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The parity problem of polymatroids without double circuits

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Abstract

According to the present state of the theory of the matroid parity problem, the existence of a good characterization to the size of a maximum matching depends on the behavior of certain substructures, called double circuits. In this paper we prove that if a polymatroid has no double circuits then a partition type min-max formula characterizes the size of a maximum matching. Applications to parity constrained orientations and to a rigidity problem are given.

1 Introduction

Polymatroid parity (a.k.a. matroid parity or matroid matching) is an involved notion of combinatorial optimization concerning parity. Early known special cases of it are the matching problem of graphs and the matroid intersection problem, which in fact motivated Lawler to introduce this notion. Jensen and Korte [6], and Lovász [11] have shown that in general the matroid parity problem is of exponential complexity under the independence oracle framework. However, Lovász gave a good characterization to the size of a maximum matching and also a polynomial algorithm for linearly represented matroids [10, 11]. Lovász [9], and Dress and Lovász [1] manifest that the solvability of the linear case is due to the fact that these matroids can be embedded into a matroid sharing the so-called *double circuit property*, or *DCP* for short. It was also shown that full linear, full algebraic, full graphic, and full transversal matroids are DCP matroids [1]. The subsequent results in the literature are so as to guarantee the DCP in particular matroid classes. The disadvantage of this approach is that, due to the embedding into a bigger matroid, in many cases the obtained min-max formula cannot be translated into a combinatorial form. However, the diversity and the importance of solvable special cases of the matroid parity problem force to search for techniques which yield combinatorial characterizations.

In this work we investigate the class of those polymatroids which have no non-trivial double circuits. We prove that in these polymatroids a partition type combinatorial

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formula characterizes the size of a maximum matching. Thereafter, two applications are presented. First we show that the parity constrained orientation problem of Király and Szabó [7] can be formulated as a polymatroid parity problem in such a way that the polymatroid in question has no non-trivial double circuits, yielding the partition type formula of [7]. Second, we deduce a result of Fekete [2] about adding a clique of minimum size to a graph resulting in a generic rigid graph in the plane.

To formulate our main result, some definitions are in order. We denote by $\mathbb{R}_+$ and $\mathbb{Z}_+$ the set of non-negative reals and non-negative integers resp. Let S be a finite ground set. A set-function $f : 2^S \to \mathbb{Z}$ is said to be **submodular** if

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) \tag{1}$$

holds whenever $X, Y \subseteq S$. The set-function f is said to be **non-decreasing** if $f(X) \leq f(Y)$ for every $\emptyset \neq X \subseteq Y \subseteq S$, and we say that f is **non-increasing** if $-f$ is non-decreasing. A non-decreasing submodular set-function $f : 2^S \to \mathbb{Z}_+$ with $f(\emptyset) = 0$ is called a **polymatroid function**. A vector $x \in \mathbb{Z}^S$ is said to be **even** if x_i is even for every $i \in S$. The even vectors $m \in \mathbb{Z}_+^S$ with $m(U) \leq f(U)$ for every $U \subseteq S$ are said to be the **matchings** of f. A matching m with maximum $m(S)$ is said to be a **maximum matching** of f, and we denote

$$\nu(f) = \max\{m(S)/2 : \ m \text{ is a matching of } f\}.$$

The **polymatroid parity** problem is to determine the value of $\nu(f)$ for a polymatroid function f. Our main result is as follows. The notion of a non-trivial compatible double circuit will be defined in Section 2.

Theorem 1.1. *Let $f : 2^S \to \mathbb{Z}_+$ be a polymatroid function having no non-trivial compatible double circuits. Then*

$$\nu(f) = \min \sum_{j=1}^{t} \left\lfloor \frac{f(U_j)}{2} \right\rfloor,$$

where the minimum is taken over all partitions $U_1, U_2, \ldots, U_t$ of S.

Recently, Pap [13] developed a deep combinatorial algorithm for finding a maximum matching in a polymatroid without non-trivial compatible double circuits. His result serves as a completion of Theorem 1.1, as we do not concern algorithmic aspects in this paper.

