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Brick Partitions of Graphs

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Abstract

For each rational number $q \geq 1$, we describe two partitions of the vertex set of a graph G, called the q-brick partition and the q-superbrick partition. The special cases when $q = 1$ are the partitions given by the connected components and the 2-edge-connected components of G, respectively. We obtain structural results on these partitions and describe their relationship to the principal partitions of a matroid.

1 Introduction

All graphs considered are finite and without loops, but may contain multiple edges. Given a graph G and a positive integer c, we use cG to denote the graph obtained from G by replacing each edge by c parallel edges.

For each positive rational number $q = b/c$ where $b \geq c$ are positive integers, we define a *q-brick* of G to be a maximal subgraph H of G such that cH has b edge-disjoint spanning trees, and a *q-superbrick* of G to be a maximal subgraph H of G such that $cH - e$ has b edge-disjoint spanning trees for all edges e of cH. (We will see that these definitions are independent of the representation of q as b/c.) We show in Section 2 that the vertex sets of the q-bricks of G partition the vertex set of G, and that the vertex sets of the q-superbricks of G form a refinement of this partition, see Figure 1. The special cases of the brick and superbrick partitions of a graph G when $q = 1$ are the partitions given by the connected components and the 2-edge-connected components of G, respectively. The brick partitions of a graph are closely related to the principal partitions of its cycle matroid. This relationship will be described in Section 3.

Our motivation for considering brick partitions is their application to the study of the flexibility of molecules. One can model a molecule as a graph in three-space in which atoms are represented by vertices and bonds by edges. The atoms in the

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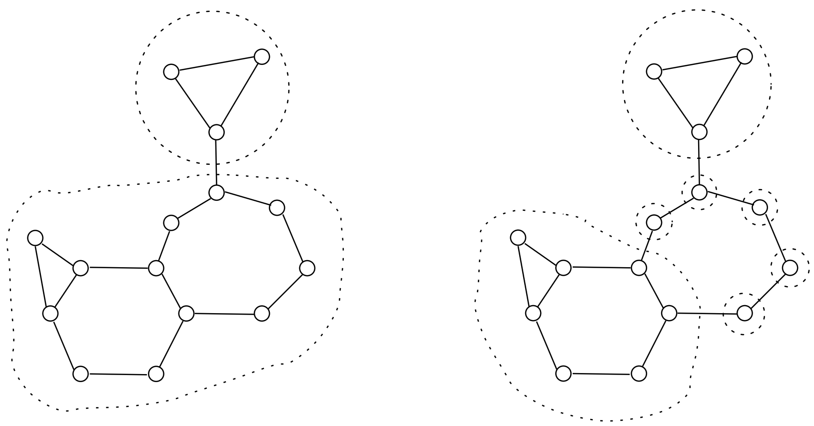

Figure 1: When $q = 1$ the above graph G has one q-brick (G itself) and two q-superbricks (the two 2-edge-connected components of G). The q-brick (and q-superbrick) partitions are successively refined when q becomes greater than (resp. equal to) 1, 6/5, 5/4, and 3/2. For $q > 3/2$ both partitions consist of the vertices of G as singleton members. The figure illustrates the q-brick partition and the q-superbrick partition of G when $q = 6/5$.

molecule are free to move subject to the constraints that both the lengths of bonds and the angles between pairs of adjacent bonds remain constant. This corresponds to allowing the vertices to move subject to the constraints that the lengths of all edges in the *square* of G, i.e. the graph G^2 obtained by joining all pairs of vertices of G of distance at most two, remain constant. (Squares of graphs are sometimes called *molecular graphs* because of this correspondence.) It is a difficult open problem to determine when an arbitrary graph is rigid in three-space, but this problem may be easier for molecular graphs. The Molecular Conjecture, due to Tay and Whiteley [19, Conjecture 1], asserts that, if G has minimum degree at least two, then G^2 is rigid in three-space if and only if $5G$ contains six edge-disjoint spanning trees. We use the q-brick and q-superbrick partitions with $q = 6/5$, in [8, 9, 11] to obtain partial results on the molecular conjecture. They are also used with $q = 3/2$ in [10] to verify a 2-dimensional version of the conjecture.

