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Operations Preserving the Global Rigidity of Graphs and Frameworks in the Plane

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Abstract

A straight-line realization of (or a bar-and-joint framework on) graph G in $\mathbb{R}^d$ is said to be globally rigid if it is congruent to every other realization of G with the same edge lengths. A graph G is called globally rigid in $\mathbb{R}^d$ if every generic realization of G is globally rigid. We give an algorithm for constructing a globally rigid realization of globally rigid graphs in $\mathbb{R}^2$. If G is triangle-reducible, which is a subfamily of globally rigid graphs that includes Cauchy graphs as well as Grünbaum graphs, the constructed realization will also be infinitesimally rigid.

Our algorithm is based on an inductive construction of globally rigid graphs which uses Henneberg 1-extensions and edge additions. We show that vertex splitting, which is another well-known operation in combinatorial rigidity, also preserves global rigidity in $\mathbb{R}^2$.

1 Introduction

We shall consider finite graphs without loops, multiple edges or isolated vertices. A d-dimensional *framework* is a pair (G,p), where $G=(V,E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^d$. Two frameworks (G,p) and (G,q) are *equivalent* if $||p(u)-p(v)|| = ||q(u)-q(v)||$ holds for all pairs u,v with $uv \in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G,p), (G,q) are *congruent* if $||p(u)-p(v)|| = ||q(u)-q(v)||$ holds for all pairs u,v with $u,v \in V$. This is the same as saying that (G,q) can be obtained from (G,p) by an isometry of $\mathbb{R}^d$.

We say that (G,p) is *globally rigid* if every framework (G,q) which is equivalent to (G,p) is congruent to (G,p). The framework (G,p) is *rigid* if there exists an $\epsilon > 0$ such that if (G,q) is equivalent to (G,p) and $||p(v)-q(v)|| < \epsilon$ for all $v \in V$ then

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(G, q) is congruent to (G, p). Intuitively, this means that if we think of a d-dimensional framework (G, p) as a collection of bars and joints where points correspond to joints and each edge to a rigid bar joining its end-points, then the framework is rigid if it has no non-trivial continuous deformations (see also [6],[15, Section 3.2]). It seems to be a hard problem to decide if a given framework is rigid or globally rigid. Indeed Saxe [12] has shown that it is NP-hard to decide if even a 1-dimensional framework is globally rigid. These problems become more tractable, however, if we assume that there are no algebraic dependencies between the coordinates of the points of the framework.

A framework (G, p) is said to be *generic* if the set containing the coordinates of all its points is algebraically independent over the rationals. It is known [15] that rigidity of frameworks in $\mathbb{R}^d$ is a generic property, that is, the rigidity of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic. We say that the graph G is *rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid. The characterization of rigid graphs in $\mathbb{R}^d$ is known only for $d \leq 2$, see [11]. Similarly, we say that a graph G is *globally rigid* in $\mathbb{R}^d$ if every generic realization of G in $\mathbb{R}^d$ is globally rigid. The characterization of globally rigid graphs in $\mathbb{R}^d$ (and the fact that global rigidity is a generic property) is known only for $d \leq 2$. See Subsection 1.1 below.

The *rigidity matrix* of the framework is the matrix $R(G, p)$ of size $|E| \times d|V|$, where, for each edge $v_iv_j \in E$, in the row corresponding to v_iv_j, the entries in the d columns corresponding to vertices i and j contain the d coordinates of $(p(v_i) - p(v_j))$ and $(p(v_j) - p(v_i))$, respectively, and the remaining entries are zeros. We say that a framework (G, p) on n vertices in $\mathbb{R}^d$ is *infinitesimally rigid* if $\operatorname{rank} R(G, p) = \max\{\operatorname{rank} R(K_n, q) : q \in \mathbb{R}^{nd}\}$, where K_n is the complete graph on n vertices. It is known that the infinitesimal rigidity of (G, p) implies rigidity, and that the reverse implication holds if the realization is generic. See [15] for a survey on rigidity.

In this paper we are concerned with the following algorithmic problem: given a graph G, how to create, in polynomial time, a globally rigid realization (G, p) in $\mathbb{R}^d$, if such a realization exists? We shall develop an algorithm for the case when $d = 2$ and G is globally rigid. We are not aware of any previous results on this problem.

One of the difficulties is due to the fact that the output of the algorithm, which is a realization of G with rational coordinates, is non-generic. However, there is no 'simple' sufficient condition for the global rigidity of a non-generic framework. As an additional illustration, consider the problem of constructing a rigid realization of a rigid graph G in $\mathbb{R}^d$. In this case infinitesimal rigidity turns out to be a 'simple' sufficient condition that is essentially expressed by polynomials of the coordinates. Based on this fact, it was shown that a rigid realization, even with integer coordinates in a small grid, can be found in polynomial time, see [5].

Another issue is the level of degeneracy of the framework (G, p) output by the algorithm. Since rather degenerate frameworks may be globally rigid (for example, if G is connected and all vertices are mapped to the same point), it is natural to impose certain additional requirements. It is natural to try to make (G, p) infinitesimally rigid, too[1].

[1]It is known, see e.g. [2], that if (G, p) is a globally rigid and infinitesimally rigid framework then

If G is triangle-reducible, which is a subfamily of globally rigid graphs that includes Cauchy graphs as well as Grünbaum graphs, the constructed realization will also be infinitesimally rigid. Our algorithm is based on a sufficient condition for global rigidity which is based on stress matrices as well as an inductive construction of globally rigid graphs which uses the 1-extension operation.

In the last part of the paper we investigate another operation that can be used in inductive constructions and show that it also preserves global rigidity in $\mathbb{R}^2$. This verifies a conjecture of Cheung and Whiteley [2].

1.1 Globally rigid graphs in two dimensions

In the rest of the paper we assume that $d = 2$, unless specified otherwise.

