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Applications of Eulerian splitting-off

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Abstract

The aim of the present paper is to show how a slight generalization of a splitting-off result of Berstimas and Teo [1] offers a common understanding of several seemingly unrelated results in graph connectivity theory.

1 Introduction

In his early seminal paper on splitting-off in Eulerian graphs [6], Lovász proved three theorems showing how the splitting operation may preserve certain connectivity properties of the graph. He suggested that the formulation of a general theorem implying these results would be possible, but, in his words, it "would be complicated".

Several years later Bertsimas and Teo [1] proved a more abstract splitting-off result during their investigation of the parsimonious property of cut-covering problems. They claimed that their result provides a unified approach to several results on edge-disjoint paths.

The present paper proposes a slight generalization of the theorem of Berstimas and Teo. This generalization implies the edge-disjoint path results mentioned in [1] (including one case where the method described in that paper fails without the generalization). We also show that this theorem is useful in proving various other results on graph connectivity, including those addressed in the paper of Lovász. The main purpose of the paper is not to provide new results, but rather to show how this abstract theorem offers a common understanding of seemingly unrelated results in different areas of graph connectivity.

The term "graph" is used for undirected graphs with possible parallel edges and loops. For a graph $G = (V, E)$ and sets $X, Y \subseteq V$ we use the following notations:

$$\begin{aligned} \delta_G(X, Y) &:= \{uv \in E : \ u \in X, v \in Y\}, \\ d_G(X, Y) &:= |\delta_G(X, Y)|, \\ \delta_G(X) &:= \delta_G(X, V - X), \\ d_G(X) &:= |\delta_G(X)|, \\ i_G(X) &:= d_G(X, X). \end{aligned}$$

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Let $p : 2^V \to \mathbb{Z}_+$ be a non-negative set function on the ground set V and let $m : V \to \mathbb{Z}_+$ be a function on V, called *degree specification*. We consider the problem of finding a graph $G = (V, E)$ for which $d_G(v) = m(v)$ for every $v \in V$ and $d_G(X) \geq p(X)$ for every $X \subseteq V$. We assume in the rest of the paper that $p(\emptyset) = 0$ and the set function p is *symmetric*, i.e. $p(X) = p(V - X)$ for every $X \subseteq V$.

An obvious necessary condition for the existence of G is $m(X) \geq p(X)$ for every $X \subseteq V$, where $m(X)$ denotes $\sum_{v \in X} m(v)$. This motivates the introduction of the *excess function* m_p:

$$m_p(X) := m(X) - p(X) \quad (X \subseteq V).$$

In addition to the requirement that $m_p(X) \geq 0$ for every $X \subseteq V$, it is also necessary for $m(V)$ (or equivalently $m_p(V)$) to be even. The result described in this section requires more: $m_p(X)$ should also be even for every $X \subseteq V$ for which $p(X) > 0$.

The set function p is called *semi-skew-supermodular* if for any 3 sets X_1, X_2, X_3 with $p(X_i) > 0$ $(i = 1, 2, 3)$ at least one of the following four possibilities holds:

- $p(X_i) + p(X_j) \leq p(X_i \cap X_j) + p(X_i \cup X_j)$ for some $i \neq j$,
- $p(X_i) + p(X_j) \leq p(X_i - X_j) + p(X_j - X_i)$ for some $i \neq j$,
- $p(X_1) + p(X_2) + p(X_3) \leq p(X_1 \cap X_2 \cap X_3) + p(X_1 - (X_2 \cup X_3)) + p(X_2 - (X_1 \cup X_3)) + p(X_3 - (X_1 \cup X_2))$,
- $p(X_1) + p(X_2) + p(X_3) \leq p(X_1 \cup X_2 \cup X_3) + p((X_2 \cap X_3) - X_1) + p((X_1 \cap X_3) - X_2) + p((X_1 \cap X_2) - X_3)$.

Obviously every skew supermodular set function is semi-skew-supermodular (a set function is *skew supermodular* if one of the first two inequalities holds for *any* two sets). In addition, the following is implied by the definition.