2 Bricks and superbricks

Let $G = (V, E)$ be a graph. For a family $\mathcal{F}$ of pairwise disjoint subsets of V let $E_G(\mathcal{F})$ denote the set, and $e_G(\mathcal{F})$ the number, of edges of G connecting distinct members of $\mathcal{F}$. The following classical result determines when a graph has a specified number of edge-disjoint spanning trees.

Theorem 2.1. *Let $H = (V, E)$ be a graph and let k be a positive integer.*
(a) The maximum size of the union of k forests in H is equal to the minimun value of

$$e_H(\mathcal{P}) + k(|V| - |\mathcal{P}|) \tag{1}$$

taken over all partitions $\mathcal{P}$ of V;
(b) H contains k edge-disjoint spanning trees if and only if

$$e_H(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$$

for all partitions $\mathcal{P}$ of V.

Theorem 2.1(a) appears in [18, Chapter 51]. It follows easily from the matroid union theorem of Nash-Williams [16] and Edmonds [4], by applying this theorem to the matroid union of k copies of the cycle matroid of H. Part (a) implies part (b), which is a well-known result of Tutte [21] and Nash-Williams [15].

We assume henceforth in this section that $q \geq 1$ is a fixed rational number, and that $q = b/c$ for integers $b \geq c > 0$. For a partition $\mathcal{P}$ of V, let

$$\mathrm{def}_{G,q}(\mathcal{P}) = q(|\mathcal{P}| - 1) - e_G(\mathcal{P})$$

denote the *deficiency* of $\mathcal{P}$ in G (with respect to q) and let

$$\mathrm{def}_q(G) = \max\{\mathrm{def}_{G,q}(\mathcal{P}) : \mathcal{P} \text{ is a partition of } V\}.$$

Note that $\mathrm{def}_q(G) \geq 0$ since $\mathrm{def}_{G,q}(\{V\}) = 0$. When q is an integer, Theorem 2.1(a) implies that $\mathrm{def}_q(G)$ is the minimum number of edges which must be added to G so that the resulting graph has q edge-disjoint spanning trees. More generally, $c\,\mathrm{def}_q(G)$ is the minimum number of edges which must be added to $c\,G$ so that the resulting graph has b edge-disjoint spanning trees. We say that $\mathcal{P}$ is a *q-tight partition of G* if $\mathrm{def}_{G,q}(\mathcal{P}) = \mathrm{def}_q(G)$. In what follows we may omit G or q (or both) from the subscript if it is clear from the context.

Lemma 2.2. *Suppose $G = (V, E)$ is a graph and $\mathcal{P}$ is a tight partition of G. Let $\mathcal{Q} \subseteq \mathcal{P}$ with $|\mathcal{Q}| \geq 2$, $P' = \bigcup_{P \in \mathcal{Q}} P$ and $H = G[P']$. Then*
(a) $\mathrm{def}_H(\mathcal{Q}) \geq 0$.
(b) Furthermore, if $\mathcal{P}$ is chosen such that $|\mathcal{P}|$ is as small as possible, then $\mathrm{def}_H(\mathcal{Q}) > 0$.

Proof: Let $\mathcal{R} = (\mathcal{P} - \mathcal{Q}) \cup \{P'\}$. Then

$$\mathrm{def}_G(\mathcal{P}) = \mathrm{def}_G(\mathcal{R}) + \mathrm{def}_H(\mathcal{Q}).$$

Since $\mathcal{P}$ is a tight partition of G we have $\mathrm{def}_G(\mathcal{P}) \geq \mathrm{def}_G(\mathcal{R})$. Hence $\mathrm{def}_H(\mathcal{Q}) \geq 0$.

Now suppose that $\mathrm{def}_H(\mathcal{Q}) = 0$. Then $\mathrm{def}_G(\mathcal{P}) = \mathrm{def}_G(\mathcal{R})$. Thus $\mathcal{R}$ is a tight partition of G with $|\mathcal{R}| = |\mathcal{P}| - |\mathcal{Q}| + 1$. Hence, if $\mathcal{P}$ is chosen such that $|\mathcal{P}|$ is as small as possible, then we must have $\mathrm{def}_H(\mathcal{Q}) > 0$. •

We say that a graph G is *q-strong* (or *strong*, when q is clear from the context) if $\mathrm{def}_q(G) = 0$. Equivalently, by Theorem 2.1(b), G is q-strong if and only if cG has b edge-disjoint spanning trees.

