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Abstract

Tensegrity frameworks are defined on a set of points in $\mathbb{R}^d$ and consist of bars, cables, and struts, which provide upper and/or lower bounds for the distance between their endpoints. The graph of the framework, in which edges are labelled as bars, cables, and struts, is called a tensegrity graph. It is said to be rigid in $\mathbb{R}^d$ if it has an infinitesimally rigid realization in $\mathbb{R}^d$ as a tensegrity framework.

We show that a graph can be labelled as a rigid tensegrity graph in $\mathbb{R}^d$ containing only cables and struts if and only if it is redundantly rigid in $\mathbb{R}^d$. When $d = 2$ we give an efficient combinatorial algorithm for finding a rigid cable-strut labelling. We also obtain some partial results on the characterization of rigid tensegrity graphs in $\mathbb{R}^2$.

1 Introduction

A *tensegrity graph* $T = (V; B, C, S)$ is a simple graph on vertex set $V = \{v_1, v_2 \dots, v_n\}$ and edge set $E = B \cup C \cup S$. The elements of E are called *members* and are labelled as *bars*, *cables*, and *struts*, respectively. A tensegrity graph containing no bars is called a *cable-strut tensegrity graph.* The *underlying graph* of T is the (unlabelled) graph $\overline{T} = (V; E)$. A d-dimensional *tensegrity framework* is a pair (T, p), where $T = (V; B, C, S)$ is a tensegrity graph and p is a map from V to $\mathbb{R}^d$. (T, p) is also called a *realization* of T. If T has neither cables nor struts then we may simply call it a graph and a realization (T, p) is said to be a *bar framework.*

An *infinitesimal motion* of a tensegrity framework is an assignment of infinitesimal velocities $u_i \in \mathbb{R}^d$ to the vertices, such that

$$\begin{aligned} (p_i - p_j)(u_i - u_j) &= 0 \quad \text{for all } ij \in B, \\ (p_i - p_j)(u_i - u_j) &\le 0 \quad \text{for all } ij \in C, \\ (p_i - p_j)(u_i - u_j) &\ge 0 \quad \text{for all } ij \in S. \end{aligned}$$

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An infinitesimal motion is *trivial*, if $(p_i - p_j)(u_i - u_j) = 0$ for all $1 \leq i, j \leq n$. The tensegrity framework (T, p) is *infinitesimally rigid* in $\mathbb{R}^d$ if all of its infinitesimal motions are trivial. A tensegrity graph T is said to be *rigid* in $\mathbb{R}^d$ if it has an infinitesimally rigid realization (T, p) in $\mathbb{R}^d$. We refer the reader to [4, 5, 8, 10] for more details on the rigidity of tensegrity frameworks.

In this paper we consider the following problems:

(a) Which graphs are the underlying graphs of rigid cable-strut tensegrity graphs in $\mathbb{R}^d$?

(b) If $G = (V, E)$ is the underlying graph of a rigid cable-strut tensegrity graph in $\mathbb{R}^d$, how can one find, in polynomial time, a cable-strut labelling $E = C \cup S$, for which $T = (V; C \cup S)$ is a rigid tensegrity graph?

(c) Which tensegrity graphs are rigid in $\mathbb{R}^d$?

We shall prove that a graph is the underlying graph of a rigid cable-strut tensegrity graph in $\mathbb{R}^d$ if and only if it has a redundantly rigid realization as a bar framework in $\mathbb{R}^d$. This leads to a complete charaterization for $d \leq 2$. When $d = 2$ we also give an efficient combinatorial algorithm for finding a rigid cable-strut labelling.

The answer to the third question is known only for $d = 1$, see [7]. For $d \geq 3$ even the special case of graphs is one of the major open problems in combinatorial rigidity. Here we make a few observations and give a solution for two special families of graphs in the case when $d = 2$.