The *1-extension* operation (which is one of the two well-known Henneberg operations [8]) *on edge uw and vertex t* deletes an edge uw from a graph G and adds a new vertex v and new edges vu, vw, vt for some vertex $t \in V(G) - \{u, w\}$.

The characterization of globally rigid graphs in $\mathbb{R}^d$ follows from results of Hendrickson [7], Connelly [4, Proof of Corollary 1.7], and Jackson and Jordán [9]. We say that G is *redundantly rigid* if $G - e$ is rigid for all edges e of G.

Theorem 1.1. *[7, 4, 9] Let (G, p) be a generic framework. Then (G, p) is globally rigid if and only if either G is a complete graph on two or three vertices, or G is 3-connected and redundantly rigid.*

A key step in the proof of the above combinatorial characterization is the following inductive construction.

Theorem 1.2. *[9, Theorem 6.15] Let G be a 3-connected and redundantly rigid graph. Then G can be obtained from K_4 by a sequence of 1-extensions and edge additions.*

2 Sufficient conditions for global rigidity of frameworks

The sufficient conditions known for the global rigidity of frameworks are in terms of stresses. Let $G = (V, E)$ be a graph, where V is the set of vertices labelled $1, 2, ..., n$. A *stress* is a map $\omega : E \to \mathbb{R}$. The stress is *non-zero* (*nowhere-zero*), if $w_{ij} \neq 0$ for at least one (for all, resp.) $ij \in E$. The *stress matrix* Ω associated with a stress ω is an n-by-n symmetric matrix defined by

$$\Omega_{ij} = \begin{cases} \sum_{ki \in E} \omega_{ki} & \text{if } i = j \\ -\omega_{ij} & \text{if } i \neq j \text{ and } ij \in E \\ 0 & \text{if } i \neq j \text{ and } ij \notin E \end{cases}$$

there exists an $\epsilon > 0$ such that if $||p(v) - q(v)|| < \epsilon$ for all $v \in V$ then (G, q) is also globally rigid. Thus infinitesimal rigidity makes the framework 'stable', too.

Let (G, p) be a framework. We say that $\omega : E \to \mathbb{R}$ is a *self stress* for a framework (G, p) if for each $i \in V$,

$$\sum_{ij \in E} \omega_{ij}(p_i - p_j) = 0.$$

It is easy to see that Ω is the stress matrix of a self stress of framework (G, p) if and only if Ω is symmetric, $\Omega_{ij} = 0$ whenever $ij \notin E$ $(i \neq j)$, and $P\Omega = 0$, where

$$P = \begin{bmatrix} p_{11} & p_{21} & \dots & p_{n1} \\ p_{12} & p_{22} & \dots & p_{n2} \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

is the *augmented configuration matrix* of p.

For completeness, we provide a proof of the following theorem, which can be extracted from [3] and [13]. We say that a framework (G, p) is *bidirectional* if there exist vectors $v_1, v_2 \in \mathbb{R}^2$ such that for each $ij \in E$ either $p_i - p_j = \lambda v_1$ or $p_i - p_j = \lambda v_2$ holds for some $\lambda \in \mathbb{R}$. Otherwise (G, p) is said to be *multidirectional.*

Theorem 2.1. *Let (G, p) be a multidirectional framework on n vertices for which there is a self-stress ω, such that the associated stress matrix Ω is positive semi-definite and has rank $n - 3$. Then (G, p) is globally rigid.*

Proof. Let $q = (q_{11}, q_{21}, \dots, q_{n1}, q_{12}, q_{22}, \dots, q_{n2}) \in \mathbb{R}^{2n}$ and let

$$H(q) = \sum_{ij \in E} \omega_{ij}(q_i - q_j)^2 = q\widehat{\Omega}q^\top$$

be a quadratic form, where $q_i = (q_{i1}, q_{i2})$ and

$$\widehat{\Omega} = \begin{bmatrix} \Omega & 0 \\ 0 & \Omega \end{bmatrix}$$

Since Ω is positive semi-definite, so is $\widehat{\Omega}$. Thus $H(q) \geq 0$ for all $q \in \mathbb{R}^{2n}$.

Claim 2.2. *If $\nabla H(q) = 0$, then $H(q) = 0$.*

Proof. Let $g(t) = H(tq) = t^2 H(q)$. Then $g'(t) = \nabla H(tq)q = t\nabla H(q)q = 0$. Hence $g(t)$ is constant and $H(q) = g(1) = g(0) = 0$. □

The gradient of this form at a point q can be written as

$$\nabla H(q) = 2\left(\sum_{1j \in E} \omega_{1j}(q_1 - q_j), \dots, \sum_{nj \in E} \omega_{nj}(q_n - q_j)\right) = 2\widehat{\Omega}q.$$

Since ω is a self-stress for p, we have $\nabla H(p) = 0$. Thus $H(p) = 0$ by Claim 2.2.

Consider a framework (G, p') that is equivalent to (G, p). First we show that p' is an affine image of p. By definition, $H(p') = H(p) = 0$. Thus, since $H(q) \geq 0$, the point p' is a local minimum of H, and hence $\nabla H(p') = 2\widehat{\Omega}p' = 0$. Let us define the following two subspaces of $\mathbb{R}^{2n}$:

$$S_1 = \{q \in \mathbb{R}^{2n} \mid q_i = Ap_i + b,\ 1 \leq i \leq n,\ A \in \mathbb{R}^{2\times 2}, b \in \mathbb{R}^2\}$$

$$S_2 = \{q \in \mathbb{R}^{2n} \mid \widehat{\Omega}q = 0\} = \ker \widehat{\Omega}$$

It is clear that $\dim S_1 = 6$. Since $\operatorname{rank} \widehat{\Omega} = 2 \operatorname{rank} \Omega = 2n - 6$, this implies $\dim S_2 = \dim \ker \widehat{\Omega} = 6$. To prove that $S_1 = S_2$, it is enough to show that $S_1 \subseteq S_2$. To see this suppose that $q \in S_1$. Then

$$(\widehat{\Omega}q)_i = \sum_{ij\in E} \omega_{ij}(q_i - q_j) = \sum_{ij\in E} \omega_{ij}(Ap_i - Ap_j) = A \sum_{ij\in E} \omega_{ij}(p_i - p_j) = 0,$$

which gives $q \in S_2$. Since $p' \in S_2$, there exist $A \in \mathbb{R}^{2\times 2}$ and $b \in \mathbb{R}^2$, such that $p'_i = Ap_i + b$ for each $1 \leq i \leq n$. Thus p' is an affine image of p, as claimed.