Lemma 2.3. *Let $G = (V, E)$ be a graph, and $\mathcal{P}$ be a tight partition of G. Choose $P \in \mathcal{P}$ and let $H = G[P]$. Then*
(a) H is strong.
(b) Furthermore, if $\mathcal{P}$ is chosen such that $|\mathcal{P}|$ is as large as possible, then $\{P\}$ is the only tight partition of H.

Proof: Let $\mathcal{Q}$ be a tight partition of H and $\mathcal{R} = (\mathcal{P} - \{P\}) \cup \mathcal{Q}$. Then $\mathcal{R}$ is a partition of V and

$$\text{def}_G(\mathcal{R}) = \text{def}_G(\mathcal{P}) + \text{def}_H(\mathcal{Q}).$$

Since $\mathcal{P}$ is a tight partition of G we have $\text{def}_H(\mathcal{Q}) \leq 0$. Since $\mathcal{Q}$ is a tight partition of H, $\text{def}_H(\mathcal{Q}) \geq 0$. Thus $\text{def}_H(\mathcal{Q}) = 0$ and H is strong. Furthermore, $\text{def}_G(\mathcal{R}) = \text{def}_G(\mathcal{P})$. Thus, if $\mathcal{P}$ is chosen such that $|\mathcal{P}|$ is as large as possible, we must have $|\mathcal{Q}| = 1$ and $\mathcal{Q} = \{P\}$. •

A subgraph H of a graph G is said to be a *q-brick* (or simply *brick*) of G if H is a maximal q-strong subgraph of G with respect to inclusion. Thus bricks are induced subgraphs.

Lemma 2.4. *Let $G = (V, E)$ be a graph, let $A, B \subseteq V$ with $A \cap B \neq \emptyset$ and suppose that $G[A]$ and $G[B]$ are strong. Then $G[A \cup B]$ is strong.*

Proof: Put $H = cG$. Let $T_1, T_2, ..., T_b$ be edge-disjoint spanning trees in $H[A]$ and $F_1, F_2, ..., F_b$ be edge-disjoint spanning trees in $H[B]$. Let $R_i = T_i - E(H[A \cap B])$, $1 \leq i \leq b$. Then $F_i \cup R_i$ are b edge-disjoint connected graphs on $H[A \cup B]$. (Each $F_i \cup R_i$ contains a spanning tree of $H[B]$, and a path from each vertex in $A - B$ to $A \cap B$.) •

It follows immediately that the bricks of a graph G are vertex disjoint. Since, by definition, a single vertex is strong, every vertex of G belongs to a brick, and hence we have:

Corollary 2.5. *The vertex sets of the bricks of a graph $G = (V, E)$ partition V.*

We shall use the term *q-brick partition* (or simply *brick partition*) of G to refer to the partition of V given by the vertex sets of the q-bricks of G.

Theorem 2.6. *Let $G = (V, E)$ be a graph and $\mathcal{P}$ be a tight partition of G such that $|\mathcal{P}|$ is as small as possible. Then $\mathcal{P}$ is the brick partition of G.*

Proof: Let $\mathcal{B}$ be the brick partition of G. If $\text{def}(G) = 0$ then G is a brick and $\mathcal{B} = \{V\} = \mathcal{P}$, so we may assume that $\text{def}(G) \geq 1$. Lemma 2.3(a) implies that each of the parts in $\mathcal{P}$ induces a strong subgraph of G. Thus $\mathcal{P}$ is a refinement of $\mathcal{B}$ by Lemma 2.4. Since each part of $\mathcal{B}$ induces a strong subgraph of G, Lemma 2.2(b) now implies that $\mathcal{B} = \mathcal{P}$. •

We say that a graph $G = (V, E)$ is *q-superstrong* (or simply *superstrong*) if $\text{def}(G) = 0$ and the only tight partition of G is $\{V\}$. Equivalently, by Theorem 2.1(b), G is superstrong if $cG - e$ has b edge-disjoint spanning trees for all $e \in E(cG)$. A subgraph H of G is said to be a *q-superbrick* (or simply *superbrick*) of G if H is a maximal q-superstrong subgraph of G with respect to inclusion. Thus superbricks are induced subgraphs.