Claim 1.1. *Semi-skew-supermodularity has the following properties:*

- *If p_1 and p_2 are skew supermodular set functions, then $\max\{p_1, p_2, 0\}$ is a semi-skew-supermodular set function.*
- *If p is semi-skew-supermodular and $G = (V, E)$ is a graph, then $\max\{p - d_G, 0\}$ is semi-skew-supermodular.*
- *If p is a non-symmetric semi-skew-supermodular set function, then $p'(X) := \max\{p(X), p(V - X)\}$ is also semi-skew-supermodular, hence the assumption of symmetry is not restrictive.*

We show that if p is semi-skew-supermodular and m_p is even-valued then the non-negativity of m_p is sufficient for the existence of G. A similar result for a slightly more restricted class of set functions appeared in [1].

Theorem 1.2. *Let $p : 2^V \to \mathbb{Z}_+$ be a symmetric and semi-skew-supermodular set function. Let $m : V \to \mathbb{Z}_+$ be a degree specification with the properties that $m(V)$ is even and $m_p(X)$ is non-negative and even-valued if $p(X) > 0$. Then there exists a graph G such that $d_G(v) = m(v)$ for every $v \in V$ and $d_G(X) \geq p(X)$ for every $X \subseteq V$.*

Proof. We prove the theorem by induction on $m(V)$. The theorem is clearly true if m is positive on at most one node, so we assume that there are at least two nodes where it is positive. A set $X \subseteq V$ is called *tight* if $p(X) > 0$ and $m_p(X) = 0$. Let u be an arbitrary node with $m(u) > 0$, and let $X_1, \dots, X_k$ be the maximal tight sets that contain u. Let $W := V - \cup_{i=1}^k X_i$.

Claim 1.3. $m(W) > 0$.

Proof. The definition of tightness implies that

$$m(X_i) = p(X_i) = p(V - X_i) \le m(V - X_i).$$

Therefore the claim follows immediately if $k \le 1$. Moreover, it also follows if $k = 2$ because $m(X_1 \cap X_2) > 0$ implies that

$$m(X_1) \le m(V - X_1) < m(X_2) + m(W) \le m(V - X_2) + m(W) < m(X_1) + 2m(W).$$

Suppose that $k \ge 3$. Since p is semi-skew-supermodular, we have one of the following four cases.

Case 1: $p(X_i) + p(X_j) \le p(X_i \cap X_j) + p(X_i \cup X_j)$ for some $1 \le i < j \le k$. Then $0 = m_p(X_i) + m_p(X_j) \ge m_p(X_i \cap X_j) + m_p(X_i \cup X_j)$, so $m_p(X_i \cup X_j) = 0$ by the non-negativity of m_p. Therefore $X_i \cup X_j$ is a tight set that contains u, but this contradicts the maximality of X_i and X_j.

Case 2: $p(X_i) + p(X_j) \le p(X_i - X_j) + p(X_j - X_i)$ for some $1 \le i < j \le k$. In this case $0 = m_p(X_i) + m_p(X_j) \ge m_p(X_i - X_j) + m_p(X_j - X_i) + 2m(X_i \cap X_j) > 0$, a contradiction.

Case 3: $p(X_1)+p(X_2)+p(X_3) \le p(X_1 \cap X_2 \cap X_3)+p(X_1-(X_2 \cup X_3))+p(X_2-(X_1 \cup X_3))+p(X_3-(X_1 \cup X_2))$. Then $0 = m_p(X_1)+m_p(X_2)+m_p(X_3) \ge m_p(X_1 \cap X_2 \cap X_3)+m_p(X_1-(X_2 \cup X_3))+m_p(X_2-(X_1 \cup X_3))+m_p(X_3-(X_1 \cup X_2))+2m(X_1 \cap X_2 \cap X_3) > 0$, a contradiction.

Case 4: $p(X_1)+p(X_2)+p(X_3) \le p(X_1 \cup X_2 \cup X_3)+p((X_2 \cap X_3)-X_1)+p((X_1 \cap X_3)-X_2)+p((X_1 \cap X_2)-X_3)$. This implies $0 = m_p(X_1)+m_p(X_2)+m_p(X_3) \ge p(X_1 \cup X_2 \cup X_3)+p((X_2 \cap X_3)-X_1)+p((X_1 \cap X_3)-X_2)+p((X_1 \cap X_2)-X_3)+2m(X_1 \cap X_2 \cap X_3) > 0$, again a contradiction.