2 Preliminaries

A *stress* of a tensegrity framework $T(p)$ is an assignment of scalars ω_{ij} to the members ij of T satisfying $\omega_{ij} \leq 0$ for cables, $\omega_{ij} \geq 0$ for struts and

$$\sum_{ij \in E} \omega_{ij}(p_i - p_j) = 0 \quad \text{for each } i \in V.$$

We say that $\omega = (\dots, \omega_{ij}, \dots) \in \mathbb{R}^E$ is a *proper stress*, if $\omega_{ij} \neq 0$ for all $ij \in C \cup S$. The following basic results on infinitesimally rigid tensegrity frameworks are due to Roth and Whiteley, see [8, Theorem 5.2(c), Corollary 5.3].

Theorem 2.1. *[8] Suppose that (T, p) is a tensegrity framework in $\mathbb{R}^d$. Then*
(i) (T, p) is infinitesimally rigid in $\mathbb{R}^d$ if and only if the bar framework $(\overline{T}, p)$ is infinitesimally rigid in $\mathbb{R}^d$ and there exists a proper stress of (T, p);
(ii) if (T, p) is infinitesimally rigid then the bar framework obtained by deleting any cable or strut of T and replacing the remaining members of T by bars is infinitesimally rigid in $\mathbb{R}^d$.

The *rigidity matrix* of a tensegrity framework (T, p) on n vertices is the matrix $R(T, p)$ of size $|E| \times dn$, where, for each member $v_i v_j \in E$, in the row corresponding to v_iv_j, the entries in the d columns corresponding to vertices i and j contain the d coordinates of $(p(v_i) - p(v_j))$ and $(p(v_j) - p(v_i))$, respectively, and the remaining entries are zeros. Note that a stress of (T, p) corresponds to a row dependency of $R(T, p)$.

A configuration $p \in \mathbb{R}^{dn}$ is a *regular point* of T if $\operatorname{rank} R(T, p) = \max\{\operatorname{rank} R(T, q) : q \in \mathbb{R}^{dn}\}$. It is *generic* if it also gives rise to a regular point for all non-empty edge-induced subgraphs of T. Note that the regular (generic) points of T form an open and dense subset of $\mathbb{R}^{dn}$. We also say that a realization (T, p) is regular (generic) if p is a regular (generic, respectively) point of T. A member e of T is *redundant* in (T, p) if $\operatorname{rank} R(T, p) = \operatorname{rank} R(T - e, p)$. The proof of the next lemma is implicit in the proof of [8, Theorem 5.4].

Theorem 2.2. *[8] Let (T, p) be a regular realization of tensegrity graph T in $\mathbb{R}^d$, let ω be a proper stress of (T, p), and let e be a redundant member in (T, p). Then the set*

$$\{q \in \mathbb{R}^{dn} : (T, q) \text{ is a regular realization of } T \text{ which has a proper stress } \omega' \text{ with } \omega'(e) = \omega(e)\}$$

is open in $\mathbb{R}^{dn}$.

Note that the infinitesimal velocities of a bar framework (G, p) are the vectors in the null space of $R(G, p)$. Hence (G, p) is infinitesimally rigid in $\mathbb{R}^d$ if and only if $\operatorname{rank} R(G, p) = \max\{\operatorname{rank}(K_n, q) : q \in \mathbb{R}^{dn}\}$, where K_n is the complete graph on n vertices. It also follows that infinitesimal rigidity of graphs is a generic property: G has an infinitesimally rigid realization if and only if all generic realizations are infinitesimally rigid. A graph G is said to be *rigid* in $\mathbb{R}^d$ if it has an infinitesimally rigid realization as a bar framework in $\mathbb{R}^d$. The characterization of rigid graphs in $\mathbb{R}^d$ is known for $d \leq 2$, see e.g. [9]. We say that G is *redundantly rigid* if $G - e$ is rigid for all $e \in E$. Clearly, G is the underlying graph of a rigid tensegrity graph if and only if G is rigid. The question becomes more interesting if the tensegrity graph must not contain bars.

Theorem 2.3. *A graph G is the underlying graph of a rigid cable-strut tensegrity graph in $\mathbb{R}^d$ if and only if G is redundantly rigid in $\mathbb{R}^d$.*

Proof. Necessity follows from Theorem 2.1(ii).

