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Packing trees with constraints on the leaf degree

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Abstract

If m is a positive integer then we call a tree on at least 2 vertices an *m-tree* if no vertex is adjacent to more than m leaves. Kaneko proved that an undirected graph $G = (V, E)$ has a spanning m-tree if and only if for every $X \subseteq V$ the number of isolated vertices of $G - X$ is at most $m|X| + (|X| - 1)^+$ — unless we are at the exceptional case of $G \simeq K_3$ and $m = 1$. As an attempt to integrate this result into the theory of graph packings, in this paper we consider the problem of packing a graph with m-trees. We use an approach different to that of Kaneko, and we deduce Gallai–Edmonds and Berge–Tutte type theorems and a matroidal result to the m-tree packing problem.

1 Introduction

If $\mathcal{H}$ is a set of undirected graphs then a subgraph Q of an undirected graph G is called an **$\mathcal{H}$-packing** if every connected component of Q is isomorphic to some member of $\mathcal{H}$. An $\mathcal{H}$-packing is **maximum** if it covers a maximum number of vertices of G, and it is an **$\mathcal{H}$-factor** if it is spanning. The **$\mathcal{H}$-packing problem** is to find a maximum $\mathcal{H}$-packing in the input graph G. Note that if $\mathcal{H} = \{K_2\}$ then we get the classical matching problem. The goal in graph packings is mainly to find new polynomial and NP-complete graph packing problems, and to obtain structural results to the polynomially solvable cases. These structural results are mostly extensions of the following basic theorems on matchings: Tutte's characterization on the existence of perfect matchings, the Berge–Tutte formula, and the Gallai–Edmonds decomposition theorem (cited in Theorem 2.1 below).

Apart from matchings, the first graph packing problem studied was the $\{K_2, C_3, C_5, \ldots\}$-packing problem. By a tricky reduction to bipartite matchings, Tutte [13] proved that a graph $G = (V, E)$ has a $\{K_2, C_3, C_5, \ldots\}$-factor if and only if $i(G - X) \leq |X|$ for every $X \subseteq V$, where $i(G - X)$ denotes the number of isolated vertices of $G - X$. For the next important result, we need to define **local** graph packings: here instead of a **global** family $\mathcal{H}$, along with the input graph G, a set $\mathcal{H}$ of *subgraphs* of G is given — the subgraphs allowed to form a component of the packing. Cornuéjols, Hartvigsen

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and Pulleyblank [2, 1] gave a polynomial algorithm and a Gallai–Edmonds type structure theorem to the local packing problem where $\mathcal{H}$ is constrained to consist of all K_2-subgraphs of G and an arbitrary set of factor-critical subgraphs of G. Recall that a connected graph is **factor-critical** if the deletion of any vertex leaves a graph with a perfect matching. Another important milestone was the result of Kaneko [8], who gave a Tutte type characterization on the existence of an $\mathcal{H}$-factor of a graph, where $\mathcal{H}$ consists of the paths of length at least 2. The extension of this surprising result was given by Hartvigsen, Hell and Szabó [6], who introduced and solved the **k-piece packing problem**, where a **k-piece** is a connected graph with maximum degree k. The above mentioned long path factor problem of Kaneko corresponds to the case $k = 2$.

The starting point of the present paper is another result of Kaneko. If T is a tree and $v \in V(T)$ then let $\mathrm{leaf}_T(v)$ denote the number of leaves of T adjacent to v in T. Let m be a positive integer. We call T an **m-tree** if T has at least two vertices and $\mathrm{leaf}_T(v) \leq m$ for all $v \in V(T)$. In [7] Kaneko considered the existence of spanning m-trees in undirected graphs, and he gave an algorithmic proof to the following characterization. If α is a number then we denote $\alpha^+ = \max\{0, \alpha\}$.