2 Preliminaries and proof of the main theorem

We recollect some important notions and results from the theory of matroids, polymatroids and matroid parity. We omit the proofs of some well-known facts, as the details can be found e.g. in [14]. If $x \in \mathbb{R}^S$ and $U \subseteq S$ then we use the notations $x(U) = \sum_{u \in U} x(u)$ and

$$x|_U = \begin{cases} x_i & \text{if } i \in U, \\ 0 & \text{if } i \notin U. \end{cases}$$

If $f : 2^S \to \mathbb{Z}_+$ is a function then we define the polyhedra

$$P(f) = \{x \in \mathbb{R}^S_+ : x(U) \le f(U) \text{ for every } U \subseteq S\}, \quad \text{and}$$
$$B(f) = \{x \in \mathbb{R}^S_+ : x(U) \le f(U) \text{ for every } U \subseteq S,\ x(S) = f(S)\}.$$

In the theory of submodular functions it is proved that if f is a polymatroid function then $P(f)$ and $B(f)$ are nonempty integer polyhedra. Recall that an even vector $m \in \mathbb{Z}^S_+$ is a matching of f if and only if $m \in P(f)$.

There is a close relation between polymatroid functions and matroids. First, if $M = (T, r)$ is a matroid and $\varphi : T \to S$ is a function then $f : 2^S \to \mathbb{Z}_+,\ X \mapsto r(\varphi^{-1}(X))$ is a polymatroid function, the **homomorphic image of** M **under** φ. Second, for any polymatroid function f it is possible to define a matroid M, the homomorphic image of which is f, in such a way that M is "most independent" in some sense. The ground set T of M is the disjoint union of sets T_i for $i \in S$ of size $|T_i| \ge f(\{i\})$. If $X \subseteq T$ then we define the vector $\chi^X \in \mathbb{Z}^S_+$ with $\chi^X_i = |X \cap T_i|$ for $i \in S$. With this notation, a set $X \subseteq T$ is defined to be independent in M if $\chi^X \in P(f)$. It is routine to prove that M is indeed a matroid with rank function

$$r(X) = \min_{Y \subseteq X}(|Y| + f(\varphi(X - Y))),$$

where $\varphi : T \to S$ maps t to i if $t \in T_i$. This M is called a **prematroid** of f. Note that a prematroid M is uniquely determined by f and by the sizes $|T_i|$, $i \in S$. If M is a matroid with rank function r then the prematroids of r are the parallel extensions of M. If we consider a prematroid M then we tacitly assume that $M = (T, r)$ and that the function $\varphi : T \to S$ is given with $t \mapsto i$ if $t \in T_i$.

If f is a polymatroid function and $x \in \mathbb{Z}^S_+$ then we define the **rank** of x as

$$r_f(x) = \min_{U \subseteq S}(x(U) + f(S - U)). \tag{2}$$

$r_f(x) = x(S)$ if and only if $x \in P(f)$. Besides, if $M = (T, r)$ is a prematroid of f and $X \subseteq T$ then

$$r_f(\chi^X) = r(X). \tag{3}$$

The **span** of $x \in \mathbb{Z}^S_+$ is defined by $\mathrm{sp}_f(x) = \{i \in S : r_f(x + \chi_i) = r_f(x)\}$. If M is a prematroid of f and $X \subseteq T$ then $\mathrm{sp}_f(\chi^X) = \{i \in S : T_i \subseteq \mathrm{sp}_M(X)\}$.