Lemma 2.7. *Let $G = (V, E)$ be a graph, let $A, B \subseteq V$ with $A \cap B \neq \emptyset$ and suppose that $G[A]$ and $G[B]$ are superstrong subgraphs of G. Then $G[A \cup B]$ is superstrong.*

Proof: Put $H = cG$ and choose e an edge of $H[A \cup B]$. Since $G[A]$ and $G[B]$ are both q-superstrong, $H[A] - e$ and $H[B] - e$ are both b-strong. Thus $(H[A] - e) \cup (H[B] - e)$ is b-strong by Lemma 2.4. Hence $G[A \cup B]$ is q-superstrong. •

It follows immediately that the superbricks of a graph G are vertex disjoint. Since, by definition, a single vertex is superstrong, every vertex of G belongs to a superbrick, and hence we have:

Corollary 2.8. *The vertex sets of the superbricks of a graph $G = (V, E)$ partition V.*

We shall use the term *superbrick partition* of G to refer to the partition of V given by the vertex sets of the superbricks of G.

Theorem 2.9. *Let G be a graph and $\mathcal{P}$ be a tight partition of G such that $|\mathcal{P}|$ is as large as possible. Then $\mathcal{P}$ is the superbrick partition of G.*

Proof: Let $\mathcal{S}$ be the superbrick partition of G. If $|\mathcal{P}| = 1$ then G is a superbrick and $\mathcal{S} = \{V\} = \mathcal{P}$, so we may assume that $|\mathcal{P}| \geq 2$. Lemma 2.3(b) implies that each of the parts in $\mathcal{P}$ induces a superstrong subgraph of G. Thus $\mathcal{P}$ is a refinement of $\mathcal{S}$ by Lemma 2.7. Since the union of two or more parts of $\mathcal{P}$ induces a subgraph of G which is not superstrong by Lemma 2.2(a), we may deduce that $\mathcal{S} = \mathcal{P}$. •

We say that a graph G is *minimally (super)strong* if G is (super)strong and $G - e$ is not (super)strong for all $e \in E(G)$.

Lemma 2.10. *Let $G = (V, E)$ be graph.*
(a) If G is minimally strong and H is a strong subgraph of G then H is minimally strong.
(b) If G is minimally superstrong and H is a superstrong subgraph of G then H is minimally superstrong.

Proof: We prove (b). The proof of (a) is similar. Let $e \in E(H)$ and consider the superbrick partition $\mathcal{S}$ of $G - e$. Since G is minimally superstrong, $|\mathcal{S}| \geq 2$ and the endvertices of e belong to different members of $\mathcal{S}$. Let $\mathcal{Q} = \{X \in \mathcal{S} : V(H) \cap X \neq \emptyset\}$ and let $\mathcal{Q}' = \{V(H) \cap X : X \in \mathcal{Q}\}$. We have $|\mathcal{Q}| \geq 2$. Suppose $H - e$ is superstrong. Then $e_{G-e}(\mathcal{Q}) \geq e_{H-e}(\mathcal{Q}') > q(|\mathcal{Q}'| - 1) = q(|\mathcal{Q}| - 1)$. Thus $\mathrm{def}_F(\mathcal{Q}) < 0$, where F is the subgraph of $G - e$ induced by $\bigcup_{X \in \mathcal{Q}} X$. Since $\mathcal{S}$ is a tight partition of $G - e$ by Theorem 2.9, this contradicts Lemma 2.2(a). Thus $H - e$ is not superstrong, as claimed. •

Lemma 2.10 is analogous to the result that every k-edge-connected subgraph of a minimally k-edge-connected graph is minimally k-edge-connected.