Theorem 1.1 (Kaneko [7]). *Let $G = (V, E)$ be a connected, undirected graph and m be a positive integer. Then G has a spanning m-tree if and only if $i(G - X) \leq m|X| + (|X| - 1)^+$ for every $X \subseteq V$ — except if $G \simeq K_3$ and $m = 1$ when this condition holds but G has no spanning m-tree.*

An important observation of Kaneko [7] is that G has a spanning m-tree if and only if G has an m-tree factor. Indeed, if forest Q is an m-tree factor of G, then Q has no isolated vertices so arbitrarily connecting the components of Q results in a spanning m-tree of G. On the other hand, a spanning m-tree is an m-tree factor by definition. Thus Theorem 1.1 of Kaneko is actually a Tutte type characterization on the m-tree packing problem.

This paper presents further structural results on the m-tree packing problem. Using the classical Gallai–Edmonds decomposition and the construction of matroid union, we prove that the m-tree packing problem is polynomial time solvable. We deduce a Gallai–Edmonds type theorem and a Berge–Tutte type minimax formula to the problem, which implies the Tutte type theorem 1.1 of Kaneko. We also prove that the vertex sets coverable by m-tree packings form the independent sets of a matroid, in other words, the m-tree packing problem is **matroidal**.

Throughout the paper all graphs are simple and undirected. If $G = (V, E)$ is a graph and $X \subseteq V$ then we denote by $\Gamma(X)$ the set of vertices in $V - X$ adjacent to X, and by $c(G)$ the number of connected components of G. We use the notation $\mathbb{N} = \{0, 1, 2, \ldots\}$.

2 Main results

Our approach is based on the classical Gallai–Edmonds structure theorem cited below.

Theorem 2.1 (Edmonds, Gallai [3, 4, 5]). *Let $G = (V, E)$ be an undirected graph. Define $D \subseteq V$ to consist of those vertices which are missed by some maximum matching of G. Let $A = \Gamma(D)$ and $C = V - (D \cup A)$. Then*

1. *every component of $G[D]$ is factor-critical,*
2. *a maximum matching of G misses $c(G[D]) - |A|$ vertices of G,*
3. *for all $\emptyset \neq A' \subseteq A$ the number of those components of $G[D]$ which are adjacent to A' is at least $|A'| + 1$,*
4. *$G[C]$ has a perfect matching.*

We need some basic results on matroid theory. For more details and introduction to matroids, we refer to Schrijver [11].

Definition 2.2. Let S be a finite ground set and $b : \mathcal{P}(S) \to \mathbb{N}$ a function. b is called **submodular** if for all $X, Y \subseteq S$ it holds that

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y). \tag{1}$$

b is **intersecting submodular** if (1) holds for all $X, Y \subseteq S$ with $X \cap Y \neq \emptyset$.

Lemma 2.3. *If $b : \mathcal{P}(S) \to \mathbb{N}$ is intersecting submodular and monotone increasing then*

$$\{F \subseteq S : |X| \leq b(X) \quad \forall X \subseteq F\}$$

forms the family of independent sets of a matroid, denoted by $M(b)$. The rank of $X \subseteq S$ in $M(b)$ is

$$\min\{|Y| + \sum_{j \in J} b(Y_j)\},$$

where the minimum is taken over $Y \subseteq X$ and partition $\{Y_j : j \in J\}$ of $X - Y$. If b is submodular then this partition is taken to be simply $\{X - Y\}$.

A **hypergraph** is a pair $H = (V, E)$, where the elements of V are the **vertices**, the elements of E are the **hyperedges**, and for every hyperedge $e \in E$ a ground set $V(e) \subseteq V$ is associated. For $F \subseteq E$ let $V(F) = \bigcup_{e \in F} V(e)$. The hypergraph H is said to be **connected** if for every $\emptyset \neq X \subsetneq V$ there exists a hyperedge intersecting both X and $V - X$. A maximal connected subhypergraph of H without empty hyperedges is called a **connected component** of H, and the number of connected components of H is denoted by $c(H)$.