Next we show that the affine map $x \mapsto Ax + b$ is a congruence. Let $C = I - A^\top A$. Since (G, p') is equivalent to (G, p), we have

$$(p'_i - p'_j)^2 = (p'_i - p'_j)^\top (p'_i - p'_j) = (p_i - p_j)^\top A^\top A(p_i - p_j) = (p_i - p_j)^\top (p_i - p_j) = (p_i - p_j)^2$$

for each $ij \in E$. Hence $(p_i - p_j)^\top C(p_i - p_j) = 0$ for each $ij \in E$. Thus either the set $\{x \in \mathbb{R}^2 \,|\, x^\top Cx = 0\}$ is the union of two lines, or $C = 0$. In the former case (G, p) would be bidirectional, contradicting a hypothesis of the theorem. Hence we must have $C = 0$ and $A^\top A = I$, which implies that A is orthogonal and p' is congruent to p. □

We note that if a framework (G, p) satisfies the conditions of Theorem 2.1 then it is in fact *universally globally rigid*, which means that it is globally rigid in $\mathbb{R}^d$ for all $d \geq 2$. The proof of this fact can be found in unpublished work of Connelly. Since we use Theorem 2.1 to verify the global rigidity of the frameworks output by our algorithm, it follows that the constructed frameworks are also universally globally rigid.

2.1 Gale transforms

Let (G, p) be a framework and suppose that the points in p affinely span $\mathbb{R}^2$. Let A be an $(n - 3) \times n$ matrix with linearly independent rows, satisfying $AP^\top = 0$. Then we say that the columns of A, treated as points $a_1, \ldots, a_n \in \mathbb{R}^{n-3}$, form the *Gale transform* of the original points $p_1, \ldots, p_n \in \mathbb{R}^2$ [13]. We say that the four-tuple (G, p, ω, A) is a *Gale-framework* if (G, p) is a framework, ω is a stress for (G, p) and $A = (a_1, \ldots, a_n) \in \mathbb{R}^{(n-3)\times n}$ is a Gale transform of p satisfying $a_i^\top a_j = -\omega_{ij}$ for all $ij \in E$ and $a_i^\top a_j = 0$ for all $i, j \in V$, $i \neq j$, $ij \notin E$. A Gale-framework is *multidirectional* if (G, p) is multidirectional.

For example, the following is a multidirectional Gale framework on K_4, given by its augmented configuration matrix P, A, and a self-stress ω. (Note that ω is nowhere-zero and the framework is infinitesimally rigid and is in general position.)

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\omega_{12} = \omega_{23} = \omega_{34} = \omega_{14} = 1$$

$$\omega_{13} = \omega_{24} = -1$$

Lemma 2.3. *Let (G, p, ω, A) be a Gale-framework on n vertices. Then ω is a self-stress for (G, p) with a positive semi-definite stress matrix of rank $n - 3$.*

Proof. Let $\Omega = A^\top A$. By the definition of Gale-frameworks and the fact that $\Omega P^\top = A^\top A P^\top = A^\top 0 = 0$ we get that Ω is the stress matrix of ω and ω is a self-stress for (G, p). Ω has rank $n - 3$ since A has $n - 3$ independent rows and it is positive semi-definite since $q^\top \Omega q = q^\top A^\top A q = (Aq)^\top (Aq) \geq 0$ for all $q \in \mathbb{R}^n$. □

By Theorem 2.1 and Lemma 2.3 we obtain:

Theorem 2.4. *Let (G, p, ω, A) be a multidirectional Gale-framework. Then (G, p) is globally rigid.*

3 Extension of frameworks and Gale frameworks

Let (G, p) be a framework and let $uw \in E(G)$ and $t \in V(G) - \{u, w\}$. The *1-extension* operation on edge uw and vertex t *with parameters* $\alpha_u, \alpha_w, \alpha_t$ consists of performing a 1-extension on G as well as extending the realization p by letting $p(v) = \alpha_u p(u) + \alpha_w p(w) + \alpha_t p(t)$, where $\alpha_u + \alpha_w + \alpha_t = 1$ and v is the new vertex added to G.

Lemma 3.1. *Let (G, p) be a multidirectional framework and (G^*, p^*) its 1-extension with parameters $\alpha_u, \alpha_w, \alpha_t$. If $\alpha_t = 0$ or $\alpha_u \alpha_w \neq 0$, then (G^*, p^*) is multidirectional.*

Proof. If p_u, p_w, p_t are collinear or $\alpha_t = 0$, then the set of edge directions of (G^*, p^*) are the same as that of (G, p). Otherwise, p_u, p_w, p_t are affinely independent and $\alpha_u, \alpha_w, \alpha_t \neq 0$. In this case the edges vu, vw, vt define three independent directions, so (G^*, p^*) is multidirectional. □