To prove sufficiency consider a generic realization (G, p) of G in $\mathbb{R}^d$. Since G is redundantly rigid, (G, p) is infinitesimally rigid and each edge of G is redundant in (G, p). By Theorem 2.1(i) it is enough to prove that there exists a stress $\omega \in \mathbb{R}^E$ of the bar framework (G, p) which is non-zero on each member. Then defining $C = \{ij \in E \mid \omega_{ij} < 0\}$ and $S = \{ij \in E \mid \omega_{ij} > 0\}$ gives rise to a rigid cable-strut tensegrity graph T with underlying graph G by Theorem 2.1.

The stresses of the bar framework (G, p) form a linear subspace of $\mathbb{R}^E$, namely

$$W(G, p) = \{\omega \in \mathbb{R}^E \mid \omega R(G, p) = 0.\}$$

For a contradiction suppose that for each $\omega \in W(G, p)$ there is a coordinate $ij \in E$ such that $\omega_{ij} = 0$. This means that $W(G, p)$ is in the union of $|E|$ linear subspaces of

$\mathbb{R}^E$. Thus, since $W(G,p)$ is linear, it must be contained entirely in one of them. Hence there exists an edge $ij \in E$ for which each $\omega \in W(G,p)$ satisfies $\omega_{ij} = 0$. However, this contradicts the fact that ij is redundant in (G,p). □

3 Redundant graphs in two dimensions

In the rest of the paper we shall assume that $d = 2$ unless specified otherwise. In this section we recall and prove some combinatorial properties of redundantly rigid graphs in $\mathbb{R}^2$. We say that $H = (V,E)$ is an *M-circuit* if for all generic realizations (H,p) of H the rows of $R(H,p)$ form a minimal linearly dependent set of vectors in $\mathbb{R}^{2n}$. We say that G is *redundant* if it has at least one edge and each edge of G is in an M-circuit. It follows that a graph G is redundantly rigid if and only if G is rigid and redundant. It is known that M-circuits are redundantly rigid graphs in two dimensions. See [6] for more details on the properties of M-circuits in $\mathbb{R}^2$.

A *j-separation* of a graph $G = (V,E)$ is a pair (G_1, G_2) of edge-disjoint subgraphs of G each with at least $j+1$ vertices such that $G = G_1 \cup G_2$ and $|V(G_1) \cap V(G_2)| = j$. We say that G is 3-connected if G has at least 4 vertices and has no j-separation for all $0 \le j \le 2$. If (G_1, G_2) is a 2-separation of G, then we say that $V(G_1) \cap V(G_2)$ is a *2-separator* of G.

Suppose that $G = (V,E)$ is a 2-connected graph and let (G_1, G_2) be a 2-separation of G with $V(G_1) \cap V(G_2) = \{u,v\}$. For $1 \le i \le 2$, let $G_i' = G_i + uv$ if $uv \notin E(G_i)$ and otherwise put $G_i' = G_i$. We say that G_1', G_2' are the *cleavage graphs* obtained by *cleaving G along* $\{u,v\}$.

Lemma 3.1. *Suppose that G is a 2-connected redundant graph. Let $\{u,v\}$ be a 2-separator of G and let $\tilde{H}_1$ and $\tilde{H}_2$ be the cleavage graphs obtained by cleaving G along $\{u,v\}$. Then one of the following holds:*
(i) $\tilde{H}_i$ is redundant for $i = 1,2$;
(ii) there is a 2-separation (H_1, H_2) of G with $V(H_1) \cap V(H_2) = \{u,v\}$ for which H_i is redundant for $i = 1,2$.

Proof. First we prove that each edge $f \in E(\tilde{H}_1) - uv$ belongs to an M-circuit in $\tilde{H}_1$. Since G is redundant, there is an M-circuit C in G which contains f. If C is a subgraph of $\tilde{H}_1$ then we are done. If not, then $\{u,v\}$ is a 2-separator of C. In this case it follows from [2, Lemma 4.2] that the cleavage graphs C_1 and C_2 obtained by cleaving C along $\{u,v\}$ are both M-circuits. Hence C_1 is an M-circuit in $\tilde{H}_1$ which contains f. By symmetry we also have that each edge $f' \in E(\tilde{H}_2) - uv$ belongs to an M-circuit in $\tilde{H}_2$.