Let $H = (V, E)$ be a hypergraph. It is easy to check that $b_T : \mathcal{P}(E) \to \mathbb{N},\ X \mapsto |V(X)|$ is submodular, and $M(b_T)$ is called the **transversal matroid** of H. Similarly, $b_M : \mathcal{P}(E) \to \mathbb{N},\ X \mapsto (|V(X)| - 1)^+$ is intersecting submodular, and $M(b_M)$ is called the **hypergraphic matroid** of H. If H is a graph then its hypergraphic matroid is equal to its cycle matroid.

Definition 2.4. Let $G = (V, E)$ be a graph and $S \subseteq V$ be a stable set. We define H_S to be the hypergraph with vertex set $\Gamma(S)$ and hyperedge set S, where $V(s) = \Gamma(s)$ for all $s \in S$. We denote the transversal (resp. hypergraphic) matroid of H_S on ground set S simply by T_S (resp. M_S).

If $S \subseteq V$ is a stable set then by Lemma 2.3, the rank of $X \subseteq S$ in the transversal matroid T_S is $\min\{|Y| + |\Gamma(X - Y)| : Y \subseteq X\}$. It follows that X is independent in T_S if and only $|\Gamma(Z)| \geq |Z|$ for all $Z \subseteq X$, which by Hall's theorem is equivalent to that X can be matched into $\Gamma(S)$ in G.

As for the hypergraphic matroid, Lovász [9] proved that $X \subseteq S$ is independent in M_S if and only we can choose at every vertex of X two incident edges, such that these $2|X|$ edges form a forest in G. Next we formulate the rank function of M_S in a convenient way.

Lemma 2.5. *Let $G = (V, E)$ be a graph and $S \subseteq V$ be a stable set. The rank $r(X)$ of $X \subseteq S$ in the hypergraphic matroid M_S is*

$$r(X) = \min\{|Y| + |\Gamma(X - Y)| - c(H_{X-Y}) : Y \subseteq X\}. \tag{2}$$

Proof. By Lemma 2.3,

$$r(X) = |Y| + \sum_{j \in J}(|\Gamma(Y_j)| - 1)^+, \tag{3}$$

for some $Y \subseteq X$ and partition $\{Y_j : j \in J\}$ of $X - Y$. If Y_i and Y_j have a common neighbor for some $i, j \in J$ then the right hand side of (3) does not increase when merging Y_i and Y_j. It cannot decrease either by Lemma 2.3. Moreover, if for some $j \in J$ the set Y_j partitions into nonempty subsets Y^1 and Y^2 such that Y^1 and Y^2 have no common neighbor, then the right hand side of (3) does not increase when splitting Y_j into Y^1 and Y^2. It cannot decrease either by Lemma 2.3. Hence the right hand side of (3) remains the same if the classes in the partition $\{Y_j : j \in J\}$ of $X - Y$ correspond to the connected components of $G[(X - Y) \cup \Gamma(X - Y)]$. Thus

$$r(X) = |Y| + |\Gamma(X - Y)| - c(H_{X-Y}). \tag{4}$$

However, (4) holds with $\leq$ for all $Y \subseteq X$ by Lemma 2.3, thus (2) follows. □

Finally, we need one more construction of matroids.

Lemma 2.6. *If M_i, $i \in I$, are matroids on ground set S with rank function r_i then*

$$\left\{\bigcup_{i \in I} F_i \,:\, F_i \text{ is independent in } M_i \text{ for all } i \in I\right\}$$

forms the family of independent sets of a matroid, the **union** *of the M_i's, denoted by $\bigvee_{i \in I} M_i$. The rank of $X \subseteq S$ in $\bigvee_{i \in I} M_i$ is*

$$\min\{|Y| + \sum_{i \in I} r_i(X - Y) : Y \subseteq X\}.$$

Claim 2.7. *Let $F = (V, E)$ be a simple factor-critical graph and m be a positive integer. Then F has an m-tree factor if and only if $|V| \geq 5$ or $F \simeq K_3$ and $m \geq 2$.*

