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Uniquely Localizable Networks with Few Anchors

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Abstract

In the network localization problem the locations of some nodes (called anchors) as well as the distances between some pairs of nodes are known, and the goal is to determine the location of all nodes. The localization problem is said to be solvable (or uniquely localizable) if there is a unique set of locations consistent with the given data. Recent results from graph rigidity theory made it possible to characterize the solvability of the localization problem in two dimensions.

In this paper we address the following related optimization problem: given the set of known distances in the network, make the localization problem solvable by computing the locations of as few nodes as possible, that is, by minimizing the number of anchor nodes. We develop a polynomial-time 3-approximation algorithm for this problem by proving new structural results in graph rigidity and by using tools from matroid theory.

1 Introduction

In the network localization problem the locations of some nodes (called anchors) as well as the distances between some pairs of nodes are known, and the goal is to determine the location of all nodes. This is one of the fundamental algorithmic problems in the theory of wireless sensor networks and it has been in the focus of a number of recent research articles, see e.g. [1, 3, 15].

The localization problem is said to be solvable (or the network is said to be uniquely localizable) if there is a unique set of locations consistent with the given data. Recent results from graph rigidity theory made it possible to characterize the solvability of the localization problem in two dimensions, assuming that the nodes are in 'general position'. In this case the solvability of the problem depends only on the combinatorial properties of the network. In the *graph of the network* vertices correspond to nodes,

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and two vertices are connected by an edge if either the corresponding distance is known, or both endvertices are anchors. See Figures 1, 2. As it was observed earlier [3, 15], a two-dimensional network in 'general position' is uniquely localizable if and only if it has at least three anchors and the graph of the network is globally rigid (or uniquely realizable).

In this paper we address the following related optimization problem: given the set of known distances in the network, make the localization problem solvable by computing the locations of as few nodes as possible, that is, by minimizing the number of anchor nodes. (This problem was also posed in [15] as an open question.)

We develop a polynomial-time 3-approximation algorithm for this problem by proving new structural results in graph rigidity and by using tools from matroid theory. Due to space limitations, we focus on the combinatorial aspects. For more details and other related results see the full version [4] and the list of references.

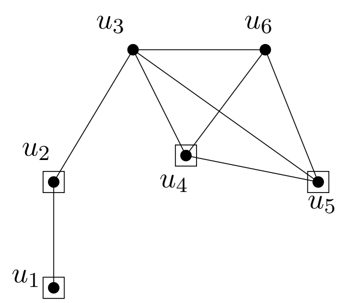

Figure 1: A uniquely localizable network with six nodes. Edges correspond to known distances between pairs of nodes. The anchor nodes u_1, u_2, u_4, u_5 are indicated by boxes. This is a smallest anchor set which can guarantee solvability for the given distances.

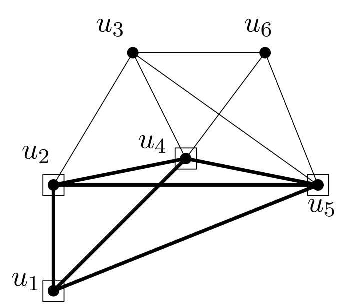

Figure 2: The graph of the network in Figure 1.

2 Rigid and globally rigid graphs

In this section we give a brief summary of the basic definitions and results concerning rigid and globally rigid graphs. See [5, 7, 13, 16] for more details on rigid graphs and frameworks. For a graph $G = (V, E)$ and a subset $X \subseteq V$ let $E_G(X)$ denote the set, and $i_G(X)$ the number, of edges in $G[X]$, that is, in the subgraph induced by X in G. For $F \subseteq E$ we shall use $i_F(X)$ to denote the number of those edges in F which are induced by X.

Let $G = (V, E)$ be a graph and let $F \subseteq E$. We say that F is *sparse* if

$$i_F(X) \leq 2|X| - 3 \text{ for all } X \subseteq V \text{ with } |X| \geq 2. \tag{1}$$

It is well-known that the sparse subsets of E form the independent sets of a matroid $\mathcal{R}(G)$ on groundset E, with rank function r.

Roughly speaking, a graph is *rigid* if, when realized as a bar-and-joint framework in 'general position', it has no non-trivial continous deformation which preserves the bar lengths. The following fundamental theorem of Laman gives a combinatorial characterization of rigidity in R^2.