Thus, if uv belongs to an M-circuit in both cleavage graphs then (i) holds. Now suppose that, say, uv is in no M-circuit in $\tilde{H}_1$. As above, this implies that if $uv \in E(G)$ then all M-circuits of G containing uv must be in $\tilde{H}_2$ and if $uv \notin E(G)$ then all M-circuits of G containing some edge of $E(\tilde{H}_1) - uv$ must be in $\tilde{H}_1 - uv$.

By moving the edge uv from one side of the 2-separation to the other, if necessary, we may assume that there is a 2-separation (H_1, H_2) of G with $V(H_1) \cap V(H_2) = \{u,v\}$ and $uv \notin E(H_1)$. The arguments above now imply that H_1 and H_2 are both redundant. Thus (ii) holds. □

We need the following result on redundant graphs. It follows by observing that the proof of [6, Theorem 3.2] goes through under the weaker hypothesis that G is redundant, and by using [6, Lemma 3.1].

Theorem 3.2. *[6] Suppose that G is a 3-connected redundant graph. Then G is redundantly rigid.*

The *1-extension* operation (on edge uw and vertex t) deletes an edge uw from a graph G and adds a new vertex v and new edges vu, vw, vt for some vertex $t \in V(G) - \{u, w\}$. The following result gives an inductive construction for 3-connected redundantly rigid graphs.

Theorem 3.3. *[6, Theorem 6.15] Let G be a 3-connected redundantly rigid graph. Then G can be obtained from K_4 by a sequence of 1-extensions and edge additions.*

4 Operations on tensegrity graphs

In this section we introduce the 'labelled generalizations' of the 1-extension and 2-sum oeartions and show that they preserve rigidity when applied to tensegrity graphs. These operations, whose unlabelled versions are well-known in combinatorial rigidity, will be used in the next section to define rigid cable-strut labellings of graphs.

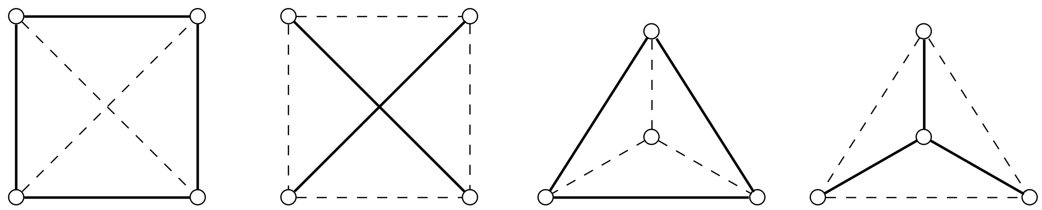

Figure 1: Infinitesimally rigid realizations of the four rigid tensegrity graphs on K_4. (In this paper we use solid (dashed) lines to denote struts (resp. cables).)

Let $T = (V; B \cup C \cup S)$ be a tensegrity graph, let $uw \in C \cup S$ be a cable or strut of T and let $t \in V - \{u, w\}$ be a vertex. The *labelled* 1-*extension* operation deletes the member uw, adds a new vertex v and new members vu, vw, vt, satisfying the condition that if uw is a cable then at least one of vu, vw is not a strut, and if uw is a strut then at least one of vu, vw is not a cable. The new member vt may be arbitrary. For example, if we consider cable-strut tensegrity graphs, this definition leads to six possible labelled 1-extensions on a strut uw, as illustrated in Figure 2.

Lemma 4.1. *Let T be a rigid tensegrity graph and let T' be a tensegrity graph obtained from T by a labelled 1-extension. Then T' is also rigid.*

Proof. Since infinitesimal rigidity (and the labelled 1-extension operation) is preserved by interchanging cables and struts, we may assume that the 1-extension is made on a strut uw of T and vertex $t \in V - \{u, w\}$. Let (T, p) be an infinitesimally rigid realization of T in $\mathbb{R}^2$. By Theorem 2.1 there is a proper stress ω of (T, p) and (G, p)