Proof. K_1 has no m-tree factor for any $m \geq 1$. The triangle K_3 has no 1-tree factor, but its every spanning tree is an m-tree for all integer $m \geq 2$. Next, assume that $|V| \geq 5$. Lovász [10] proved that every factor-critical graph F has an **odd ear decomposition**, that is a sequence $F_0, \ldots, F_k$, where F_0 is a singleton, $F_k = F$ and F_{i+1} is constructed from F_i by adding a path of odd length with its end vertices residing in F_i. Consider such an odd ear decomposition. F_1 is an odd circuit. If it has at least 5 vertices, then take a Hamiltonian path of it together with a perfect matching of $F - V(F_1)$. This is an m-tree factor of F for all $m \geq 1$. If F_1 is a triangle then, as $|V| \geq 5$ and F is simple, $|V(F_2)| \geq 5$. It is easy to see that F_2 has a Hamiltonian path, moreover, $F - V(F_2)$ has a perfect matching. These together give an m-tree factor of F for all $m \geq 1$. □

Lemma 2.8. *Let $G = (V, E)$ be a graph and $S \subseteq V$ a stable set in G. If Q is an m-tree packing of G then Q covers at most $(m + 1)\,|\Gamma(S)| - c(H_S)$ vertices in S.*

Proof. Let $S^1 \subseteq S$ consist of those vertices which are leaves in Q, and $S^2 \subseteq S$ of those which have degree at least 2 in Q. Clearly, $|S^1| \leq m\,|\Gamma(S)|$. Let $X = S^2 \cup \Gamma(S)$. By definition, the number of edges of Q spanned by X is at least $2|S^2|$, but, as Q is a forest, this number is at most $|X| - c(G[X]) = |X| - c(H_{S^2})$. The last equality follows from the fact that the vertices of S^2 are not isolated in $G[X]$. Thus $2|S^2| \leq |X| - c(H_{S^2}) \leq |S^2 \cup \Gamma(S)| - c(H_S)$. Summarizing, Q covers $|S^1 \cup S^2| \leq (m+1)\,|\Gamma(S)| - c(H_S)$ vertices in S. □

Our main result is a Gallai–Edmonds type structure theorem to the m-tree packing problem, analogous to the classical version, Theorem 2.1.

Theorem 2.9. *Let $G = (V, E)$ be a connected, undirected graph and m be a positive integer. Assume that if $G \simeq K_3$ then $m \geq 2$. Define $D_m \subseteq V$ to consist of those vertices which are missed by some maximum m-tree packing of G, and let $A_m = \Gamma(D_m)$. Then*

1. *D_m is stable in G,*
2. *a maximum m-tree packing of G misses $|D_m| - (m + 1)|A_m| + c(H_{D_m})$ vertices of G.*

Proof. Consider the decomposition $V = D \,\dot\cup\, A \,\dot\cup\, C$ defined in Theorem 2.1, and let $S \subseteq D$ be the set of isolated vertices of $G[D]$, see Figure 1. Let $N = \bigvee\{T_S : 1 \leq i \leq m\} \vee M_S$, a matroid on ground set S, and let $S' \subseteq S$ consist of those elements which are not coloops in N. We prove that $D_m = S'$.

By Lemma 2.3, $r_{T_S}(S') = \min\{|Y| + |\Gamma(S' - Y)| \,:\, Y \subseteq S'\}$. By complementary slackness, if Y gives equality here, then the elements of Y are coloops in $T_S|_{S'}$, and hence also in $\bigvee\{T_S|_{S'} \,:\, 1 \leq i \leq m\} \vee M_S|_{S'} = N|_{S'}$, which however has no coloops by definition. Thus $Y = \emptyset$ and $r_{T_S}(S') = |\Gamma(S')|$. Analogous argument gives that $r_{M_S}(S') = |\Gamma(S')| - c(H_{S'})$, using Lemma 2.5. Besides, by Lemma 2.6, $r_N(S') =$

