and has turned out to be a useful concept in graph connectivity, see

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e.g. [1]. Let $G = (V, E)$ be a graph. A pair (U, D) with $U \subseteq V$ and $D \subseteq E$ is a *mixed cut* in G if $G - U - D$ is not connected. We say that G is 6-*mixed-connected* if $2|U| + |D| \geq 6$ for all mixed cuts (U, D) in G. Equivalently, G is 6-mixed-connected if G is 6-edge-connected, $G - v$ is 4-edge-connected for all $v \in V$, and $G - \{u, v\}$ is 2-edge-connected for all pairs $u, v \in V$. It follows that 6-vertex-connected graphs are 6-mixed-connected and 6-mixed-connected graphs are 3-vertex-connected.

The following characterization of rigidity, which can be deduced from Laman's result [7], is a slight reformulation of [8, Corollary 4], see [4, Corollary 2.5]. A *cover* of $G = (V, E)$ is a collection $\mathcal{X} = \{X_1, X_2, ..., X_t\}$ of subsets of V such that $\{E(G[X_1], E(G[X_2], ..., E(G[X_t])\}$ partitions E, where $E(G[X])$ denotes the set of edges in the subgraph $G[X]$ of G induced by X.

Theorem 1.1. *[8] Let $G = (V, E)$ be a graph. Then G is rigid if and only if for all covers $\mathcal{X}$ of G we have $\sum_{X \in \mathcal{X}} (2|X| - 3) \geq 2|V| - 3$.*

As it is noted in [8], 6-vertex-connectivity of $G = (V, E)$ not only implies that G is rigid, but also implies the stronger result that $G - F$ is rigid for all $F \subseteq E$ with $|F| \leq 3$. We shall extend this stronger form to 6-mixed-connected graphs.

Theorem 1.2. *Let $G = (V, E)$ be a 6-mixed-connected graph. Then $G - F$ is rigid for all $F \subseteq E$ with $|F| \leq 3$.*

Proof: We proceed by contradiction. Suppose that the graph G and specified edge set F is a counterexample chosen such that $|V| + |F|$ is as small as possible and, subject to this condition, $|E|$ is as large as possible. Since $G - F$ is not rigid, Theorem 1.1 implies that there exists a cover $\mathcal{X}$ of $G - F$ such that $\sum_{X \in \mathcal{X}} (2|X| - 3) \leq 2|V| - 4$. Let Y_f be the vertex set of f for each $f \in F$. The minimality of $|F|$ implies that $Y_f \not\subseteq X$ for all $f \in F$ and $X \in \mathcal{X}$. Hence $\mathcal{Y} = \mathcal{X} \cup \{Y_f \ : \ f \in F\}$ is a cover of G which satisfies

$$\sum_{Y \in \mathcal{Y}} (2|Y| - 3) \leq 2|V| - 1. \tag{1}$$

The maximality of $|E|$ implies that $G[Y]$ is a complete graph $K_{|Y|}$ for all $Y \in \mathcal{Y}$.

For each $v \in V$, let $c(v)$ be the number of sets in $\mathcal{Y}$ which contain v.

Claim 1.3. *Suppose $c(v) = 1$ for some $v \in V$ and let Y be the member of $\mathcal{Y}$ which contains v. Then $G[Y] = K_7$.*