Let (G, p, ω, A) be a Gale framework. The *1-extension* operation *with parameters* $\alpha_u, \alpha_w, \alpha_t, \beta$, where $\beta \neq 0$ and $\alpha_u + \alpha_w + \alpha_t = 1$, consists of performing a 1-extension of (G, p) with parameters $\alpha_u, \alpha_w, \alpha_t$ as well as replacing ω and A by ω^* and A^* by letting

$$\omega^*_{ij} = \begin{cases} \omega_{ij} & \text{if } \ ij \in E - \{uw, ut, wt\} \\ \omega_{ij} - \beta^2 \alpha_i \alpha_j & \text{if } \ ij \in \{ut, wt\} \\ \beta^2 \alpha_j & \text{if } \ i = v \text{ and } j \in \{u, w, t\} \end{cases}$$

$$A^* = \begin{bmatrix} a_1 & \dots & a_u & a_w & a_t & \dots & a_n & 0 \\ 0 & \dots & \beta\alpha_u & \beta\alpha_w & \beta\alpha_t & \dots & 0 & -\beta \end{bmatrix}$$

Lemma 3.2. *Let (G, p, ω, A) be a Gale-framework and let $(G^*, p^*, \omega^*, A^*)$ be its 1-extension with parameters $\alpha_u, \alpha_w, \alpha_t, \beta$. If $\alpha_u\alpha_w = \omega_{uw}/\beta^2$, and if $\alpha_t = 0$ whenever $\{ut, wt\} \nsubseteq E$, then $(G^*, p^*, \omega^*, A^*)$ is a Gale-framework.*

Proof. Let a_i^* denote the columns of A^*, $1 \le i \le n+1$. It is easy to check that A^* is a Gale-transform of p^* and $a_i^{*\top} a_j^* = -\omega_{ij}^*$ if $ij \in E^*$. Let us suppose now that $ij \notin E^*$ for some $i, j \in V^*$, $i \neq j$. Then either $i = v$ and $j \notin V - \{u, w, t\}$, or $i \in \{u, w, t\}$ and $j \in V - \{u, w, t\}$ and $ij \notin E$, or $ij \in E - \{ut, wt\}$, or $ij = uw$. In the first case $a_i^{*\top} a_j^* = 0^\top a_j + \beta 0 = 0$. In the second case $a_i^{*\top} a_j^* = a_i^\top a_j + \beta\alpha_i 0 = a_i^\top a_j = 0$. In the third case $a_i^{*\top} a_j^* = a_i^\top a_t + \beta^2\alpha_i\alpha_t = 0$. In the last case $a_i^{*\top} a_j^* = a_u^\top a_w + \beta^2\alpha_u\alpha_w = -\omega_{uw} + \omega_{uw} = 0$. □

Our algorithm will create a globally rigid realization of G by iteratively constructing a multidirectional Gale framework on each graph in an inductive construction of G using edge additions and 1-extensions. To this end we shall use the following specific operations on Gale frameworks. Let (G, p, ω, A) be a Gale framework.

- **Edge addition** In this case G^* is obtained from G by an edge addition.

  Let $p^* = p$, $A^* = A$, $\omega_{ij}^* = \omega_{ij}$ if $ij \in E$ and $\omega_{uw}^* = 0$.

- **1-Extension**

  In this case G^* is obtained from G by a 1-extension on edge uw and vertex t. We define $(G^*, p^*, \omega^*, A^*)$ by defining the parameters $\beta, \alpha_u, \alpha_w, \alpha_t$. This will also determine $p(v)$. We consider three cases.

  Case 1 $\omega_{uw} = 0$.

  Let $\beta = 1$, $\alpha_t = 0$ and $\alpha_u = 0$ or $\alpha_w = 0$.

  (Note: It means that in this case $p(v) = p(w)$ or $p(v) = p(u)$.)

  Case 2 $\omega_{uw} \neq 0$ and $\{ut, wt\} \nsubseteq E$.

  Let $\alpha_t = 0$ and let α_u, α_w be chosen so that $\alpha_u\alpha_w$ has the same sign as ω_{uw}. Let $\beta^2 = \frac{\omega_{uw}}{\alpha_u\alpha_w}$.

  (Note: now $p(v) = \alpha_u p(u) + \alpha_w p(w)$ lies on the line of $p(u)p(w)$ or $p(v) = p(u) = p(w)$. If $\omega_{uw} > 0$ then $p(v)$ lies on the segment $[p(u), p(w)]$ (eg. $\alpha_u = \alpha_w = \frac{1}{2}$) and if $\omega_{uw} < 0$ then it lies in its complement (eg. $\alpha_u = 2$, $\alpha_w = -1$).)

  Case 3 $\omega_{uw} \neq 0$ and $\{ut, wt\} \subseteq E$.

  Let $\alpha_u, \alpha_w, \alpha_t$ be chosen so that $\alpha_u\alpha_w$ has the same sign as ω_{uw}, and so that $\alpha_t \notin \{0, \frac{\omega_{ut}}{\omega_{uw}}\alpha_u, \frac{\omega_{wt}}{\omega_{uw}}\alpha_w\}$. Let $\beta^2 = \frac{\omega_{uw}}{\alpha_u\alpha_w}$.

  (Note: For example, if $\omega_{uw} > 0$ we can define $\alpha_u = \alpha_w = \alpha_t = \frac{1}{3}$ and if $\omega_{uw} < 0$ then let $\alpha_u = 3$, $\alpha_w = \alpha_t = -1$. Consider the case when $p(u), p(w), p(t)$

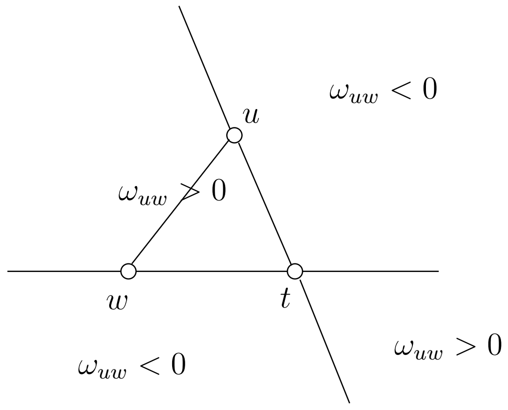

Figure 1: The possible placements of p_v in Case 3.

are not collinear. If $\omega_{uw} > 0$ then $p(v)$ can be placed anywhere in the angle $p(u)p(t)p(w)$ and in its mirror image to $p(t)$ (the lines $p(u)p(t)$ and $p(w)p(t)$ are excluded). Otherwise $p(v)$ can be in the two other angles defined by the lines through $p(u)p(t)$ and $p(w)p(t)$, but not on the lines themselves (see Figure 1). By excluding the three values for α_t we have excuded three lines.