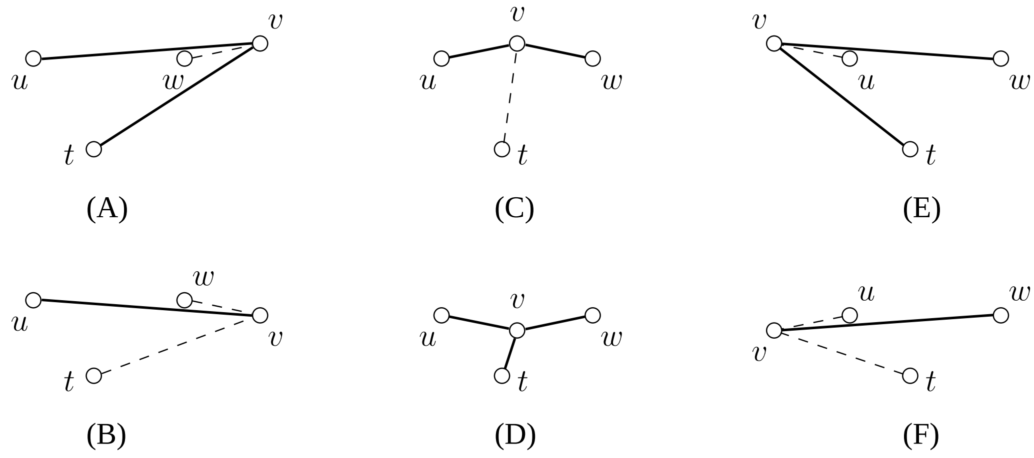

Figure 2: The six possible labelled 1-extensions on the strut uw and the feasible positions of v.

is an infinitesimally rigid bar framework, where $G = \overline{T}$ is the underlying graph of T. By Theorem 2.2 we may assume that $p(u), p(w), p(t)$ are not collinear. In the rest of the proof we shall also assume that the new members vu, vw, vt are all struts. The proof is similar for each of the six possible labelled 1-extensions.

Let us extend the configuration p by putting $p(v) = \alpha p(u) + (1-\alpha)p(w)$ for some $0 < \alpha < 1$. Let G' be the underlying graph of T', which can be obtained from G by a 1-extension. We can also extend the stress ω of (T, p) to (T', p) by defining $\omega_{vu} = \omega_{uw}/(1-\alpha)$, $\omega_{vw} = \omega_{uw}/\alpha$ and $\omega_{vt} = 0$.

Since $p(u), p(w), p(t)$ are not collinear, the bar framework (G', p) is infinitesimally rigid, see [9, Theorem 2.2.2]. Furthermore, the extended stress is nearly proper on (T', p): the only member with a zero stress is vt. This implies that (T'', p) is infinitesimally rigid, where T'' is obtained from T' by replacing the strut vt by a bar.

To obtain a proper stress we need to modify the realization a bit by replacing $p(v)$ by a point in the interior of the triangle $p(u)p(w)p(t)$. By Theorem 2.2 this can be done without destroying the infinitesimal rigidity of (T'', p). Consider a proper stress ω' of this modified realization of T''. Since we have three members incident with v, and vu, vw are struts, we must have a positive stress on vt. Thus we may replace vt by a strut and obtain the required infinitesimally rigid realization of T'.

The other labelled 1-extensions can be treated in a similar manner by appropriately defining $\alpha \in \mathbb{R} - \{0, 1\}$ and moving $p(v)$ out of the line of $p(u)p(w)$ in such a way that signs of the stresses on the members incident to v are as required. See Figure 2. $\square$

We shall also need an operation that glues together two tensegrity graphs along a pair of members. Let $T_1 = (V_1; B_1, C_1, S_1)$ and $T_2 = (V_2; B_2, C_2, S_2)$ be two tensegrity graphs with $V_1 \cap V_2 = \emptyset$ and let $u_1v_1 \in S_1$ and $u_2v_2 \in C_2$ be two designated members, a strut in T_1 and a cable in T_2. The 2-*sum* of T_1 and T_2 (along the strut-cable pair u_1v_1 and u_2v_2) is the tensegrity graph obtained from $T_1 - u_1v_1$ and $T_2 - u_2v_2$ by identifying u_1 with u_2 and v_1 with v_2. See Figure 3. We denote a 2-sum of T_1 and T_2 by $T_1 \oplus_2 T_2$.