Proof: Let $G[Y] = K_p$. Since G is 6-mixed-connected, $d_G(v) \geq 6$. Since $c(v) = 1$, this implies that $p \geq 7$. Let $G' = G - v$, $Y' = Y - v$ and $\mathcal{X}' = (\mathcal{X} - Y) \cup Y'$. Then $\mathcal{X}'$ covers $G' - F$ and $\sum_{X \in \mathcal{X}'} (2|X| - 3) \leq 2|V - v| - 4$. Thus $G' - F$ is not rigid. The minimality of $|V| + |F|$ now implies that G' is not 6-mixed-connected. Thus $G' = (V', E')$ has a mixed cut (U, D) with $U \subseteq V'$, $D \subseteq E'$, and $2|U| + |D| \leq 5$. Let $H = G' - U - D$, H_1 be a connected component of H, and $H_2 = H - H_1$. Since G is 6-mixed-connected, v is adjacent to at least one vertex of each of H_1 and H_2 in G. Let x, y_1, y_2 be the number of neighbours of v in U, H_1, and H_2, respectively. The fact that $G[Y] = K_p$ implies that $x + y_1 + y_2 = d_G(v) = p - 1$, and that the number of edges of G' from H_1 to H_2 is at least $y_1 y_2$. Thus $2x + y_1 y_2 \leq 2|U| + |D| \leq 5$. This, and the fact that $x + y_1 + y_2 = p - 1 \geq 6$ gives $p = 7$. •

Let $\mathcal{Z} = \{Y \in \mathcal{Y} \;:\; c(v) = 1 \text{ for some } v \in Y\}$ and $B = \bigcup_{Z \in \mathcal{Z}} Z$. For each $v \in V$, let $T(v)$ be the multi-set containing the sizes of the distinct members of $\mathcal{Y}$ which contain v, and let $b(v)$ be the number of elements of $\mathcal{Z}$ which contain v.

Claim 1.4. *Suppose $Z \in \mathcal{Z}$. Then*

$$\sum_{v \in Z} \frac{1}{b(v)} \sum_{a \in T(v)} \left(2 - \frac{3}{a}\right) \geq 2 \sum_{v \in Z} \frac{1}{b(v)} .$$

Proof: Let:

$$\begin{aligned}
Z_1 &= \{v \in Z \;:\; c(v) = 1\} \\
Z_2 &= \{v \in Z \;:\; b(v) = 1 \text{ and } T(v) = \{7, 2\}\} \\
Z_3 &= \{v \in Z \;:\; c(v) \geq 2,\; b(v) = 1, \text{ and } T(v) \neq \{7, 2\}\} \\
Z_4 &= \{v \in Z \;:\; b(v) \geq 2\}.
\end{aligned}$$

The definitions of Z_1, Z_2 and Claim 1.3 imply:

$$\sum_{v \in Z_1} \frac{1}{b(v)} \sum_{a \in T(v)} \left(2 - \frac{3}{a}\right) = \frac{11}{7}|Z_1| \tag{2}$$

$$\sum_{v \in Z_2} \frac{1}{b(v)} \sum_{a \in T(v)} \left(2 - \frac{3}{a}\right) = \frac{29}{14}|Z_2|. \tag{3}$$

Similarly, the definitions of Z_3, Z_4 and Claim 1.3 imply:

$$\sum_{v \in Z_3} \frac{1}{b(v)} \sum_{a \in T(v)} \left(2 - \frac{3}{a}\right) \geq \frac{18}{7}|Z_3| \tag{4}$$

$$\sum_{v \in Z_4} \frac{1}{b(v)} \sum_{a \in T(v)} \left(2 - \frac{3}{a}\right) \geq \frac{11}{7}|Z_4|. \tag{5}$$

We also have:

$$2 \sum_{v \in Z} \frac{1}{b(v)} \leq 2|Z_1| + 2|Z_2| + 2|Z_3| + |Z_4| . \tag{6}$$

We may use (in)equalities (2) to (6) and the fact that $|Z_1| = 7 - |Z_2| - |Z_3| - |Z_4|$ to obtain:

$$\begin{aligned}
\sum_{v \in Z} \frac{1}{b(v)} \sum_{a \in T(v)} \left(2 - \frac{3}{a}\right) &\geq 2 \sum_{v \in Z} \frac{1}{b(v)} - \frac{3}{7}|Z_1| + \frac{1}{14}|Z_2| + \frac{4}{7}\left(|Z_3| + |Z_4|\right) \\
&= 2 \sum_{v \in Z} \frac{1}{b(v)} + \frac{1}{2}|Z_2| + |Z_3| + |Z_4| - 3. \qquad (7)
\end{aligned}$$