Lemma 3.3. *Suppose that (G, p, ω, A) is a multidirectional Gale framework for which (G, p) is infinitesimally rigid, ω is nowhere-zero, and the points $p(v)$, $v \in V$, are in general position. Let $(G^*, p^*, \omega^*, A^*)$ be obtained from (G, p, ω, A) by a 1-extension as decribed in Case 3. Then $(G^*, p^*, \omega^*, A^*)$ is a multidirectional Gale framework, for which (G^*, p^*) is infinitesimally rigid and ω^* is nowhere-zero.*

Proof. By Lemmas 3.1 and 3.2 $(G^*, p^*, \omega^*, A^*)$ is a multidirectional Gale-framework. Since ω is nowhere-zero, we have $\omega_{uw} \neq 0$. Thus we must have $\alpha_u \neq 0$ and $\alpha_w \neq 0$. Hence $\omega^*_{vi} = \beta^2 \alpha_i \neq 0$ for $i \in \{u, w, t\}$. Furthermore, the choice of α_t implies that $\omega^*_{ut} = \omega_{ut} - \beta^2 \alpha_u \alpha_t = \omega_{ut} - \omega_{uw} \alpha_t / \alpha_w \neq 0$. Similarly, $\omega^*_{wt} \neq 0$. Thus ω^* is a nowhere-zero stress.

To show that (G^*, p^*) is infinitesimally rigid first observe that $(G - uw, p)$ is infinitesimally rigid, since ω is nowhere-zero. Moreover, the addition of the new point $p(v)$ preserves infinitesimal rigidity, since $p(u)$, $p(w)$ and $p(t)$ are in general position and $\alpha_t \neq 0$, so $p(v)$ is not on the line through $p(u)p(w)$. □

4 Globally rigid realizations

Given a graph $G = (V, E)$ we say that a 1-extension on the edge uw and vertex t is a *triangle-split* if $\{ut, wt\} \subseteq E$ (that is, if u, w, t induce a triangle of G). A graph will be called *triangle-reducible* if it can be obtained from K_4 by a sequence of triangle-splits. We note that triangle-reducible graphs are 3-connected redundantly rigid planar graphs with $2|V| - 2$ edges.

Theorem 4.1. *Let G be a globally rigid graph on at least four vertices. Then one can construct, in polynomial time, a globally rigid realization (G, p). Furthermore, if*

G is triangle-reducible, the constructed realization can be chosen to be infinitesimally rigid, too.

Proof. Let $K_4 = H_1, H_2, ..., H_m = G$ be an inductive construction of G from K_4 using edge-additions and 1-extensions. Such a sequence exists by Theorem 1.2. Furthermore, if G is triangle-reducible, we may assume that H_{i+1} is obtained from H_i by a triangle-split, $1 \leq i \leq m-1$. These inductive constructions can be obtained in polynomial time, see [1] and Lemma 4.2 below.

Let $(H_1, p_1, \omega_1, A_1)$ be a multidirectional Gale framework on $H_1 = K_4$. If G is triangle-reducible, we choose one with a nowhere-zero stress and for which (H_1, p_1) is infinitesimally rigid and is in general position. The example in Subsection 2.1 satisfies all these conditions.

To compute a globally rigid framework on G we follow the inductive construction and perform edge additions and 1-extensions as described in Cases 1-3, to create multidirectional Gale-frameworks $(H_i, p_i, \omega_i, A_i)$ for $1 \leq i \leq m$. By Lemmas 3.1, 3.2, and Theorem 2.4, the framework (H_m, p_m) will be a globally rigid realization of G.

If, in addition, G is triangle-reducible, we only perform 1-extensions, as described in Case 3, with the additional property that the points in each framework (H_i, p_i), $1 \leq i \leq m$, are in general position. In this case Lemma 3.3 implies that (H_m, p_m) will also be infinitesimally rigid.

Observe that the algorithm does not need to compute the Gale transforms A_i but updates the stress and the realization. Without giving an explicit upper bound, we note that the numbers (the values of the self-stress and the coordinates of the vertices) occuring in the algorithm can always be chosen to be of polynomial size. $\square$

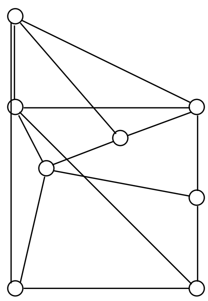

Figure 2: A globally rigid realization of a globally rigid graph produced by the algorithm.

We remark that even though the realization given by the algorithm will affinely span $\mathbb{R}^2$, it may be rather degenerate: the positions of several vertices may coincide and certain edges may have length zero. (For example, if a 1-extension is performed on an edge whose stress is zero, the position of the new vertex will be on one of the endpoints of the edge.) This can be overcome in the case of triangle-reducible graphs, for which our algorithm outputs an infinitesimally rigid realization. We believe that, possibly by using a different sufficient condition for global rigidity, it will be possible to obtain such 'non-degenerate' and 'stable' realizations for all globally rigid graphs.

4.1 Testing triangle-reducibility

In this subsection we show that testing triangle-reducibility (and finding an inductive construction for triangle-reducible graphs) can be done efficiently in a greedy fashion. Let $G = (V, E)$ be a graph. If G is triangle-reducible and $|V| > 4$ then there must be a vertex v with neighbors x, y, z spanning exactly two edges in G. The following lemma says that we can eliminate any such vertex and get another triangle-reducible graph. Thus triangle-reducibility can be tested with a simple greedy algorithm which also provides a sequence of triangle-splits which generates G.