Let $M = (T, r)$ be a matroid. A set $C \subseteq T$ is said to be a **circuit** if $r(C - x) = r(C) = |C| - 1$ for every $x \in C$. A set $D \subseteq T$ is a **double circuit** if $r(D - x) = r(D) = |D| - 2$ for every $x \in D$. If D is a double circuit then the dual of $M|D$ is a matroid of rank 2 without loops, that is a line, showing that there exists a **principal partition** $D = D_1 \dot\cup D_2 \dot\cup \ldots \dot\cup D_d$, $d \ge 2$, such that the circuits of D are exactly the sets of the form $D - D_i$, $1 \le i \le d$. We say that D is **non-trivial** if $d \ge 3$, and **trivial** otherwise. A trivial double circuit is simply the direct sum of two circuits.

Analogously, we define circuits and double circuits of the polymatroid function $f : 2^S \to \mathbb{Z}_+$. For a vector $x \in \mathbb{R}^S_+$ let $\mathrm{supp}(x) = \{i \in S : x_i > 0\}$. A vector $c \in \mathbb{Z}^S_+$ is a **circuit** of f if $r_f(c - \chi_i) = r_f(c) = c(S) - 1$ for every $i \in \mathrm{supp}(c)$. A vector $w \in \mathbb{Z}^S_+$ is a **double circuit** of f if $r_f(w - \chi_i) = r_f(w) = w(S) - 2$ for every $i \in \mathrm{supp}(w)$.

The exact relation between matroidal and polymatroidal double circuits is as follows, which can be seen easily from (3).

Lemma 2.1. *Let M be a prematroid of f, $D \subseteq T$ and $\chi^D = w$. Then D is a double circuit of M if and only if w is a double circuit of f.*

Let M be a prematroid of f and w be a double circuit of f such that there is a set $D \subseteq T$ with $\chi^D = w$. By Lemma 2.1, D is a double circuit of M, thus it has a principal partition $D = D_1 \dot\cup D_2 \dot\cup \dots \dot\cup D_{d'}$. We define the principal partition of w as follows. Due to the structure of prematroids it is easy to check that $\mathrm{supp}(w)$ has a partition $W_0 \dot\cup W_1 \dot\cup \dots \dot\cup W_d$ with the property that each set D_j is either a singleton belonging to some T_i with $w_i \geq 2$ and $i \in W_0$, or is equal to $D \cap \bigcup_{i \in W_h} T_i$ for some $1 \leq h \leq d$. Note that a partition $W_0 \dot\cup W_1 \dot\cup \dots \dot\cup W_d$ of $\mathrm{supp}(w)$ is the principal partition of w if and only if $w - \chi_i$ is a circuit of f and $w_i \geq 2$ whenever $i \in W_0$, moreover, $w|_{W-W_i}$ is a circuit of f for each $1 \leq i \leq d$. A double circuit w is said to be **compatible** if $W_0 = \emptyset$, and it is **non-trivial** if D is non-trivial.

We shortly mention what is the double circuit property (DCP). If $M = (T, r)$ is a prematroid of the polymatroid function f and $Z \subseteq T$ then $\varphi(M/Z)$ is called a **contraction** of f. A polymatroid function f is said to have the DCP if whenever w is a non-trivial compatible double circuit in a contraction f' of f with principal partition $W_1 \dot\cup \dots \dot\cup W_d$ then $f'(\bigcap_{1 \leq i \leq d} \mathrm{sp}(w|_{W-W_i})) > 0$, (Dress, Lovász [1]). A polymatroid function without non-trivial compatible double circuits has not necessarily the DCP, as its contractions may have many non-trivial compatible double circuits.

Note that every polymatroid function has double circuits, say $(f(\{i\}) + 2)\chi_i$ for some $i \in S$. However, these are not compatible, because $W_0 = \{i\}$.