This shows that $k \ge 3$ is impossible, which concludes the proof of the claim. □

The claim implies that there is a node $u \ne w \in W$ with $m(w) > 0$. Let $m'(u) := m(u) - 1$, $m'(w) := m(w) - 1$, and let $m'(v) := m(v)$ for all other nodes. Clearly m' is non-negative. Let $p'(X) := p(X) - 1$ if $|X \cap \{u, w\}| = 1$ and $p(X) > 0$, and let $p'(X) := p(X)$ otherwise. Then m' and p' have the following properties:

- p' is non-negative and symmetric,
- p' is semi-skew-supermodular by Claim 1.1,
- $m'(X) - p'(X)$ is even-valued if $p'(X) > 0$,
- $m'(X) - p'(X)$ is non-negative because $|X \cap \{u, w\}| \le 1$ for any tight set X.

By induction there exists a graph G' with $d_{G'}(v) = m'(v)$ for every $v \in V$ and $d_{G'}(X) \ge p'(X)$ for every $X \subseteq V$. Let G be the graph obtained by adding the edge uw to G'. Then G satisfies the conditions of the theorem. □

2 Applications

We present four applications of Theorem 1.2. We start with the parsimonious property that was the original motivation for the result of Bertsimas and Teo. The second application concerns the covering of graphs by edge-disjoint forests, and it is a slight extension of a result in [3]. The third one offers an simple proof for some known results on edge-disjoint paths, extending the method of [1]. The fourth application is a proof of a theorem of Karzanov and Lomonosov [5] on multiflows. The proof is essentially the same as the one by Frank, Karzanov and Sebő [2].

2.1 The parsimonious property

The main motivation for the work of Bertsimas and Teo was the study of the so-called parsimonious property of linear relaxations of cut-covering problems. Let $G = (V, E)$ be an undirected graph with a cost function $c : E \to \mathbb{Z}_+$ on the edges, and let $h : 2^V \to \mathbb{Z}_+$ be a symmetric set function. A node $v \in V$ is called *subadditive* if $h(X) + h(v) \geq h(X + v)$ for every $X \subseteq V - v$. Consider the following linear program:

$$\begin{aligned} &\min cx \\ &x(\delta_G(Z)) \geq h(Z) \quad \text{for every } Z \subseteq V \\ &x(e) \geq 0 \quad \text{for every } e \in E. \end{aligned} \tag{1}$$

We say that a node $v \in V$ has the *parsimonious property* if the linear system (1) has an optimal solution x^* with $x^*(\delta_G(v)) = h(v)$. This property has several structural consequences, and it is useful in the analysis of approximation algorithms (see e.g. [4]). The following theorem is a generalization of the result in [1] where a property stronger than subadditivity was required.

Theorem 2.1. *If G is the complete graph, c satisfies the triangle inequality, and h is semi-skew-supermodular, then the linear system (1) has an optimal solution x^* with the following property:*

$$x^*(\delta_G(v)) = h(v) \quad \textit{for every subadditive node } v.$$

Proof. Let x be an optimal solution where $x(\delta(v)) > h(v)$ for a minimum number of subadditive nodes. Suppose that there exists such a node s. Let k be a positive even integer for which $kx(e)$ is an even integer for every $e \in E$. Let $G_s = (V, E_s)$ denote the graph obtained from G by deleting the edges incident to s. Let us introduce the following set function $p : 2^V \to \mathbb{Z}_+$ and degree specification $m : V \to \mathbb{Z}_+$.

$$p(Z) := \max\{k(h(Z) - x(\delta_{G_s}(Z))), 0\}, \tag{2}$$

$$m(v) := \begin{cases} kx(\delta_G(v, s)) & \text{if } v \neq s, \\ kh(s) & \text{if } v = s. \end{cases} \tag{3}$$

The set function p is symmetric and it is semi-skew-supermodular by Claim 1.1. By the definition of k, $m(v)$ is even for every $v \in V$ and $p(Z)$ is even for every $Z \subseteq V$. This means that $m_p(Z)$ is even for every $Z \subseteq V$.