Lemma 4.2. *Let $G = (V, E)$ be a triangle-reducible graph with $|V| \geq 5$ and let $v \in V$ with $N(v) = \{x, y, z\}$. Suppose that $xz, yz \in E$ and $xy \notin E$. Then $G' = G - v + xy$ is triangle-reducible.*

Proof. Let $K_4 = G_0, G_1, \ldots, G_n = G$ be a sequence of graphs, where G_{i+1} is obtained from G_i by a triangle-split, $0 \leq i \leq n-1$. Consider the first graph G_k in the sequence which contains v. It is easy to see that, by modifying G_0 and G_1, if necessary, we may assume that $k \geq 1$. Thus v is created by a triangle split operation on G_{k-1}. Since a triangle split does not decrease the degree of any vertex, and does not add new edges connecting existing vertices, it follows that v has degree three and $N_l(v)$ induces exactly two edges in G_l for all $k \leq l \leq n$, where $N_i(v)$ denotes the set of neighbours of v in some G_i.

Let $N_k(v) = \{u, w, t\}$ and suppose that $ut, wt \in E(G_k)$ and $uw \notin E(G_k)$. Next observe that as long as t remains a neighbour of v, the other two neighbours of v must be non-adjacent. In fact, t must remain a neighbour of v in the rest of the sequence.

Claim 4.3. *$vt \in E(G_l)$ for all $k \leq l \leq n$.*

Proof. Let $i \geq k$ be the largest index for which $vt \in E(G_i)$. For a contradiction suppose that $i \leq n-1$. Let $N_i(v) = \{u_i, w_i, t\}$. It follows from the previous observation that we must have $u_i w_i \notin E(G_i)$. Since $vt \notin E(G_{i+1})$, it follows that G_{i+1} is obtained from G_i by 'splitting' the edge vt by a new vertex t' of degree three. Hence $N_{i+1}(v) = \{u_i, w_i, t'\}$ induces at most one edge in G_{i+1}. This contradicts the fact that the neighbours of v induce exactly two edges in G_l for all $k \leq l \leq n$. □

It follows from Claim 4.3 that $N_i(v) = \{u_i, w_i, t\}$ and $u_i w_i \notin E(G_i)$ for all $k \leq i \leq n$. Thus $z = t$ holds. Let $G'_i = G_i - v + u_i w_i$, $k \leq i \leq n$. Next we show, by induction on i, that G'_i is triangle-reducible. Since $G'_k = G_{k-1}$, it is true for $i = k$. Suppose that $N_{i+1}(v) = N_i(v)$, i.e. the triangle-split, applied to G_i, leaves the neighbour set of v unchanged. Then G'_{i+1} can be obtained from G'_i by the same triangle split, and hence, by induction, G'_{i+1} is also triangle-reducible. Otherwise G_{i+1} is obtained from G_i by 'splitting' the edge vu_i (or vw_i). Then, without loss of generality, we have $G_{i+1} = G_i + u_{i+1} - vu_i + \{u_{i+1}v, u_{i+1}u_i, u_{i+1}t\}$. Then $w_{i+1} = w_i$ and $G'_{i+1} = G'_i + u_{i+1} - u_i w_i + \{u_{i+1}w_i, u_{i+1}u_i, u_{i+1}t\}$. So G'_{i+1} can be obtained from G'_i by a triangle-split. By induction, this gives that G'_{i+1} is triangle-reducible. Thus $G' = G'_n$ is triangle-reducible, which completes the proof. □

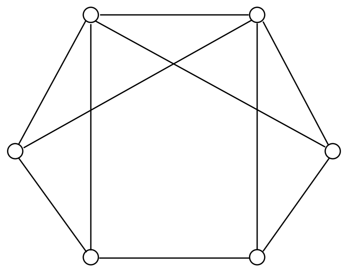

Figure 3: A Cauchy-polygon on 6 vertices.

4.2 Cauchy and Grünbaum graphs

Another sufficient condition for global rigidity, due to Connelly, is based on stresses as well as convexity. Here we formulate a 2-dimensional version of his result for bar-and-joint frameworks, which can be deduced from Corollary 1 and Theorem 5 of [3].

Theorem 4.4. *[3] Let (G, p) be a framework whose edges form a convex polygon P in $\mathbb{R}^2$ with some chords. Suppose that there is a non-zero self-stress ω for (G, p) for which $\omega_{ij} \geq 0$ if $ij \in E$ is an edge on the boundary of P and $\omega_{ij} \leq 0$ if $ij \in E$ is an edge which is a chord of P. Then (G, p) is globally rigid.*

The *Cauchy-graphs* C_n and *Grünbaum graphs* G_n are both defined on vertex set $\{1, \ldots, n\}$ and both contain the edges $\{i, i+1\}$, $i = 1, 2, \ldots, n$ (modulo n). In addition, the Cauchy graph contains the chords $\{i, i+2\}$, $i = 1, \ldots, n-2$, and the Grünbaum graph has the edges $1, 3$ and $2, i$ for $i = 4, \ldots, n$.

A *Cauchy-polygon* (*Grünbaum polygon*) is a framework (C_n, p) ((G_n, p)), where the positions $p_1, \ldots, p_n$ of the vertices are in general position and, in this order, they form the set of vertices of a convex polygon in the plane. See Figure 3.

It is easy to check that Cauchy-graphs as well as Grünbaum-graphs are triangle-reducible. One can also show, by induction on n, that any given Cauchy-polygon (C_n, p) (or Grünbaum-polygon (G_n, p)) can be obtained as the output of our algorithm. This gives a different proof of the first part of the next theorem.

Theorem 4.5. *(i) [3, Lemma 4, Theorem 5] Every Cauchy-polygon (C_n, p) is globally rigid.*
(ii) Every Grünbaum-polygon (G_n, p) is globally rigid.