Lemma 2.2. *If $w \in \mathbb{Z}_+^S$ is a double circuit of the polymatroid function $f : 2^S \to \mathbb{Z}_+$ with principal partition $W = W_0 \dot\cup W_1 \dot\cup \dots \dot\cup W_d$ then $f(W) = w(W) - 2$ and $f(W - W_i) = w(W - W_i) - 1$ for $1 \leq i \leq d$.*

Proof. We prove that if $x \in \mathbb{Z}_+^S$ is a vector with the property that $r_f(x) = r_f(x - \chi_i)$ for all $i \in \mathrm{supp}(x)$ then $f(\mathrm{supp}(x)) = r_f(x)$. By definition, $r_f(x) = x(U) + f(S - U)$ for some $U \subseteq S$. Note that $r_f(x - \chi_i) \leq (x - \chi_i)(U) + f(S - U) = r_f(x) - 1$ for all $i \in \mathrm{supp}(x) \cap U$. Thus $\mathrm{supp}(x) \cap U = \emptyset$. Concluding, $f(S - U) = r_f(x) \leq f(\mathrm{supp}(x)) \leq f(S - U)$, because f is non-decreasing. If x is a circuit or a double circuit then $r_f(x) = r_f(x - \chi_i)$ for all $i \in \mathrm{supp}(x)$, so we are done. □

Next we explore how two polymatroid operations alter double circuits.

Translation

If $f : 2^S \to \mathbb{Z}_+$ is a function and $n \in \mathbb{Z}^S$ then define $f + n : 2^S \to \mathbb{Z}_+$ by $X \mapsto f(X) + n(X)$. If f is a polymatroid function and $n \in \mathbb{Z}_+^S$ then $f + n$ is clearly a polymatroid function, too.

Claim 2.3. *If $n \in \mathbb{Z}^S$ and f and $f + n$ are polymatroid functions then a vector w is a double circuit of f with $W = \mathrm{supp}(w)$ if and only if $w + n|_W$ is a double circuit of $f + n$. In this case their principal partition coincide.*

Proof. Let $y \in \mathbb{Z}_+^S$ and assume that $y' := y + n|_{\mathrm{supp}(y)} \in \mathbb{Z}_+^S$. Note that in (2) we can always choose U with $U \cup \mathrm{supp}(y) = S$, thus $r_{f+n}(y') - y'(S) = r_f(y) - y(S)$. Hence by symmetry, it is enough to prove that if w is a double circuit of f with support W then $\mathrm{supp}(w + n|_W) = W$. Otherwise $w_i + n_i \le 0$ for some $i \in W$, thus by Lemma 2.2 we would have

$$w(W - i) \ge w(W) + n_i = f(W) + n_i + 2 \ge f(W - i) + 2,$$

which is impossible. □

Deletion or upper bound

Let $u \in \mathbb{Z}_+^S$ be a bound vector and define $f \backslash u = \varphi(r_{M|Z})$ where M is a prematroid of f and $Z \subseteq T$ with $\chi^Z = u$. The matroid union theorem asserts that $(f\backslash u)(X) = \min_{Y \subseteq X}(u(Y) + f(X - Y))$ for $X \subseteq S$. If M is a matroid with rank function r then $r\backslash u$ is the rank function of $M|\mathrm{supp}(u)$.

Claim 2.4. *Let $u \in \mathbb{Z}_+^S$. If $w \in \mathbb{Z}_+^S$ is a double circuit of $f' := f\backslash u$ then w is either a double circuit of f with the same principal partition, or trivial, or non-compatible.*

Proof. Let $M = (T, r)$ be a prematroid of f and let $Z \subseteq T$ with $\chi^Z = u$. If $w \le \chi^Z$ then w is a double circuit of f with the same principal partition by Lemma 2.1. Observe that $w_i \le f'(\{i\}) + 2$ and $f'(\{i\}) \le u_i$ for every $i \in S$. Thus if $w \not\le \chi^Z$ then there exists an $i \in S$ such that $w_i - f'(\{i\}) \in \{1, 2\}$. If $w_i = f'(\{i\}) + 2$ then $r_{f'}(w_i \chi_i) = w_i - 2$, thus $W_0 = \mathrm{supp}(w) = \{i\}$, implying that w is non-compatible. If $w_i = f'(\{i\}) + 1$ then $w_i \chi_i$ is a circuit of f' thus if $W_0 \ne \emptyset$ then w is non-compatible, and if $W_0 = \emptyset$ then w is trivial. □

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. It is easy to see that $\nu(f) \le \sum_{j=1}^t \left\lfloor \frac{f(U_j)}{2} \right\rfloor$ holds for every partition $U_1, U_2, \dots, U_t$ of S. For the other direction we use a deep and important result of Lovász on 2-polymatroids, which can be translated to polymatroids as follows.