Let X be a subset of V. If $X \subseteq V-s$, then $m(X)-p(X) \geq k(x(\delta_G(X))-h(X)) \geq 0$ since x is a solution of (1). If $s \in X$, then $m(X)-p(X) \geq m(X-s)+k(h(s)-h(X)+x(\delta_{G_s}(X))) \geq m(X-s)+k(-h(X-s)+x(\delta_{G_s}(X))) = k(x(\delta_G(X-s))-h(X-s)) \geq 0$. We can conclude that m_p is non-negative.

By Theorem 1.2 there exists a graph $G^* = (V, E^*)$ such that $d_{G^*}(v) = m(v)$ for every $v \in V$ and $d_{G^*}(X) \geq p(X)$ for every $X \subseteq V$. Let $x^*(e)$ denote the multiplicity of the edge e in G^*, and let x' be the vector obtained by

$$x'(e) := \begin{cases} x^*(e)/k & \text{if } e \text{ is incident to } s, \\ x(e) + x^*(e)/k & \text{if } e \text{ is not incident to } s. \end{cases}$$

The triangle inequality implies that $cx' \leq cx$. We claim that x' is a solution of (1). It suffices to check that $x'(\delta_G(Z)) \geq h(Z)$ for every $Z \subseteq V - s$. Here $x'(\delta_G(Z)) = x(\delta_{G_s}(Z))+x^*(\delta_G(Z))/k$, and our construction guarantees that $x^*(\delta_G(Z))/k \geq h(Z)-x(\delta_{G_s}(Z))$, hence $x'(\delta_G(Z)) \geq h(Z)$ as required.

Finally let us observe that $x'(\delta_G(s)) = h(s)$ and $x'(\delta_G(v)) = x(\delta_G(v))$ for all other nodes. This means that x' is an optimal solution where the number of subadditive nodes with $x'(\delta_G(v)) > h(v)$ is strictly less than in the case of x, which contradicts the choice of x. □

2.2 Augmentation of k-forest-coverable graphs

The following theorem of Nash-Williams [7] characterizes graphs that can be covered by k forests.

Theorem 2.2. *The edge-set of a graph $G = (V, E)$ can be covered by k forests if and only if $i_G(X) \leq k(|X| - 1)$ for every non-empty subset X of V.* □

Given a graph $G = (V, E)$ that can be covered by k forests and weights on the edges of the complete graph on V, we may want to find an edge set F of maximum weight for which the graph $G' = (V, E + F)$ can still be covered by k forests. This is an easy problem in the sense that it can be solved by finding the maximum weight independent set in a matroid. In fact, the following, more general problem can be solved using the weighted matroid intersection algorithm:

Given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ that can both be covered by k forests and weights on the edges of the complete graph on V, find an edge set F of maximum weight such that the graphs $G'_1 = (V, E_1 + F)$ and $G'_2 = (V, E_2 + F)$ can still be covered by k forests.

What happens if we also want to prescribe the number of new edges incident to each node? The weighted problem cited above becomes NP-complete, even for $E_1 = E_2 = \emptyset$ and $k = 1$. To see this, let the weight of the edges of a given graph G^* be 1, and the weight of all other edges be 0; let furthermore the degree specification be 1 on two designated nodes v_1 and v_2, and 2 on all other nodes. Then the maximum weight of a feasible edge set is $|V|$ if and only if G^* contains a Hamiltonian path between v_1 and v_2.

In the following, we show that the non-weighted degree-prescribed problem can be solved, and there is a simple necessary and sufficient condition for the existence of F. This extends the result in [3] that dealt with the case $E_1 = E_2$. An interesting point about this application of Theorem 1.2 is that parity is not part of the definition of the problem; as we shall see, an even-valued excess function appears implicitly.

Theorem 2.3. *Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs that can be covered by k forests, and let $m : V \to \mathbb{Z}_+$ be a degree specification with $m(V)$ even. There exists an edge set F for which $d_F(v) = m(v)$ for every $v \in V$ and both $G'_1 = (V, E_1 + F)$ and $G'_2 = (V, E_2 + F)$ can be covered by k forests if and only if*

$$\max\left\{m(X) - \frac{m(V)}{2}, 0\right\} \le k(|X|-1) - \max\{i_{G_1}(X), i_{G_2}(X)\} \text{ for every } \emptyset \ne X \subseteq V. \tag{4}$$

Proof. It is clear that $\max\{i_{G'_1}(X), i_{G'_2}(X)\} \le k(|X|-1)$ must hold for every $\emptyset \ne X \subseteq V$, so the fact that $i_F(X) \ge \max\{m(X) - m(V)/2, 0\}$ implies the necessity of the condition.