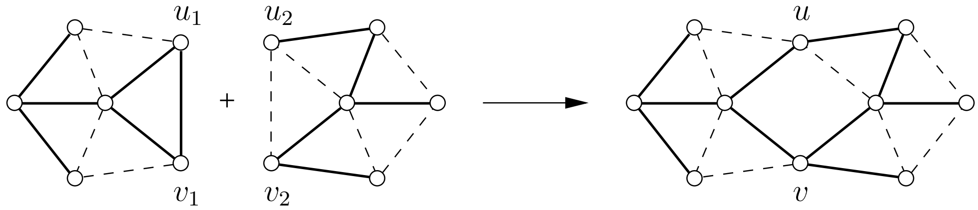

Figure 3: The 2-sum of two tensegrity graphs along the pair u_1v_1, u_2v_2.

Since we shall apply the 2-sum operation to non-rigid tensegrity graphs as well, we first prove the following lemma.

Lemma 4.2. *Let (T_1, p_1) and (T_2, p_2) be regular realizations of tensegrity graphs T_1, T_2 with a proper stress. Then $T = T_1 \oplus_2 T_2$ also has a regular realization with a proper stress.*

Proof. By Theorem 2.2 we may assume that (T_i, p_i) is generic for $i = 1, 2$. Let ω_i be a proper stress of (T_i, p_i), $i = 1, 2$. By scaling, translating, and rotating the frameworks, if necessary, we may assume that $p_1(u_1) = p_2(u_2)$ and $p_1(v_1) = p_2(v_2)$. These operations will not destroy genericity and ω_i remains a proper stress in the realization of T_i for $i = 1, 2$. By scaling the stresses we can also assume that $\omega_1(u_1v_1) = -\omega_2(u_2v_2) = 1$. Since the realizations are generic, it follows from Theorem 2.1(ii) that u_1v_1 and u_2v_2 are both redundant.

Let T' be the tensegrity graph obtained from T_1, T_2 by identifying u_1 with u_2, and v_1 with v_2. Consider the realization (T', p) of T' obtained by merging the frameworks (T_i, p_i), $i = 1, 2$, along the points $p_1(u_1), p_1(v_1)$. We can find generic realizations of T' arbitrarily close to (T', p) without changing the positions of $p(u), p(v)$. Now we can use Theorem 2.2, applied to each (T_i, p_i), and the fact that u_iv_i is redundant in (T_i, p_i), $i = 1, 2$, to deduce that there is an $\epsilon > 0$ for which any regular realization of T' in the ϵ-neighbourhood has a proper stress whose value is equal to 1 on the strut u_1v_1 and -1 on the cable u_2v_2. Since the stresses on u_1v_1 and u_2v_2 cancel each other, we have that (T, p) is a regular realization of $T = T_1 \oplus_2 T_2$ with a proper stress. This proves the lemma. □

Theorem 2.1 and the glueing lemma [9, Lemma 3.1.4] gives the following corollary.

Lemma 4.3. *Suppose that T_1 and T_2 are rigid tensegrity graphs. Then $T = T_1 \oplus_2 T_2$ is also rigid.*

5 Cable-strut labellings of redundant graphs

In this section we give a new proof of the two-dimensional version of Theorem 2.3. Based on this new approach, we can develop an efficient combinatorial algorithm for finding a rigid cable-strut labelling of a redundantly rigid graph. The proof will imply that every redundant graph can be obtained from disjoint copies of K_4 (which is the smallest redundant graph) by some simple operations. By starting with one of the four possible rigid cable-strut labellings of each of these K_4's (see Figure 1) we can obtain a rigid cable-strut labelling of G by using the labelled versions of these operations.

Theorem 5.1. *Let $G = (V, E)$ be a redundant graph in $\mathbb{R}^2$. Then the edge set of G has a cable-strut labelling $E = C \cup S$ for which the tensegrity graph $T = (V; C \cup S)$ has a regular realization with a proper stress. Furthermore, such a cable-strut labelling of E can be found in polynomial time.*

Proof. We prove the theorem by induction on $|V|$. Since G is redundant and the smallest M-circuit is K_4, we must have $|V| \geq 4$ with equality only if $G = K_4$. The statement is straightforward for K_4 (see Figure 1), so we may assume that $|V| \geq 5$ and that the theorem holds for all redundant graphs containing less vertices than G.