$\min\{|Y| + m \cdot r_{T_S}(S' - Y) + r_{M_S}(S' - Y) \,:\, Y \subseteq S'\}$. Again, if Y gives equality here, then the elements of Y are coloops in $N|_{S'}$ by complementary slackness. Thus $Y = \emptyset$ and

$$r_N(S') = m \cdot r_{T_S}(S') + r_{M_S}(S') = (m+1)\,|\Gamma(S')| - c(H_{S'}). \tag{5}$$

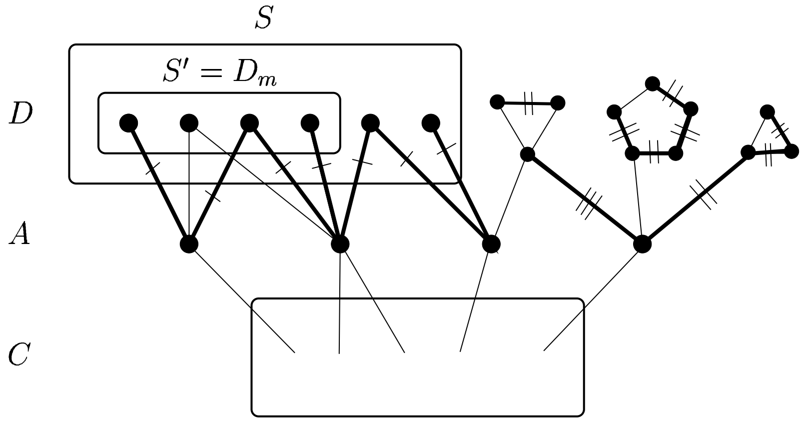

Figure 1: The decomposition of G, $m = 1$

Claim 2.10. *For every base J of N, the graph $G[J \cup (V - S)]$ has an m-tree factor.*

Proof. In subsequent steps we construct an m-tree factor Q of $G[J \cup (V - S)]$. By definition, $J = \bigcup\{J_i \,:\, 1 \le i \le m\} \cup J_M$, where J_i, $1 \le i \le m$, are independent in T_S and J_M is independent in M_S. By the remark before Lemma 2.5, J_M gives rise to an isolated vertex free forest F with $2|J_M|$ edges, such that the vertices in J_M has degree 2 in F. In addition, every set J_i can be matched into A for $1 \le i \le m$, and the union of these matchings give a collection of disjoint $\le m$-star subgraphs of G, with leaves in J and centers in A. Adding these stars to F we get a forest Q covering J. The edges of the actual forest Q are thick with one stripe in Figure 1.

Observe that $\mathrm{leaf}_Q(v) \le m$ for $v \in A$, $\mathrm{leaf}_Q(v) \le 2$ for $v \in S$ and $\mathrm{leaf}_Q(v) = 0$ elsewhere. Thus Q is an m-tree packing, unless $m = 1$ and Q has a 2-star component with center $v \in S$. In this latter case delete from Q a leaf in every such 2-star component, in order to obtain an m-tree packing.

Due to Theorem 2.1, *2*, A can be matched into distinct components of $G[D]$, so fix such a matching P. Next we modify Q in such a way that $E(Q) \cap E(P)$ strictly increases at every step. While there exists a vertex $y \in A \setminus V(Q)$, take the edge $yv \in E(P)$ matching y into $v \in D$ residing in component K of $G[D]$. If K is not a singleton then add yv to Q. Do the same if K is a singleton, except if v has already m leaves in Q (note that this can happen only if $m \le 2$). In this case delete from Q a leaf adjacent to v and add yv to Q. In the end of this procedure, Q is an m-tree packing covering $J \cup A$. The newly added edges of the actual Q are thick with three stripes in Figure 1.