Theorem 2.5 (Lovász [9])**.** *If $f : 2^S \to \mathbb{Z}_+$ is a polymatroid function then at least one of the following cases holds.*

1. *$f(S) = 2\nu(f) + 1$.*
2. *$S = S_1 \dot\cup S_2$ where $S_i \ne \emptyset$ and $\nu(f) = \nu(f|_{2^{S_1}}) + \nu(f|_{2^{S_2}})$.*
3. *There exists an $i \in S$ such that for all maximum matchings m we have $i \in \mathrm{sp}_f(m)$.*
4. *There exists a certain substructure, called ν-*double flower* in f, which we do not define here, but which always contains a non-trivial compatible double circuit.*

We argue by induction on the pair $(|S|, |K(f)|)$, where

$$K(f) = \{s \in S : s \in \mathrm{sp}_f(m) \text{ for all maximum matchings } m \text{ of } f\}.$$

If $S = \emptyset$ then the statement is trivial. If $K(f) = \emptyset$ then either *1.* or *2.* holds in Theorem 2.5. If *1.* holds then the trivial partition will do, while if *2.* holds then we can use our induction hypothesis applied to $f|_{2^{S_1}}$ and $f|_{2^{S_2}}$.

Next, let $K(f) \neq \emptyset$. We prove that if m is a maximum matching of $f + 2\chi_s$ then $m(s) \geq 2$. Indeed, assume that $m(s) = 0$. As m is a maximum matching, there exists a set $s \in U \subseteq S$ with $m(U) \geq (f + 2\chi_s)(U) - 1$. Thus $m(U - s) = m(U) \geq (f+2\chi_s)(U) - 1 \geq f(U-s)+1$, which is a contradiction. It is also clear that $m+2\chi_s$ is a matching of $f+2\chi_s$ for all matchings m of f. Therefore, m is a maximum matching of f if and only if $m + 2\chi_s$ is a maximum matching of $f + 2\chi_s$.

Let $s \in K(f)$. Clearly, $\nu(f) \leq \nu(f + \chi_s) \leq \nu(f + 2\chi_s) = \nu(f) + 1$ and we claim that in fact, $\nu(f + \chi_s) = \nu(f)$ holds. Indeed, if $\nu(f + \chi_s) = \nu(f) + 1$ and m is a maximum matching of $f + \chi_s$ then m is also a maximum matching of $f + 2\chi_s$, thus $m(s) \geq 2$. Then $m - 2\chi_s$ is a maximum matching of f and, as $s \in \mathrm{sp}_f(m - 2\chi_s)$, there exists a set $s \in U \subseteq S$ with $(m - 2\chi_s)(U) = f(U)$. This implies $m(U) = f(U) + 2$, contradicting to that m is a matching of $f + \chi_s$.

So if m is a maximum matching of f then m is a maximum matching of $f + \chi_s$, too, and clearly, $\mathrm{sp}_f(m) = \mathrm{sp}_{f+\chi_s}(m) - s$. Thus we have $K(f + \chi_s) \subseteq K(f) - s$. By Lemma 2.3, $f + \chi_s$ has no non-trivial compatible double circuits, so we can apply induction to $f + \chi_s$. This gives a partition $U_1, U_2, \ldots, U_t$ of S such that

$$\nu(f + \chi_s) = \sum_{j=1}^{t} \left\lfloor \frac{(f + \chi_s)(U_j)}{2} \right\rfloor .$$

But then,

$$\nu(f) = \nu(f + \chi_s) = \sum_{j=1}^{t} \left\lfloor \frac{(f + \chi_s)(U_j)}{2} \right\rfloor \geq \sum_{j=1}^{t} \left\lfloor \frac{f(U_j)}{2} \right\rfloor .$$