Note that our algorithm may also generate non-convex globally rigid realizations of Cauchy-graphs, see Figure 4. Thus, in this sense, it gives an extension of Theorem 4.5(i).

5 Vertex splitting

Motivated by the usefulness of the 1-extension operation and by the following conjecture, in this section we investigate the effect of another operation on global rigidity in $\mathbb{R}^2$.

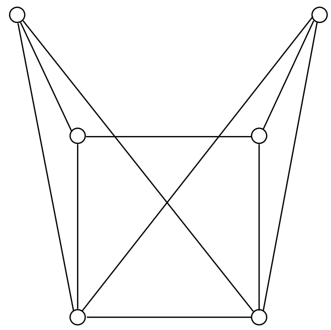

Figure 4: A non-convex globally and infinitesimally rigid realization of the Cauchy-graph C_6.

Conjecture 5.1. *[2] If G is globally rigid in $\mathbb{R}^d$ and G' is obtained from G by a (d-dimensional) vertex-splitting operation, so that each of the split vertices has degree at least $d+1$, then G' is globally rigid in $\mathbb{R}^d$.*

Given a graph $G = (V, E)$, an edge $uv \in E$ and a bipartition F_1, F_2 of the edges incident to v (except uv), the (2-dimensional) *vertex-splitting* operation replaces vertex v by two new vertices v_1 and v_2, replaces the edge uv by three new edges uv_1, uv_2, v_1v_2, and replaces all edges $wv \in F_i$ by an edge wv_i, $i = 1, 2$. See Figure 5. In this section we will prove that this operation preserves global rigidity in $\mathbb{R}^2$, provided it does not create vertices of degree two.

We need the following refinement of the well-known inductive construction of *minimally rigid graphs*[2], which uses 0-extensions and 1-extensions. A *0-extension* adds a new vertex v and new edges vu, vw for two distinct vertices $u, w \in V(G)$. We say that a sequence $H_1, H_2, ..., H_m$ is a *Henneberg sequence* of a minimally rigid graph G if H_1 is an edge, $H_m = G$, and H_{i+1} is obtained from H_i by a 0- or 1-extension, $1 \leq i \leq m-1$. It is well-known that both extensions preserve rigidity. The existence of a Henneberg sequence for each minimally rigid graph G follows from the facts that G has $2|V(G)| - 3$ edges and has minimum degree at least two, and hence it has at least one vertex of degree two or three (if $|V(G)| \geq 3$). Furthermore, if v is a vertex of degree two or three in G, it is always possible to perform the inverse of the 0- or 1-extension operation at v so that the resulting graph remains minimally rigid. See e.g. Section 2.1 of [9]. A similar argument can be used to deduce the following somewhat stronger result. It follows by observing that the small degree vertex v can be chosen to be distinct from the end-vertices of a designated edge and a designated vertex, since (i) G has at least three vertices of degree at most three, and (ii) if the graph has at least four vertices and has exactly three vertices of degree at most three, then these vertices must have degree two, and they must be pairwise non-adjacent. Thus we have:

[2] A rigid graph G is minimally rigid if $G - e$ is not rigid for all $e \in E(G)$. Equivalently, G is minimally rigid if it has $2|V(G)| - 3$ edges and each of its subgraphs on a set X of vertices, $|X| \geq 2$, contains at most $2|X| - 3$ edges.

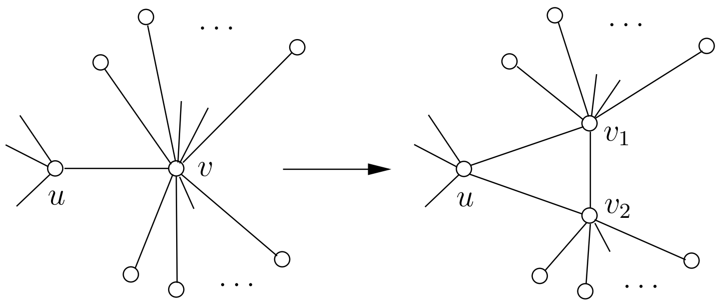

Figure 5: The vertex-splitting operation on edge uv and vertex v.

Lemma 5.2. *Let $G = (V, E)$ be a minimally rigid graph and let $uv \in E$ be a designated edge. If $|V| \geq 3$ then let w be a designated vertex which is different from u, v. Then (i) there exists a Henneberg sequence which starts with the edge uv and generates G, (ii) if $|V| \geq 3$ then there exists a Henneberg sequence which starts with the triangle uvw and generates G.*

It is well-known that vertex-splitting preserves rigidity [15]. The following proof method, however, is new, and will be used to deal with redundant rigidity.

Lemma 5.3. *Let G be a rigid graph and let G' be obtained from G by a vertex splitting operation. Then G' is rigid.*

Proof. Let G' be obtained from G by a vertex splitting at edge uv and with bipartition F_1, F_2. Let H be a minimally rigid spanning subgraph of G which contains the edge uv and consider a Henneberg sequence $H_1, \ldots, H_m$ of H with $H_1 = uv$. We define a bipartition F_1^j, F_2^j of the edges incident to v (except uv) in each H_j, starting with H_m. Let $F_i^m = F_i$, $i = 1, 2$. If $j < n$ and $wv \in E(H_j) \cap E(H_{j+1})$ then let wv belong to the same partition as in H_{j+1}. If $wv \in E(H_j) - E(H_{j+1})$ then $H_{j+1} = H_j + y - wv + \{yv, yw, yt\}$, for some y and t. In this case let wv belong to the same partition as yv in H_{j+1}. Now consider the triangle uv_1v_2 and apply the 'same' Henneberg sequence in such a way that every time a new edge incident to v is added in H_j, it is connected to either v_1 or v_2, according to the bipartition F_1^j, F_2^j. The graph H' obtained this way is a minimally rigid spanning subgraph of G'. Thus G' is rigid. □

Lemma 5.4. *Let G be a redundantly rigid graph and let G' be obtained from G by a vertex splitting at edge uv and with bipartition F_1, F_2 such that F_1 and F_2 are both non-empty (or equivalently, in such a way that v_1, v_2 have degree at least three in G'). Then G' is redundantly rigid.*

Proof. We have to show that $G' - xy$ is rigid for all edges $xy \in E(G')$. This follows from Lemma 5.3 for all edges $xy \in E(G') - \{uv_1, uv_2, v_1v_2\}$, since $G' - xy$ can be obtained from the rigid graph $G - xy$ by a vertex splitting operation.