Now consider a non-singleton component K of $G[D]$. Observe that $|V(Q) \cap V(K)| \le 1$. If $V(Q) \cap V(K) = \{v\}$ then add a perfect matching of $K - v$ to Q. If $V(Q) \cap V(K) = \emptyset$ then add an m-tree factor of K to Q, guaranteed by Claim 2.7 — unless $K \simeq K_3$

and $m = 1$. In this latter case a vertex $v \in V(K)$ is joined by an edge to some $y \in A$ in G, by the condition that G is connected and $G \simeq K_3$ implies $m \geq 2$. Thus add vy and a 2-star of K with v as a leaf to Q. The new edges of Q are thick with two stripes in Figure 1.

The vertex set of the actual m-tree packing Q is exactly $J \cup (V - S - C)$, so supplement it with a perfect matching of $G[C]$, guaranteed by Theorem 2.1, to finish the proof. ☐

By Lemma 2.8, every m-tree packing of G misses at least $d := |S'| - (m+1)\,|\Gamma(S')| + c(H_{S'})$ vertices of S'. If J is a base of N, then $G[J \cup (V - S)]$ has size $|V| - d$ by (5). The m-tree factors of $G[J \cup (V - S)]$, guaranteed by Claim 2.10, are thus maximum. Hence by the definition of S', the relation $S' = D_m$ follows, implying *2*. ☐

As one can obtain the classical Gallai–Edmonds decomposition in polynomial time, and there are polynomial algorithms for the matroid union problem, this proof is algorithmic. Thus one can find a maximum m-tree packing of G in polynomial time.

We mention that the relation between the decompositions for distinct m's is that $D_1 \supseteq D_2 \supseteq \ldots$ and $A_1 \supseteq A_2 \supseteq \ldots$. In addition, if $G \neq K_1$ then $D_{n-1} = A_{n-1} = \emptyset$ for $n = |V|$.

As opposed to the classical Gallai–Edmonds theorem 2.1, the graph $G[C_m]$ (with $C_m = V - (D_m \cup A_m)$) may fail to have an m-tree factor, as shown in Figure 2.

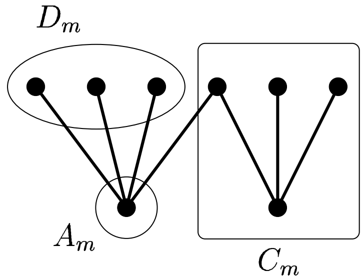

Figure 2: $G[C_m]$ has no m-tree factor, $m = 2$

Theorem 2.11. *Let $G = (V, E)$ be a connected, undirected graph, m be a positive integer, and assume that if $G \simeq K_3$ then $m \geq 2$. Then a maximum m-tree packing of G misses $|S| - (m+1)\,|\Gamma(S)| + c(H_S)$ vertices of G, where the maximum runs on stable sets $S \subseteq V$.*

Proof. One direction is implied by Lemma 2.8, and the other by Theorem 2.9, *2*, which shows that equality is attained by $S = D_m$. ☐

From Theorem 2.11 it is easy to deduce Theorem 1.1 of Kaneko.

Proof of Theorem 1.1. We assume that $G \simeq K_3$ implies $m \geq 2$. Suppose first that there exists a set $X \subseteq V$ violating the condition of Theorem 1.1. Let $S \subseteq V$ consist of the isolated vertices of $G - X$. Clearly, $X \supseteq \Gamma(S)$. Note that if $\Gamma(S) = \emptyset$ then $c(H_S) = 0$, while if $\Gamma(S) \neq \emptyset$ then $c(H_S) \geq 1$. In both cases, we have

$$|S| = i(G - X) > m\,|X| + (|X| - 1)^+ \geq (m+1)\,|\Gamma(S)| - c(H_S).$$

Thus by Lemma 2.8, every m-tree packing of G misses at least one vertex of S.