□

3 A parity constrained orientation theorem

Frank, Jordán and Szigeti [4] proved that the existence of a k-rooted-connected orientation with prescribed parity of in-degrees can be characterized by a partition type condition. Recently, Király and Szabó [7] proved that the connectivity requirement in this parity constrained orientation problem can be given by a more general non-negative intersecting supermodular function. It is well-known that all these problems can be formalized as polymatroid parity problems. In this section we show that it is possible to formalize the problem of Király and Szabó in such a way that the arising polymatroid function has no non-trivial double circuits. So Theorem 1.1 can be applied to yield the result in [7].

$H = (V, \mathcal{E})$ is called a **hypergraph** if V is a finite set and $\emptyset \notin \mathcal{E}$ is a collection of multisets of V, the set of **hyperedges** of H. If in every hyperedge $h \in \mathcal{E}$ we designate a vertex $v \in h$ as the **head vertex** then we get a **directed hypergraph** $D = (V, \mathcal{A})$, called an **orientation** of H. For a set $X \subseteq V$, let $\delta_D(X)$ denote the number of directed hyperedges **entering** X, that is the set of hyperedges with head in X and at least one vertex in $V - X$.

Let $p : 2^V \to \mathbb{Z}_+$ be a function with $p(\emptyset) = p(V) = 0$. An orientation D of a hypergraph $H = (V, \mathcal{E})$ **covers** p if $\delta_D(X) \geq p(X)$ for every $X \subseteq V$. In a *connectivity orientation problem* the question is the existence of an orientation covering p. When we are talking about *parity constrained orientations*, we are looking for connectivity orientations such that the out-degree at each vertex is of prescribed parity. Now define $b : 2^V \to \mathbb{Z}$ by

$$b(X) = \sum_{h \in \mathcal{E}} h(X) - |\mathcal{E}[X]| - p(X) \quad \text{for} \quad X \subseteq V, \tag{4}$$

where $\mathcal{E}[X]$ denotes the set of hyperedges $h \in \mathcal{E}$ with $h \cap (V - X) = \emptyset$, and h equivalently stands for the hyperedge and its multiplicity function. It is clear that if $x : V \to \mathbb{Z}_+$ is the out-degree vector of an orientation covering p then $x \in B(b)$. The contrary is also easy to prove (see e.g. [5]).

Lemma 3.1. *Let $H = (V, \mathcal{E})$ be a hypergraph, $p : 2^V \to \mathbb{Z}_+$ be a function with $p(\emptyset) = p(V) = 0$, and $x : V \to \mathbb{Z}_+$. Then H has an orientation covering p such that the out-degree of each vertex $v \in V$ is $x(v)$ if and only if $x \in B(b)$.*

The function $b : 2^V \to \mathbb{Z}$ is said to be **intersecting submodular** if (1) holds whenever $X \cap Y \neq \emptyset$. Similarly, $p : 2^V \to \mathbb{Z}$ is **intersecting supermodular** if $-p$ is intersecting submodular. If $b : 2^V \to \mathbb{Z}_+$ is a non-negative, non-decreasing intersecting submodular function then we can define a polymatroid function $\widehat{b} : 2^V \to \mathbb{Z}_+$ by

$$\widehat{b}(X) = \min \left\{ \sum_{i=1}^{t} b(X_i) \ : \ X_1 \dot\cup X_2 \dot\cup \ldots \dot\cup X_t = X \right\} \text{ for } X \subseteq V,$$

which is called the **Dilworth truncation** of b.

Lemma 3.2. *If $p : 2^V \to \mathbb{Z}_+$ is intersecting supermodular with $p(V) = 0$ then p is non-increasing.*

Proof. If $\emptyset \neq X \subseteq Y \subseteq V$ then $p(Y) \leq p(Y) + p((V - Y) \cup X) \leq p(X) + p(V) = p(X)$. □

The following theorem can be proved using basic properties of polymatroids (Frank [3]).