Let D be the set of edges of G from Z_2 to $V - Z$. The definition of Z_2 implies that $|D| = |Z_2|$. Since $Z_1 \neq \emptyset$, $(Z_3 \cup Z_4, D)$ is a mixed cut of G. (Note that $V - Z \neq \emptyset$, since otherwise $G = K_7$ and $F = \emptyset$ would follow, contradicting the fact that K_7 is rigid.) Hence

$$2(|Z_3| + |Z_4|) + |D| = 2(|Z_3| + |Z_4|) + |Z_2| \geq 6. \tag{8}$$

The claim now follows from (7) and (8). •

We can now complete the proof of the theorem. For each $v \in V$ let $\mathcal{Y}_v$ be the set of all elements of $\mathcal{Y}$ which contain v. We have

$$\sum_{v\in V}\sum_{Y\in\mathcal{Y}_v}\left(2-\frac{3}{|Y|}\right) = \sum_{Y\in\mathcal{Y}}\sum_{v\in Y}\left(2-\frac{3}{|Y|}\right) = \sum_{Y\in\mathcal{Y}}|Y|\left(2-\frac{3}{|Y|}\right) = \sum_{Y\in\mathcal{Y}}(2|Y|-3) \le 2|V|-1 \tag{9}$$

by (1). On the other hand

$$\sum_{v\in V}\sum_{Y\in\mathcal{Y}_v}\left(2-\frac{3}{|Y|}\right) = \sum_{v\in B}\sum_{Y\in\mathcal{Y}_v}\left(2-\frac{3}{|Y|}\right) + \sum_{v\in V-B}\sum_{Y\in\mathcal{Y}_v}\left(2-\frac{3}{|Y|}\right).$$

By Claim 1.4

$$\sum_{v\in B}\sum_{Y\in\mathcal{Y}_v}\left(2-\frac{3}{|Y|}\right) = \sum_{Z\in\mathcal{Z}}\sum_{v\in Z}\frac{1}{b(v)}\sum_{a\in T(v)}\left(2-\frac{3}{a}\right) \ge 2\sum_{Z\in\mathcal{Z}}\sum_{v\in Z}\frac{1}{b(v)} = 2|B|.$$

Furthermore, for each $v \in V-B$, we have $c(v) = |\mathcal{Y}_v| \ge 2$, and $\sum_{Y\in\mathcal{Y}_v}(|Y|-1) \ge 6$, since $d_G(v) \ge 6$. These inequalities imply that

$$\sum_{Y\in\mathcal{Y}_v}\left(2-\frac{3}{|Y|}\right) \ge 2.$$

Thus

$$\sum_{v\in V-B}\sum_{Y\in\mathcal{Y}_v}\left(2-\frac{3}{|Y|}\right) \ge 2|V-B|.$$

Thus

$$\sum_{v\in V}\sum_{Y\in\mathcal{Y}_v}\left(2-\frac{3}{|Y|}\right) \ge 2|B| + 2|V-B| = 2|V|.$$

This contradicts (9) and completes the proof of the theorem. •

We showed in[4, Theorem 7.1] that a graph on at least four vertices is globally rigid in two dimensions if and only if it is 3-vertex-connected and redundantly rigid, i.e. it remains rigid after deleting any of its edges. Let $G = (V, E)$ be a 6-mixed-connected graph. By definition, $G - e$ is 3-vertex-connected for all $e \in E$. Theorem 1.2 implies that $G - e$ is redundantly rigid for all $e \in E$. Thus we obtain:

Theorem 1.5. *Let $G = (V, E)$ be a 6-mixed-connected graph. Then $G - e$ is globally rigid for all $e \in E$.*

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