The proof of sufficiency relies on Theorem 1.2. We define the following set functions.

$$q_1(X) := m(X) + 2i_{G_1}(X) - 2k(|X|-1) \quad (\emptyset \ne X \subseteq V), \tag{5}$$
$$q_2(X) := m(X) + 2i_{G_2}(X) - 2k(|X|-1) \quad (\emptyset \ne X \subseteq V), \tag{6}$$
$$p(X) := \max\{q_1(X), q_1(V-X), q_2(X), q_2(V-X), 0\}. \tag{7}$$

The set function p is non-negative and symmetric. Since $m(X) \equiv m(V-X) \mod 2$, $m(X) - p(X)$ is even for every $X \subseteq V$ for which $p(X) > 0$.

Claim 2.4. *If condition (4) holds, then p_m is non-negative.*

Proof. First we consider the case when $p(X) = q_1(X)$ (the case when $p(X) = q_2(X)$ can be proved analogously). In this case $m_p(X) = 2k(|X|-1) - 2i_{G_1}(X)$, which is non-negative since $i_{G_1}(X) \le k(|X|-1)$ by (4).

Now suppose that $p(X) = q_1(V-X)$ (the case $p(X) = q_2(V-X)$ is analogous). By definition, $m_p(X) = m(X) - m(V-X) + 2k(|V-X|-1) - 2i_{G_1}(V-X) = 2[m(V)/2 - m(V-X) + k(|V-X|-1) - i_{G_1}(V-X)]$, which is non-negative because inequality (4) for $V-X$ states that $m(V-X) - m(V)/2 \le k(|V-X|-1) - i_{G_1}(V-X)$. □

Claim 2.5. *The set function p is semi-skew-supermodular.*

Proof. Both q_1 and q_2 are supermodular, so the set functions $\max\{q_1(X), q_1(V-X)\}$ and $\max\{q_2(X), q_2(V-X)\}$ are skew supermodular. Now Claim 1.1 implies that p is semi-skew-supermodular. □

By Theorem 1.2 there exists a graph $G^* = (V, E^*)$ for which $d_{G^*}(v) = m(v)$ for every $v \in V$ and $d_{G^*}(X) \ge p(X)$ for every $X \subseteq V$. By definition, $d_{G^*}(X) = m(X) - 2i_{G^*}(X)$, so $m(X) - 2i_{G^*}(X) \ge p(X)$. Since $p(X) \ge \max\{q_1(X), q_2(X)\}$, we obtain that

$$m(X) - 2i_{G^*}(X) \ge m(X) + 2\max\{i_{G_1}(X), i_{G_2}(X)\} - 2k(|X|-1).$$

Let $G'_1 = (V, E_1 + E^*)$ and $G'_2 = (V, E_2 + E^*)$. The above inequality can be rewritten as $\max\{i_{G'_1}(X), i_{G'_2}(X)\} \leq k(|X| - 1)$. According to Theorem 2.2 this implies that both G'_1 and G'_2 can be covered by k forests. □

2.3 Edge-disjoint paths: the Eulerian case

Let $G = (V, E)$ and $H = (V, F)$ be undirected graphs on the same ground set. Our goal is to find a family $\{P_f : f \in F\}$ of edge-disjoint paths in G such that for every $f \in F$ the end-nodes of the path P_f are the end-nodes of f.

We say that *the cut condition holds* if $d_G(X) \geq d_H(X)$ for every $X \subseteq V$. The cut condition is clearly necessary for the existence of the required edge-disjoint paths. It is known that if H is a double star, a K_4 or a C_5, possibly with multiple parallel edges, and $G + H$ is Eulerian, then the cut condition is also sufficient (see e.g. [8]). In this section we give a short proof of this result using Theorem 1.2. A somewhat similar proof was sketched in [1], but the splitting off theorem used there is not strong enough to deal with the K_4 case.

Theorem 2.6. *Let $G = (V, E)$ and $H = (V, F)$ be undirected graphs such that*

- *H is a double star, a K_4 or a C_5, possibly with multiple parallel edges,*
- *$G + H$ is Eulerian,*
- *$d_G(X) \geq d_H(X)$ for every $X \subseteq V$.*

Then there exists a family $\{P_f : f \in F\}$ of edge-disjoint paths in G such that for every $f \in F$ the end-nodes of the path P_f are the end-nodes of f.