Theorem 3.3. *Let $H = (V, \mathcal{E})$ be a hypergraph and $p : 2^V \to \mathbb{Z}_+$ be an intersecting supermodular function with $p(\emptyset) = p(V) = 0$. Define b as in (4). Then H has an orientation covering p if and only if*

$$b(V) \leq \sum_{j=1}^{t} b(U_j) \tag{5}$$

holds for every partition $U_1, U_2, \ldots, U_t$ of V.

Remark 3.4. It follows that if H has an orientation covering p then b is non-negative. Indeed, if $\emptyset \subseteq U \subseteq V$ then $b(U) + b(V - U) \geq b(V)$ by Theorem 3.3, implying that $b(U) \geq b(V) - b(V - U) \geq 0$. In the last inequality we used that b is non-decreasing by Lemma 3.2. So we can define the Dilworth-truncation $\widehat{b}$. Thus (5) is equivalent to that $\widehat{b}(V) = b(V)$ and hence that $B(\widehat{b}) = B(b)$.

Let $H = (V, \mathcal{E})$ be a hypergraph and $T \subseteq V$. Our goal is to find an orientation of H covering p, where the set of odd out-degree vertices is as close as possible to T.

Theorem 3.5 (Király and Szabó [7]). *Let $H = (V, \mathcal{E})$ be a hypergraph, $T \subseteq V$, $p : 2^V \to \mathbb{Z}_+$ be an intersecting supermodular function with $p(\emptyset) = p(V) = 0$, and assume that H has an orientation covering p. Define b as in* (4). *For an orientation D of H let $Y_D \subseteq V$ denote the set of odd out-degree vertices in D. Then*

$$\min \{|T \triangle Y_D| \ : \ D \text{ is an orientation of } H \text{ covering } p\} =$$
$$\max \left\{ b(V) - \sum_{j=1}^{t} b(U_j) + |\{j : b(U_j) \not\equiv |T \cap U_j| \bmod 2\}| \right\}, \quad (6)$$

where the maximum is taken over partitions $U_1, U_2, \ldots, U_t$ of V.

Proof. For every $v \in T$ add a loop $2\chi_v$ to $\mathcal{E}$, resulting in the hypergraph $H' = (V, \mathcal{E}')$. Define b' as in (4), w.r.t. H'. As there is a straightforward bijection between the orientations of H and H', we have

$$\min\{|T \triangle Y_D| : D \text{ is an ori. of } H \text{ cov. } p\} = \min\{|Y_{D'}| : D' \text{ is an ori. of } H' \text{ cov. } p\}.$$

Furthermore,

$$b(V) - \sum_{j=1}^{t} b(U_j) + |\{j : b(U_j) \not\equiv |T \cap U_j| \bmod 2\}| = b'(V) - \sum_{j=1}^{t} b'(U_j) + |\{j : b'(U_j) \text{ odd}\}| .$$

Thus we can assume that $T = \emptyset$.

By Lemma 3.1, the integer vectors of $B(b)$ are exactly the out-degree vectors of the orientations of H covering p. Thus the $\geq$ direction is easy to check. Now we prove the other direction. As H has an orientation covering p, Remark 3.4 implies that we can define the polymatroid function $f = \widehat{b}$. We state that it is enough to prove that

$$\nu(f) = \min \sum_{i=1}^{s} \left\lfloor \frac{f(V_i)}{2} \right\rfloor , \quad (7)$$

where the minimum is taken over all partitions $V_1, V_2, \ldots, V_s$ of V. Indeed, using the definition of the Dilworth-truncation and that $b(V) = f(V)$ by Theorem 3.3, we get

$$\min\{|Y_D| : D \text{ is an ori. of } H \text{ covering } p\} = f(V) - 2\nu(f) =$$
$$= b(V) - \min \left\{ \sum_{i=1}^{s} f(V_i) - |\{i : f(V_i) \text{ is odd}\}| \ : \ V_1, \ldots, V_s \text{ partitions } V \right\} \leq$$

$$\leq b(V) - \min\left\{\sum_{j=1}^{t} b(U_j) - |\{j : b(U_j) \text{ is odd}\}| \;:\; U_1,\ldots,U_t \text{ partitions } V\right\}.$$