It is straightforward to obtain efficient algorithms for testing whether a graph $G = (V, E)$ is q-(super)strong, and for determining the q-(super)brick partition of G by using well-known algorithms for packing trees, or more generally, packing independent sets in a matroid, see [8] for more details. In particular, if q is an integer, $\mathcal{M}_q(G)$ is the matroid union of q copies of the cycle matroid of G and E_0 is the set of edges of E which lie in no circuits of $\mathcal{M}_q(G)$, then the q-superbricks of G are just the connected components of $G - E_0$.

3 Principal partitions

We use $\mathcal{M} = (E, r)$ to denote a matroid with groundset E and rank function r. Recall that the dual matroid $\mathcal{M}^* = (E, r^*)$ of $\mathcal{M}$ is determined by the dual rank function $r^*(X) = |X| - r(E) + r(E - X)$ for all $X \subseteq E$.

Consider the following optimization problem for $\mathcal{M}$.

Problem 1. Given a matroid $\mathcal{M} = (E, r)$ and a rational number $p \geq 0$, find $X \subseteq E$ to minimize $p\, r(X) + r^*(E - X)$.

Substituting for the dual rank function and putting $q = p + 1$, we may reformulate Problem 1 as:

Problem 2. Given a matroid $\mathcal{M} = (E, r)$ and a rational number $q \geq 1$, find $X \subseteq E$ to minimize $q\, r(X) - |X|$.

The special case of Problem 1 when $p = 1$ and $\mathcal{M}$ is the cycle matroid of a connected graph $G = (V, E)$ has an application to the electrical network represented by G. We would like to determine a minimum set of edges $I_1 \cup I_2$ such that if we measure voltage differences on the edges in I_1 and current along the edges of I_2 then we can use Kirchoff's laws and Ohm's law to determine voltage differences and current for every edge of G. We can construct such sets by solving Problem 1 with $p = 1$, and taking I_1 and I_2 to be maximal subsets of X and $E - X$ which are independent in $\mathcal{M}$ and $\mathcal{M}^*$, respectively. Different formulations of this special case of Problems 1 and 2 were solved independently by Kishi and Kajitani [12], Ohtsuki, Ishizaki, and Watanabe [17], and Iri [7]. In particular, Kishi and Kajitani showed that there is an ordered partition (F^+, F^0, F^-) of E, such that F^+ is the unique smallest solution to Problem 1, and $F^+ \cup F^0$ is the unique largest solution to Problem 1. They called this the *principal partition* of E. Their result was extended to integer p by Bruno and Weinberg [1], and to rational p by Tomizawa [20] and Narayanan and Vartak [14], as follows. Let $q \geq 1$ be a rational number. We say that $F \subseteq E$ is a *q-minimizer* in a matroid $\mathcal{M} = (E, r)$ if $qr(F) - |F|$ is as small as possible. It was proved that, for each rational $q \geq 1$, there is a unique ordered partition (F_q^+, F_q^0, F_q^-) of E, called the *q-principal partition* of E, such that F_q^+ is the smallest and $F_q^+ \cup F_q^0$ is the largest q-minimizer in $\mathcal{M}$. (Thus Kishi and Kajitani's principal partition is the 2-principal partition of $\mathcal{M}$.)

We shall see that there is a close relationship between the q-brick partitions of a graph G and the q-principal partition of its cycle matroid. We first reformulate Problem 2 for the case when $\mathcal{M} = (E, r)$ is the cycle matroid of a graph $G = (V, E)$. For each $X \subseteq E$ let $c(X)$ be the number of components in the graph (V, X). Then $r(X) = |V| - c(X)$ and Problem 2 becomes:

Problem 3. Given a graph $G = (V, E)$ and a rational number $q \geq 1$, find $X \subseteq E$ to minimize $q(|V| - c(X)) - |X|$.

Lemma 3.1. *Let $G = (V, E)$ be a graph and $q \geq 1$ be a rational number. (a) Suppose $X \subseteq E$ is a q-minimizer in the cycle matroid of G and let $H = (V, X)$. Then the components of H are induced subgraphs of G and their vertex sets form a q-tight*

partition of G.
(b) Suppose $\mathcal{P}$ is a q-tight partition of G and let $Y = E - E_G(\mathcal{P})$. Then Y is a q-minimizer in the cycle matroid of G.