First suppose that G has at least two blocks (i.e. maximal 2-connected subgraphs), denoted by $H_1, H_2, ..., H_t$. Since M-circuits are 2-connected, each block is redundant. Thus, by induction, we can find a cable-strut labelling of each block H_i and a regular realization of the corresponding tensegrity graph T_i with a proper stress. Let T be the tensegrity graph on G whose cable-strut labelling is induced by the T_i's. Since a proper stress remains a proper stress after translating a framework, and since the blocks of G are edge-disjoint, we may obtain a realization (T, p) of T with a proper stress by simply translating and merging the realizations of the T_i's at the cut-vertices of G. Since the regular realizations of T form a dense open set, we can use Theorem 2.2, applied to each of the realizations of the T_i's, to make the realization regular. This shows that G has the required labelling.

Hence we may assume that G is 2-connected. If G is 3-connected then G can be obtained from K_4 by 1-extensions and edge additions by Theorem 3.3. Thus we can obtain a rigid cable-strut tensegrity graph T with underlying graph G by starting with a rigid cable-strut labelling of K_4 and using labelled 1-extensions as well as cable or strut additions, following the inductive construction of G. Theorem 4.1 implies that the labelled graph is indeed rigid. (The addition of new members clearly preserves rigidity.) Since an infinitesimally rigid realization of T is regular and has a proper stress by Theorem 2.1, the existence of the required cable-strut labelling of G follows.

It remains to consider the case when G is 2-connected and has a 2-separation $\{u, v\}$. Let $\tilde{H}_1, \tilde{H}_2$ be the cleavage graphs obtained by cleaving G along $\{u, v\}$. By Lemma 3.1 either G can be obtained as the edge-disjoint union of two redundant graphs with two vertices in common or both cleavage graphs are redundant. In the former case we can proceed as in the case of 1-separations: by induction, we can find good cable-strut labellings and realizations of the smaller graphs. These labellings induce a cable-strut labelling T of G. Furthermore, by first rotating, translating, and scaling the frameworks, if necessary, we can merge the realizations to obtain a realization of T with a proper stress. By perturbing this realization, and using Theorem 2.2, we can make the realization regular, too.

In the latter case we can also find, by induction, good cable-strut labellings and realizations of the cleavage graphs. We may assume that these realizations are generic by Theorem 2.2. Then, after interchanging cables and struts in one of the cleavage graphs, if necessary, we take the 2-sum of the labelled cleavage graphs to obtain a good labelling T' and a regular realization (T', p) of $G' = G - uv$ with a proper stress, by Lemma 4.2. If $uv \notin E(G)$ then this provides a desired labelling of G. Now suppose that $uv \in E(G)$. By Theorem 2.2 we may assume that p is chosen so that $(T' + uv, p)$ is generic. Then uv is redundant in $(T' + uv, p)$, and hence there is a stress ω' of $(T' + uv, p)$ whose value on uv is not zero. By adding ω' to ω with a small coefficient we obtain a proper stress of $(T' + uv, p)$. Thus adding a new cable (or strut) uv to the labelled graph T' gives the required labelling of G. This completes the proof of the first part of the theorem.

To see that there is an efficient algorithm for finding a good cable-strut labelling note that the proof shows that G can be reduced to disjoint copies of K_4's by applying the inverse operations of 1-extensions, edge additions, 2-sums, and merging along at most two vertices. This provides an inductive construction of G. Such a construction can be obtained in polynomial time, since each of the required subroutines (e.g. finding small separators, testing rigidity, and testing redundantness) can be performed in polynomial time, see e.g. [1]. We omit the details. By following the steps of the construction we can find a good labelling of G by applying labelled 1-extensions, cable or strut additions, interchanging cables and struts in certain subgraphs, taking 2-sums, and merging. □

Theorem 2.1 and Theorem 5.1 implies:

Theorem 5.2. *Let $G = (V, E)$ be a redundantly rigid graph in $\mathbb{R}^2$. Then the edge set of G has a cable-strut labelling $E = C \cup S$ for which the tensegrity graph $T = (V; C \cup S)$ is rigid. Furthermore, such a cable-strut labelling of E can be found in polynomial time.*

We remark that the proof of Theorem 5.1 implies that one can find rigid cable-strut labellings of redundantly rigid graphs with various structural properties. For example, consider a 3-connected M-circuit G. Then it can be shown that if G has at least five vertices then G has a rigid cable-strut labelling in which the cables as well as the struts induce a spanning tree of G. One can also verify that if G contains a triangle then G has a rigid cable-strut labelling in which the cables induce a single triangle and all other members are struts. We omit the details.