Theorem 2.1. *[8] A graph $G = (V, E)$ is rigid in R^2 if and only if $r(E) = 2|V| - 3$.*

Thus the matroid $\mathcal{R}(G)$ is called the *rigidity matroid* of G. We say that a graph $G = (V, E)$ is *M-independent* if E is independent in $\mathcal{R}(G)$. We have the following formula for the rank of a set of edges of G. A *cover* of $G = (V, E)$ is a collection of subsets $\mathcal{X} = \{X_1, X_2, \dots, X_t\}$ of V, each of size at least two, such that $\{E_G(X_1), E_G(X_2), \dots, E_G(X_t)\}$ partitions E. The cover is *non-trivial* if $t \geq 2$. The *value* of the cover is equal to $val(\mathcal{X}) = \sum_{i=1}^{t}(2|X_i| - 3)$.

Theorem 2.2. *[11] Let $G = (V, E)$ be a graph. The rank of a non-empty set $E' \subseteq E$ of edges in $\mathcal{R}(G)$ is given by*

$$r(E') = \min val(\mathcal{X}),$$

where the minimum is taken over covers $\mathcal{X}$ of (V, E').

Given a graph $G = (V, E)$, a subgraph $H = (W, C)$ is said to be an *M-circuit* in G if C is a circuit (i.e. a minimal dependent set) in $\mathcal{R}(G)$. In particular, G is an *M-circuit* if E is a circuit in $\mathcal{R}(G)$. Using Theorem 2.1 we may deduce that G is an M-circuit if and only if $|E| = 2|V| - 2$ and $E - e$ is sparse for all $e \in E$. $G - e$ is rigid for all $e \in E$.

Given a matroid $\mathcal{M} = (E, \mathcal{I})$, we define a relation on E by saying that $e, f \in E$ are related if $e = f$ or if there is a circuit C in $\mathcal{M}$ with $e, f \in C$. It is well-known that this is an equivalence relation. The equivalence classes are called the *components* of $\mathcal{M}$. If $\mathcal{M}$ has at least two elements and only one component then $\mathcal{M}$ is said to be *connected*.

We say that a graph $G = (V, E)$ is *M-connected* if its rigidity matroid $\mathcal{R}(G)$ is connected. For example, complete graphs K_m with $m \geq 4$ and complete bipartite graphs $K_{3,m}$ with $m \geq 4$ are M-connected. Note that M-connected graphs are rigid [7]. The *M-components* of G are the subgraphs of G induced by the components of $\mathcal{R}(G)$. For example, the graph in Figure 1 has three M-components: the edges u_1u_2, u_2u_3, and the K_4. Note that the M-components of G are induced subgraphs. For more examples and basic properties of M-connected graphs see [7].

Globally rigid graphs were characterized in [7], see also [6].

Theorem 2.3. *[7] A graph on at least four vertices is globally rigid in R^2 if and only if it is 3-connected and M-connected.*

3 The M-connected relaxation of the anchor minimization problem

The previous discussions and Theorem 2.3 imply that the *anchor minimization problem* can be reformulated as the following purely combinatorial problem: given a graph $G = (V, E)$, find a smallest set $P \subseteq V$, $|P| \geq 3$, for which $G + K(P)$, the graph obtained from G by adding a complete graph on vertex set P, is 3-connected and M-connected.

To find an approximate solution for the anchor minimization problem we first neglect the 3-connectivity condition and consider its 'M-connected relaxation'. (Note that the complexity status of both problems is still open.)

The following lemma is easy to prove by standard matroid techniques.

Lemma 3.1. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid with components $E_1, E_2, \ldots, E_t$. Then (i) $r(\mathcal{M}) = \sum_1^t r(E_i)$, and (ii) if $r(\mathcal{M}) = \sum_1^q r(F_i)$ for some partition $F_1, F_2, \ldots, F_q$ of E and E_i is a component of $\mathcal{M}$ for some $1 \leq i \leq t$, then $E_i \subseteq F_j$ for some $1 \leq j \leq q$.*

The next lemma can be deduced from Theorem 2.2, Lemma 3.1, and the fact that the M-connected components are induced rigid subgraphs.

Lemma 3.2. *$G = (V, E)$ is M-connected if and only if $val(\mathcal{X}) \geq 2|V| - 2$ for all non-trivial covers $\mathcal{X}$ of G.*

The following key lemma characterizes the feasible solutions of the 'M-connected relaxation'.

Lemma 3.3. *Let $G = (V, E)$ be a graph, let $\mathcal{H} = \{H_1, H_2, ..., H_t\}$ be the M-components of G, and let $P \subseteq V$ with $|P| \geq 4$. Then $G + K(P)$ is M-connected if and only if*

$$2|V| - 2 \leq 2|Z| - 3 + \sum_{H_i \in \mathcal{H}: V(H_i) \cap (V - Z) \neq \emptyset} (2|V(H_i)| - 3) \tag{2}$$

holds for all $Z \subset V$ with $P \subseteq Z$, $Z \neq V$.