Proof. Suppose for contradiction that $G = (V, E)$ and $H = (V, F)$ form a counterexample for which $|E|$ is minimal. If $uv \in E$, then $uv \notin F$ since otherwise we could obtain a smaller counterexample by deleting uv from both E and F.

The use of Theorem 1.2 depends on the following lemma.

Lemma 2.7. *If H is a double star, a K_4 or a C_5, possibly with parallel edges, then the set function d_H is semi-skew-supermodular.*

Proof. Suppose for contradiction that d_H is not semi-skew-supermodular on sets X_1, X_2, X_3. Then $d_H(X_i)+d_H(X_j) > d_H(X_i \cap X_j)+d_H(X_i \cup X_j)$ for every $1 \leq i < j \leq 3$ and $d_H(X_i)+d_H(X_j) > d_H(X_i - X_j)+d_H(X_j - X_i)$ for every $1 \leq i < j \leq 3$. It follows that for every $1 \leq i < j \leq 3$ there is an edge between $X_i \cap X_j$ and $V - (X_i \cup X_j)$, and an edge between $X_i - X_j$ and $X_j - X_i$ in H. We denote the former edges by $u_i v_i$ (i=1,2,3) and the latter edges by $w_i z_i$ (i=1,2,3), e.g. $u_1 \in X_2 \cap X_3$ and $v_1 \in V - (X_2 \cup X_3)$.

Let us examine the position of the edges $u_i v_i$. These cannot be node-disjoint, since H does not contain 3 node-disjoint edges. By the symmetry of p (and of the definition of semi-skew-supermodularity) we may assume that $u_1 = u_2 =: u$, so $u \in X_1 \cap X_2 \cap X_3$. The node v_1 (and similarly v_2) cannot be in $V - (X_1 \cup X_2 \cup X_3)$ since then the edge uv_1 together with the 3 edges $w_i z_i$ would contain either 3 disjoint edges, or a triangle and a disjoint edge, both of which is impossible in H. Thus $v_1 \in X_1 - (X_2 \cup X_3)$ and $v_2 \in X_2 - (X_1 \cup X_3)$.

Next we consider the edge u_3v_3. If $u_3 \neq u$, then u_3v_3 is disjoint from u_1v_1, u_2v_2 and w_3z_3, so again we have either 3 disjoint edges or a triangle and a disjoint edge, which is impossible. So $u_3 = u$, and consequently $v_3 \in X_3 - (X_1 \cup X_2)$. Since u is connected to 3 different nodes, H is not a C_5.

Suppose that $H = K_4$ (with possible multiple edges) with nodes u, v_1, v_2, v_3. Since we know exactly the positions of the nodes, it is easy to check that $d_H(X_1)+d_H(X_2)+d_H(X_3) \leq d_H(X_1 \cap X_2 \cap X_3) + d_H(X_1 - (X_2 \cup X_3)) + d_H(X_2 - (X_1 \cup X_3)) + d_H(X_3 - (X_1 \cup X_2))$, contradicting our assumption that d_H is not semi-skew-supermodular on sets X_1, X_2, X_3.

The remaining case is when H is a double star, i.e. the edges of H can be covered by two nodes. One of these nodes must be u (since otherwise we cannot even cover the edges uv_1, uv_2, uv_3). This means that the other node should cover the edges w_1z_1, w_2z_2, w_3z_3, but these cannot be covered by a single node. $\square$

Suppose that there is a node $s \in V$ in our minimal counterexample where $d_H(s) < d_G(s)$. Let G_s denote the graph obtained from G by deleting the edges incident to s. Let us define a set function $p : 2^V \to \mathbb{Z}_+$ and a degree specification $m : V \to \mathbb{Z}_+$ by

$$p(X) := \max\{d_H(X) - d_{G_s}(X), 0\} \quad \text{for every } X \subseteq V, \tag{8}$$

$$m(v) := \begin{cases} d_G(v, s) & \text{if } v \neq s, \\ d_H(s) & \text{if } v = s. \end{cases} \tag{9}$$

We show that m and p have the required properties.