It remains to prove that $G' - uv_1$ and $G' - v_1v_2$ is rigid. (By symmetry the rigidity of $G' - uv_2$ will also follow.) First consider $G' - uv_1$. Let $vy \in F_2$ and let H be a minimally rigid spanning subgraph of $G - vy$ which contains uv. Such an H exists, since G is redundantly rigid. Consider a Henneberg sequence $H_1, H_2, ..., H_m$ of H for which H_2 is the triangle uvy (recall Lemma 5.2). Define the bipartition F_1^j, F_2^j of the edges incident to v in each H_j as in Lemma 5.3. Now apply the 'same' Henneberg sequence, but by replacing the starting triangle uvy by the minimally rigid graph $K_{u,v_1,v_2,y} - uv_1$ and then, as above, in such a way that every time a new edge incident to v is added to H_j, it is connected to either v_1 or v_2, according to the bipartition F_1^j, F_2^j. (Here $K_{u,v_1,v_2,y}$ denotes the complete graph on vertex set u, v_1, v_2, y.) The graph H' obtained this way is a minimally rigid spanning subgraph of $G' - uv_1$. Thus $G' - uv_1$ is rigid.

The case of $G' - v_1v_2$ is similar. The only difference is that the starting triangle uvy is replaced by the minimally rigid graph $K_{u,v_1,v_2,y} - v_1v_2$. □

Lemma 5.5. *Let G be a 3-connected graph and let G' be obtained from G by a vertex splitting at edge uv and with bipartition F_1, F_2 such that F_1 and F_2 are both non-empty (or equivalently, in such a way that v_1, v_2 have degree at least three in G'). Then G' is 3-connected.*

Proof. For a contradiction suppose that G' is not 3-connected. Then there is a small 'separator', i.e. a set $S \subset V(G')$ with $|S| \leq 2$ for which $G' - S$ is disconnected. Since each vertex has degree at least three in G', it follows that each connected component of $G' - S$ contains at least two vertices. Furthermore, since u, v_1, v_2 induce a triangle in G', there is exactly one component of $G' - S$ which intersects $\{u, v_1, v_2\}$. Thus by 'contracting' the edge v_1v_2 in G' (i.e. by performing the inverse of vertex splitting) we obtain a graph H with a separator of size at most two. Since $G = H$, this is a contradiction. □

Lemma 5.4, 5.5, and Theorem 1.2 now implies an affirmative answer to the two-dimensional version of Conjecture 5.1. (The one-dimensional case is easy to verify.)

Theorem 5.6. *Let G be a globally rigid graph and let G' be obtained from G by a vertex splitting at edge uv and with bipartition F_1, F_2 in such a way that v_1, v_2 have degree at least three in G'. Then G' is globally rigid.*

5.1 Diamond split

There is a second form of vertex splitting in two dimensions. Let uv, vw be two adjacent edges and let F_1, F_2 be a bipartition of the edges incident to v (except uv, vw). The operation *diamond split* deletes the edges uv, vw, and replaces them by a four-cycle (diamond): it adds two new vertices v_1, v_2, edges uv_1, uv_2, wv_1, wv_2, and replaces each edge $zv \in F_i$ by an edge zv_i, for $i = 1, 2$. See Figure 6.

Whiteley [16] asked whether the diamond-split operation also preserves the redundant rigidity or global rigidity of a graph G, provided the new vertices have degree at least three. It is not difficult to show that if G has $2|V(G)| - 2$ edges then diamond-split preserves redundant rigidity (that is, it takes an 'M-circuit' to an 'M-circuit'). In general, however, this is not always the case. See Figure 7. The diamond-split operation may also destroy 3-connectivity, if u, v, w form a separating set of size three in G. Thus, in general, it does not preserve global rigidity either.

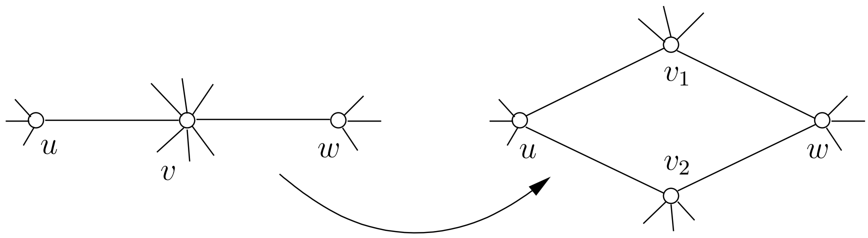

Figure 6: The diamond split operation.

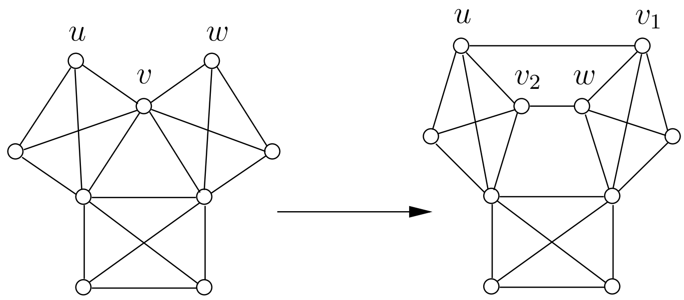

Figure 7: Diamond split may destroy redundant rigidity.

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