Proof: (a) Suppose some $e \in E - X$ is incident with two vertices in the same component of H. Let $X' = X \cup \{e\}$. Then $c(X) = c(X')$ and X' contradicts the fact that X is a q-minimizer. Thus each component of H is an induced subgraph of G.

Let $\mathcal{P}$ be a q-tight partition of G and $Y = E - E_G(\mathcal{P})$. Each part of $\mathcal{P}$ induces a q-strong (and hence connected) subgraph of G by Lemma 2.3(a). Thus

$$\mathrm{def}_{G,q}(\mathcal{P}) = q(|\mathcal{P}| - 1) - e_G(\mathcal{P}) = q(c(Y) - 1) + |Y| - |E|.$$

Similarly, if $\mathcal{Q}$ is the partition of V given by the vertex sets of the components of H, then

$$\mathrm{def}_{G,q}(\mathcal{Q}) = q(|\mathcal{Q}| - 1) - e_G(\mathcal{Q}) = q(c(X) - 1) + |X| - |E|.$$

Since X is a q-minimizer, we have

$$\mathrm{def}_{G,q}(\mathcal{Q}) - \mathrm{def}_{G,q}(\mathcal{P}) = q\, c(X) + |X| - q\, c(Y) - |Y| \geq 0.$$

Thus $\mathcal{Q}$ is a tight partition of G.

(b) Let X be a q-minimizer in the cycle matroid of G and $\mathcal{Q}$ be the partition of G given by the vertex sets of the components of $H = (V, X)$. By (a), $\mathcal{Q}$ is a tight partition of G. Thus $\mathrm{def}_{G,q}(\mathcal{Q}) = \mathrm{def}_{G,q}(\mathcal{P})$. This implies that $q\, c(X) + |X| = q\, c(Y) + |Y|$ and hence Y is a q-minimizer in the cycle matroid of G. •

Lemma 3.1 defines a bijection between q-minimizers and q-tight partitions.

Lemma 3.2. *Let $G = (V, E)$ be a graph, $q \geq 1$ be a rational number and (F_q^+, F_q^0, F_q^-) be the q-principal partition of the cycle matroid of G. Then:*
(a) the q-bricks of G are the components of $H^0 = (V, F_q^+ \cup F_q^0)$, and
(b) the q-superbricks of G are the components of $H^+ = (V, F_q^+)$.

Proof: We prove (a). The proof of (b) is similar. Let $H_1, H_2, \ldots, H_m$ be the components of H^0, $\mathcal{Q}$ be the partition of V defined by $H_1, H_2, \ldots, H_m$, and $\mathcal{P}$ be the brick partition of G. Then $\mathcal{Q}$ is a tight partition of G and each subgraph H_i is a q-strong induced subgraph of G by Lemmas 3.1(a) and 2.3(a). Hence $\mathcal{Q}$ is a refinement of $\mathcal{P}$. Let $X = E - E_G(\mathcal{P})$. By Lemma 3.1(b), X is a q-minimizer in the cycle matroid of G. Since $F_q^+ \cup F_q^0$ is the largest q-minimizer and $F_q^+ \cup F_q^0 \subseteq X$, we must have $F_q^+ \cup F_q^0 = X$. Thus the components of H^0 and the bricks of G have the same edge sets. The fact that the bricks of G are connected now implies that they are the same as the components of H^0. •

A weaker version of Lemma 3.2(a) was proved by Lin in [13] for $q = 2$, where the author defined the 'maximal subgraph of G in which each component contains two edge-disjoint spanning trees' and showed that this subgraph gives rise to a 2-tight partition.

3.1 Further Remarks

The maximum value of q such that a graph G is q-strong was first considered by Gusfield [6]. It was called the *strength* of G by Cunningham in [3] and extended to matroids. A unified approach to strength and principal partitions in matroids is given by Catlin et al in [2]. For integer values of q, inductive constructions for q-strong and q-superstrong graphs have been given by Nash-Williams [15] and Frank and Király [5], respectively.

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