- p is symmetric and non-negative. It is semi-skew-supermodular by Lemma 2.7 and Claim 1.1.
- $m(V)$ is even because $d_G(s) + d_H(s)$ is even.
- If $p(X) > 0$, then $m(X) - p(X)$ is even (it suffices to check this for $s \notin X$, and then $m(X) - p(X) = d_G(X) - d_H(X)$).
- To see that $m(X) - p(X)$ is non-negative, we distinguish two cases. If $s \notin X$, then $m(X) - p(X) \geq d_G(X) - d_H(X) \geq 0$. If $s \in X$, then $m(X) - p(X) \geq m(X) + d_{G_s}(X - s) - d_H(X) \geq m(X) + d_{G_s}(X - s) - d_H(X - s) - d_H(s) = d_G(X - s) - d_H(X - s) \geq 0$.

By Theorem 1.2 there exists a graph $G^* = (V, E^*)$ with $d_{G^*}(v) = m(v)$ for every $v \in V$ and $d_{G^*}(X) \geq p(X)$ for every $X \subseteq V$. Let $G' := G_s + G^*$, and let E' denote the edge-set of G'. The graph $G' + H$ is Eulerian since $d_{G'}(v) = d_G(v)$ for every $v \in V - s$ and $d_{G'}(s) = d_H(s)$. Furthermore, $d_{G'}(X) = d_{G_s}(X) + d_{G^*}(X) \geq d_{G_s}(X) + p(X) \geq d_H(X)$ for every $X \subseteq V$, so G' and H satisfy the cut condition.

Since $|E'| < |E|$, the pair (G', H) is not a counter-example to the theorem. Thus there is a family of edge-disjoint paths $\{P'_f : f \in F\}$ that satisfy the demands. We can obtain a family of edge-disjoint walks $\{W_f : f \in F\}$ in G by the following transformation: if an edge uv in P'_f is in E^* and $u, v \neq s$, then we replace this edge by the two edges us and sv. It is easy to check that we only use edges that are in E, and we use parallel edges at most as many times as they appear in E.

We can reduce each walk W_f to a path P_f in G, thereby obtaining a family of edge-disjoint paths $\{P_f : f \in F\}$ in G that satisfy the demands. This contradicts the assumption that G is a counterexample.

We proved that $d_G(v) = d_H(v)$ for every $v \in V$ in a minimal counterexample and no edge can appear in both E and F. The theorem is obviously true if $E = F = \emptyset$. Let uv be an edge in $E - F$. Then $d_G(\{u,v\}) = d_G(u) + d_G(v) - 2 < d_H(u) + d_H(v) = d_H(\{u,v\})$, which means that the set $\{u,v\}$ violates the cut condition, contradicting the assumption that (G,H) is a counterexample. □

2.4 Multiflows in inner Eulerian graphs

Let $G = (V,E)$ be an undirected graph and $T \subseteq V$ a set of terminal nodes. We say that the pair (G,T) is *inner Eulerian* if $d_G(v)$ is even for every $v \in V - T$. A *T-path* is a path with both end-nodes in T. If $\mathcal{P}$ is a family of T-paths and $Z \subseteq T$, then $d_{\mathcal{P}}(Z)$ denotes the number of paths in $\mathcal{P}$ that have exactly one end-node in Z. Let

$$\lambda_G(Z) := \min\{d_G(X) : \ X \subseteq V, \ X \cap T = Z\}.$$

If $\mathcal{P}$ is a family of edge-disjoint T-paths, then obviously $d_{\mathcal{P}}(Z) \leq \lambda_G(Z)$ for every $Z \subseteq T$. Generalizing a result of Lovász [6], Karzanov and Lomonosov [5] proved that equality can be attained on any given family $\mathcal{L}$ of subsets of T that is *3-cross-free*: it has no three members that are pairwise crossing on the ground set T.