6 Rigid cable-strut tensegrity graphs

As we noted earlier, the characterization of the rigid (cable-strut) tensegrity graphs is still open, even in two dimensions. In one dimension it turns out that a cable-strut tensegrity graph T is rigid if and only if its underlying graph is rigid (i.e. connected) and each of its M-connected components[1] (i.e. blocks) contains at least one cable and

[1]A graph is M-connected if each pair of its edges belongs to an M-circuit. The M-connected components of a graph G are the maximal M-connected subgraphs of G. Thus M-circuits are M-connected graphs. It is known that M-connected graphs are redundantly rigid and 3-connected redundantly rigid graphs are M-connected. It is also known that the 2-sum of two M-connected graphs is M-connected. See [6] for more details.

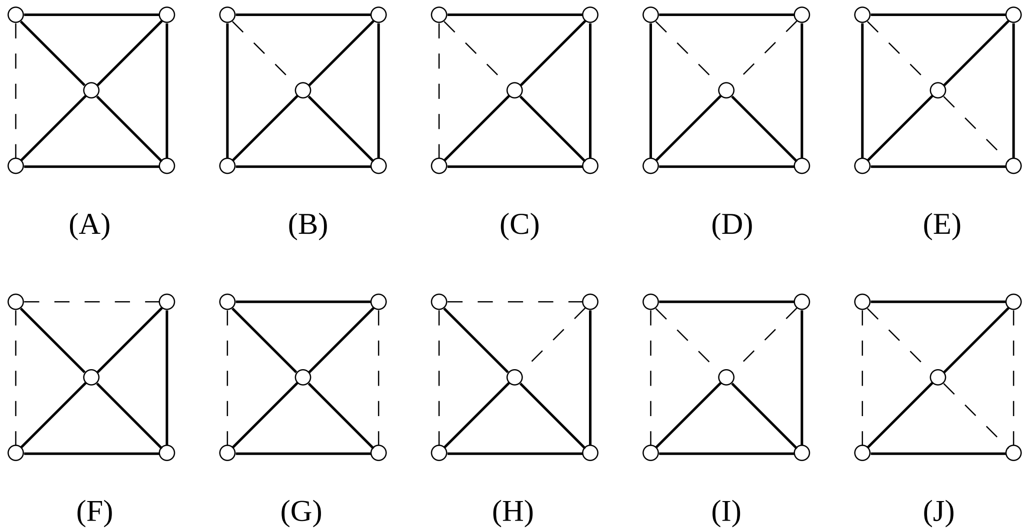

Figure 4: The non-rigid tensegrity graphs on W_5.

at least one strut, see [7]. (Note that by using this result the solution of the labelling problem in one dimension is straightforward.)

The same conditions, however, are not sufficient to guarantee the rigidity of a cable-strut tensegrity graph T in two dimensions. This follows by observing that a cable-strut tensegrity graph with just one cable (or strut) can never be rigid. In addition, there is no lower bound on the number of cables and struts which would imply the rigidity of a tensegrity graph, even if its underlying graph is M-connected. This follows by observing that the 2-sum of a rigid cable-strut tensegrity graph with an M-connected underlying graph and a tensegrity graph on K_4 which contains only struts, is not rigid. However, the following may be true.

Conjecture 6.1. *There exists a (smallest) integer k such that every tensegrity graph T containing at least k cables and at least k struts, and with a 3-connected and redundantly rigid underlying graph, is rigid in $\mathbb{R}^2$.*

We have a complete characterization of rigid tensegrity graphs whose underlying graph is either a complete graph K_n or a wheel W_n. From these results it follows that the conjecture is true in these special cases. Already the wheels show that if k exists, it must be greater or equal to five. See Figure 4.

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