On the other hand, assume that G has no m-tree factor. Then, by Theorem 2.11, there exists a stable set $S \subseteq V$ with $|S| > (m+1)\,|\Gamma(S)| - c(H_S)$. If $\Gamma(S) = \emptyset$ then G is a singleton thus $X = \emptyset$ violates the condition of Theorem 1.1. So we assume that $\Gamma(S) \neq \emptyset$, which implies that $k := c(H_S) \geq 1$ and that H_S has no empty hyperedges. The edge sets of the components of H_S give a partition $\{S_i : 1 \leq i \leq k\}$ of S. If $|S_i| \leq (m+1)\,|\Gamma(S_i)| - 1$ for all $1 \leq i \leq k$ then

$$|S| = \sum_{i=1}^{k} |S_i| \leq (m+1) \sum_{i=1}^{k} |\Gamma(S_i)| - k = (m+1)\,|\Gamma(S)| - c(H_S),$$

a contradiction. So choose an index $1 \leq i \leq k$ with $|S_i| > (m+1)\,|\Gamma(S_i)| - 1$. As $X := \Gamma(S_i) \neq \emptyset$, we have

$$i(G - X) \geq |S_i| > (m+1)\,|X| - 1 = m\,|X| + (|X| - 1)^+,$$

in other words, X violates the condition of Theorem 1.1. □

Theorem 2.12. *Let $G = (V, E)$ be a connected, undirected graph and m be a positive integer. Then $\mathcal{I} := \{U \subseteq V :$ there exists an m-tree packing of G covering $U\}$ is the collection of independent sets of a matroid.*

Proof. If $G \simeq K_3$ and $m = 1$ then $\mathcal{I}$ is the collection of independent sets of the uniform matroid $U_{3,2}$. Otherwise recall the matroid N in the proof of Theorem 2.9. Add the vertices of $V - S$ as coloops to $N|_S$ resulting in matroid M on ground set V. We prove that $\mathcal{I}$ is equal to the collection of independent sets of M.

If $J \cup (V - S)$ is a base of M, then exactly as in the proof of Theorem 2.9, one can obtain an m-tree packing of G with vertex set $J \cup (V - S)$. On the other hand, let Q be an m-tree packing of G. Let $D^1 \subseteq D_m$ consist of those vertices which are leaves in Q, and $D^2 \subseteq D_m$ of those which have degree at least 2 in Q. Clearly, D^1 is independent in $\bigvee\{T_{D_m} : 1 \leq i \leq m\}$. Moreover, D^2 is independent in M_{D_m}, as $|\Gamma(X)| - 1 \geq |X|$ for all $\emptyset \neq X \subseteq D^2$. Thus $V(Q)$ is independent in M. □

We point out that it is possible to modify the above considerations to solve the following **local leaf constrained tree packing problem**: given a connected, undirected graph $G = (V, E)$ and a function $m : V \to \mathbb{N}$, find a forest T of G of maximum size, with the property that $\text{leaf}_T(v) \leq m(v)$ for all $v \in V(T)$. It can be proved that such a maximum forest misses $|S| - (m+1)(\Gamma(S)) + c(H_S)$ vertices of G, where the maximum runs on stable sets $S \subseteq V$ — except if $G = K_3$ and $m \equiv 1$. Theorems 2.9 and 2.12 are easy to modify to apply to this problem as well.

In [12] an attempt is made to provide a unifying framework to all polynomial graph packing problems known at the time of writing. This framework embraces a skeleton of an Edmonds type alternating forest algorithm, a Berge–Tutte type and a Gallai–Edmonds type theorem and a matroidal result. To convert these structural statements into exact results on a specific packing problem, we need this algorithm to work and also some properties to be satisfied. The Gallai–Edmonds and Berge–Tutte type theorems 2.9 and 2.11 to the m-tree packing problem are analogous to known structural results on other polynomial packing problems. However, an alternating forest algorithm to m-tree packings is not known yet, and hence it is not clear whether there is a way to adapt the framework of [12] to the m-tree packing problem.

It is interesting to note that the family of m-tree packing problems do not include the classical matching problem, contrary to the other known polynomially solvable families of packing problems.

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