Proof. First suppose that $G + K(P)$ is M-connected. Since $\mathcal{H}$ is a cover of G and $P \subseteq Z$, $Z \cup \{H_i \in \mathcal{H} : V(H_i) \cap (V - Z) \neq \emptyset\}$ is a cover of $G + K(P)$. This cover is non-trivial, since $Z \neq V$. Thus (2) follows from Lemma 3.2.

To prove the other direction suppose, for a contradiction, that (2) holds but $G' = G + K(P)$ is not M-connected. Let $\mathcal{H}' = \{H'_1, H'_2, ..., H'_q\}$ denote the M-components of G'. Since complete graphs on at least four vertices are M-connected, and $|P| \geq 4$, it follows that $G'[P]$ is M-connected. Thus there is an M-component H'_1, say, for which $P \subseteq V(H'_1)$.

Now consider a graph H on vertex set V. We claim that $H = H'_j$ for some M-component H'_j of G' with $2 \leq j \leq q$, if and only if $H = H_i$ for some M-component H_i of G with $V(H_i) \cap (V - V(H'_1)) \neq \emptyset$. To see this focus on an M-component $H'_j \in \mathcal{H}'$ with $j \geq 2$. Since $P \subseteq V(H'_1)$ and $E_{G'}(H'_1) \cap E_{G'}(H'_j) = \emptyset$, it follows that $G[V(H'_j)]$ is M-connected. Thus, since G' is a supergraph of G and H'_1 is an induced subgraph, we must have $H'_j = H_i$ for some $H_i \in \mathcal{H}$ with $V(H_i) \cap (V - V(H'_1)) \neq \emptyset$. Now let $H_i \in \mathcal{H}$ with $V(H_i) \cap (V - V(H'_1)) \neq \emptyset$. Since H'_1 is M-connected in G' and M-components are edge-disjoint, it follows that $V(H_i)$ induces a maximal M-connected subgraph in G'. This proves the claim.

By using the previous claim and Lemma 3.1(i), and by applying (2) with $Z = V(H'_1)$, we obtain

$$2|V|-3 \geq r(G') = 2|V(H'_1)|-3+\sum_{H_i\in\mathcal{H}:V(H_i)\cap(V-V(H'_1))\neq\emptyset}(2|V(H_i)|-3) \geq 2|V|-2, \quad (3)$$

a contradiction. □

Let $\tilde{\mathcal{H}} = (V, \mathcal{E})$ be the hypergraph obtained from $\mathcal{H}$ by replacing each set H_i by $2|V(H_i)| - 3$ copies of $V(H_i)$, $1 \leq i \leq t$. For some $X \subseteq V$ let $e_{\tilde{\mathcal{H}}}(X)$ denote the number of hyperedges $e \in \mathcal{E}$ with $e \cap X \neq \emptyset$. By letting $S = V - Z$ in Lemma 3.3 and using the above definitions we obtain:

Lemma 3.4. *Let $G = (V, E)$ be a graph, let $\mathcal{H} = \{H_1, H_2, ..., H_t\}$ be the M-components of G, and let $P \subseteq V$ with $|P| \geq 4$. Then $G + K(P)$ is M-connected if and only if*

$$e_{\tilde{\mathcal{H}}}(S) \geq 2|S| + 1 \quad (4)$$

holds for all non-empty subsets $S \subseteq V - P$.

4 The matroid matching problem

Let $\mathcal{M}$ be a matroid on gound-set S and suppose that S is partitioned into a set A of pairs. A subset $M \subseteq A$ is a *matroid matching* if the union of the pairs in M is independent in $\mathcal{M}$. A set $P \subseteq S$ is a *co-matching* if $S - P$ is the union of pairs of a matroid matching. In the *matroid matching problem* the goal is to find a largest matroid matching, see [14, Chapter 43], or equivalently, to find a smallest co-matching. Lovász [10] has shown that this problem may require exponential time in general but can be solved polynomially if the matroid is presented by a set of vectors in some linear space.

We claim that the 'M-connected relaxation', i.e. the problem of finding a smallest set P for which $G+K(P)$ is M-connected, is a special case of the smallest co-matching problem. To see this consider the bipartite graph G^* obtained from the bipartite incidence graph of $\tilde{\mathcal{H}}$ by splitting each vertex $u_i \in V$ into two vertices u'_i, u''_i. Let U denote the color class containing the split vertices. See Figure 3. It is not difficult to see that there is a one-to-one correspondence between the subsets $P \subseteq V$ satisfying (4) and the subsets $P' \subseteq U$ for which $U - P'$ consist of pairs of split vertices and which satisfy the *strong Hall condition* in G^*. (A subset of U is said to satisfy the strong Hall condition if all non-empty subsets of U satisfy the Hall condition with strict inequality.)