Theorem 2.8 ([5]). *Let (G,T) be inner Eulerian and let $\mathcal{L}$ be a 3-cross-free family of subsets of T. Then there is a family $\mathcal{P}$ of edge-disjoint T-paths for which $d_{\mathcal{P}}(Z) = \lambda_G(Z)$ for every $Z \in \mathcal{L}$.*

Proof. The proof is essentially a reformulation of the proof in [2]. We assume that $\mathcal{L}$ is symmetric, i.e. $Z \in \mathcal{L}$ if and only if $T - Z \in \mathcal{L}$. This does not affect the 3-cross-free property. We prove the theorem by induction on the number of edges incident to $V - T$. If there is no such edge, then the family of T-paths consisting of all edges of G as individual paths satisfies the conditions of the theorem. We can therefore assume that there is a node $s \in V - T$ with $d_G(s) > 0$. Let G_s be the graph obtained from G by deleting the edges incident to s. Let us define the following degree specification and set function on V:

$$m(v) := \begin{cases} d_G(v,s) & \text{if } v \neq s, \\ 0 & \text{if } v = s, \end{cases} \tag{10}$$

$$p(X) := \begin{cases} \max\{\lambda_G(X \cap T) - d_{G_s}(X), 0\} & \text{if } X \cap T \in \mathcal{L}, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

The set function p is non-negative and symmetric, while the degree specification m is non-negative and $m(V)$ is even since $d_G(s)$ is even.

Claim 2.9. *$p_m(X)$ is even whenever $p(X) > 0$.*

Proof. It suffices to check this for $X \subseteq V - s$. Observe that there is a set $Y \subseteq V$ such that $Y \cap T = X \cap T$ and $\lambda_G(X \cap T) = d_G(Y)$. Thus $m(X) - p(X) = d_G(X) - \lambda_G(X \cap T) = d_G(X) - d_G(Y) \equiv d_G(X \Delta Y) \mod 2$, and $d_G(X \Delta Y)$ is even since $X \Delta Y \subseteq V - T$ and (G, T) is inner Eulerian. □

Claim 2.10. *The set function p is semi-skew-supermodular.*

Proof. Let X_1, X_2, X_3 be 3 subsets of V with $p(X_i) > 0$ $(i = 1, 2, 3)$. Since $X_i \cap T \in \mathcal{L}$ $(i = 1, 2, 3)$, there are indices $i \neq j$ for which $X_i \cap T$ and $X_j \cap T$ are not crossing. This means that the pair $(X_i \cap T, X_j \cap T)$ is the same as one of the following three pairs: $((X_i \cap X_j) \cap T, (X_i \cup X_j) \cap T)$, $((X_i - X_j) \cap T, (X_j - X_i) \cap T)$, or $(T - (X_i - X_j), T - (X_j - X_i))$. It follows that the set function

$$f(X) := \begin{cases} \lambda_G(X \cap T) & \text{if } X \cap T \in \mathcal{L}, \\ 0 & \text{if } X \cap T \notin \mathcal{L} \end{cases}$$

is semi-skew-supermodular (here we used the fact that $\mathcal{L}$ is symmetric). Now Claim 1.1 implies that p is semi-skew-supermodular. □

By Theorem 1.2 there exists a graph $G^* = (V, E^*)$ with $d_{G^*}(v) = m(v)$ for every $v \in V$ and $d_{G^*}(X) \geq p(X)$ for every $X \subseteq V$. Let $G' := G_s + G^*$ and let E' denote the edge-set of G'. If $X \cap T \in \mathcal{L}$ for some $X \subseteq V$, then $d_{G'}(X) = d_{G_s}(X) + d_{G^*}(X) \geq d_{G_s}(X) + p(X) \geq \lambda_G(X \cap T)$. This implies that $\lambda_{G'}(Z) = \lambda_G(Z)$ for every $Z \in \mathcal{L}$. By induction there is a family $\mathcal{P}'$ of edge-disjoint T-paths in G' for which $d_{\mathcal{P}'}(Z) = \lambda_G(Z)$ for every $Z \in \mathcal{L}$.

We can obtain a family $\mathcal{W}$ of edge-disjoint walks in G by the following transformation: if an edge uv of a path is in E^*, then we replace this edge by the two edges us and sv. It is easy to check that we only use edges that are in E, and we use parallel edges at most as many times as they appear in E.

We can reduce each walk in $\mathcal{W}$ to a path in G, thus obtaining a family $\mathcal{P}$ of edge-disjoint paths in G that satisfy $d_{\mathcal{P}}(Z) = \lambda_G(Z)$ for every $Z \in \mathcal{L}$. This concludes the proof of the theorem. □

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