We say that a hypergraph $H' = (V, \mathcal{E}')$ is a *hyperforest*) if $|\cup \mathcal{F}| \geq |\mathcal{F}| + 1$ for all $\emptyset \neq \mathcal{F} \subseteq \mathcal{E}'$. Lorea [9] proved that in a hypergraph $H = (V, \mathcal{E})$ the edge sets of the hyperforest subhypergraphs of H form a family of independent sets of a matroid on ground set $\mathcal{E}$. This matroid is the *hypergraphic matroid* $\mathcal{M}(H)$ of H. It is easy to see that a subhypergraph of H is a hyperforest if and only if the corresponding set of vertices in the bipartite incidence graph of H satisfies the strong Hall condition.

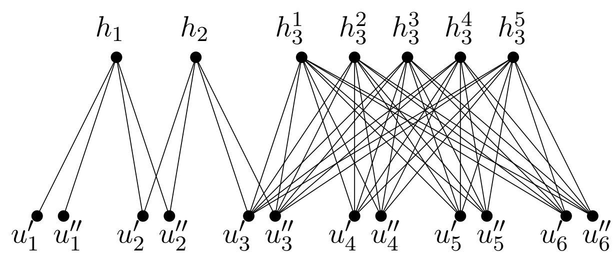

Figure 3: The bipartite graph G^* obtained from the graph G in Figure 1.

The above discussion, the construction of G^*, and Lemma 3.4 imply that the problem of finding a smallest set P for which $G + K(P)$ is M-connected can be formulated as finding a smallest co-matching in a hypergraphic matroid.

Hypergraphic matroids are known to be linear, but it is not known how to find a suitable linear representation. Thus the complexity status of the matroid matching (and co-matching) problems in hypergraphic matroids is still open. However, as we shall see, the greedy algorithm provides a good approximation.

5 The approximation algorithm for the anchor minimization problem

Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid with rank function r and suppose that S is partitioned into a set A of pairs. A subset $M \subseteq A$ is a *matroid matching* if the union of the pairs in M belongs to $\mathcal{I}$. A set $P \subseteq S$ is a *co-matching* if $S - P$ is the union of pairs of a matroid matching.

Lemma 5.1. *Let P' be an inclusionwise minimal co-matching and let P be a smallest co-matching in $\mathcal{M}$. Then $|P'| \leq 2|P|$.*

Proof. Let $X = S - P'$. The minimality of P' implies that $r(X + e) \leq r(X) + 1$ for all pairs $e \in E$ with $e \subseteq P'$. Thus $r(S) \leq |X| + |S - X|/2$, and hence $|P'| = |S - X| \leq 2|S| - 2r(S)$. On the other hand, since $S - P$ is independent, we have $|P| \geq |S| - r(S)$. This gives $|P'| \leq 2|P|$. □

Note that if $H = G + K(P)$ is M-connected then it must be 2-connected. The following simple lemmas imply that a smallest set P' for which $H + K(P')$ is 3-connected can be found efficiently.

Let $H = (V, E)$ be a graph. For some $X \subseteq V$ let $N(X)$ denote the set of neighbours of X and let $S(X) = X \cup N(X)$. We say that $X \subset V$ is *tight* if $|N(X)| = 2$ and $S(X) \neq V$.

Lemma 5.2. *Let $H = (V, E)$ be 2-connected and let $X, Y \subset V$ be distinct minimal tight sets in G. Then $X \cap Y = \emptyset$.*

Lemma 5.3. *Let $H = (V, E)$ be 2-connected and let $P' \subseteq V$. Then $H + K(P')$ is 3-connected if and only if $P' \cap X \neq \emptyset$ for all minimal tight sets X of H.*

It follows from Lemmas 5.2 and 5.3 that every inclusionwise minimal set P' for which $H + K(P')$ is 3-connected is in fact a smallest set.

Thus in the second phase of the algorithm we find a smallest set P' for which $H + K(P')$ is 3-connected. It is easy to see that $P \cup P'$ will be a feasible solution of the anchor minimization problem whose size is not more than twice the size of an optimal solution.

6 Concluding remarks

In this paper we considered the M-connected pinning problem and the anchor minimization problem. For these minimization problems we gave polynomial-time 2-approximation (resp. 3-approximation) algorithms, by using new structural results on rigid graphs and matroid theoretical tools.

The complexity status of both problems remains open. In the full version of this paper [4] we also give a randomized polynomial-time algorithm which can optimally solve the former problem and leads to a 2-approximation algorithm for the latter. We note that in a recent paper Makai [12] gave a good characterization (a minimax formula) for the maximum size of a matroid matching in a hypergraphic matroid. This indicates that the M-connected pinning problem might turn out to be polynomially solvable.

Further open problems include the unit disk graph case and the pinning extension problem for both.

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