Thus by Theorem 1.1 it is enough to prove that f has no non-trivial compatible double circuits. The next lemma, with the choice $\alpha(v) = \sum_{h\in\mathcal{E}} h(v)$ for $v \in V$, does the job. □

Lemma 3.6. *Let $H = (V, \mathcal{E})$ be a hypergraph, $\alpha : V \to \mathbb{Z}_+$ and $p : 2^V \to \mathbb{Z}_+$ an intersecting supermodular function with $p(\emptyset) = 0$. Suppose moreover that $b : 2^V \to \mathbb{Z}$ defined by (4) is non-negative and non-decreasing. Then the polymatroid function $f := \widehat{b}$ has no non-trivial compatible double circuits.*

Proof. Assume that $w : V \to \mathbb{Z}_+$ is a non-trivial compatible double circuit of f with principal partition $W = W_1 \dot\cup W_2 \dot\cup \ldots \dot\cup W_d$. Clearly,

$$b(W) \geq w(W) - 2.$$

Let $1 \leq i < j \leq d$ and $Z = W - W_i$. As $w|_Z$ is a circuit of f, Lemma 2.2 yields that $w(Z) - 1 = f(Z) = \min \sum\{b(X_i) \;:\; X_1,\ldots,X_k \text{ partitions } Z\}$. However, if a non-trivial partition with $k \geq 2$ gave equality here, then we would have $f(Z) = \sum b(X_i) \geq \sum f(X_i) \geq \sum w(X_i) = w(Z) = f(Z) + 1$, using that $w|_{X_i} \in P(f)$. Thus

$$w(W - W_i) - 1 = b(W - W_i) \quad \text{and similarly,} \quad w(W - W_j) - 1 = b(W - W_j).$$

By applying intersecting submodularity to $W - W_i$ and $W - W_j$, and using that $w|_{W-W_i-W_j} \in P(f)$, we get

$$0 \geq b(W) - b(W - W_i) - b(W - W_j) + b(W - W_i - W_j) \geq$$
$$\geq (w(W) - 2) - (w(W - W_i) - 1) - (w(W - W_j) - 1) + w(W - W_i - W_j) = 0,$$

so equality holds throughout. As a corollary, each hyperedge $e \in \mathcal{E}[W]$ is spanned by one of the W_i's, and

$$\binom{d-1}{2}(b(W)+2) = \binom{d-1}{2} w(W) = \sum_{1\leq i<j\leq d} w(W - W_i - W_j) = \sum_{1\leq i<j\leq d} b(W - W_i - W_j). \tag{8}$$

On the other hand,

$$\binom{d-1}{2}\alpha(W) = \sum_{1\leq i<j\leq d} \alpha(W - W_i - W_j),$$

as α is modular, and

$$\binom{d-1}{2} p(W) \leq \sum_{1\leq i<j\leq d} p(W - W_i - W_j),$$

as p is non-negative and non-increasing. Finally,

$$\binom{d-1}{2}|\mathcal{E}[W]| = \binom{d-1}{2}\sum_{i=1}^{d} |\mathcal{E}[W_i]| = \sum_{1\leq i<j\leq d} |\mathcal{E}[W - W_i - W_j]|.$$

By the definition of b, the last 3 equalities together contradict (8). □

The next corollary of Theorem 3.5 is formally very similar to the characterization in Theorem 3.3.

Theorem 3.7. *Let $H = (V, \mathcal{E})$ be a hypergraph and $p : 2^V \to \mathbb{Z}_+$ an intersecting supermodular function with $p(\emptyset) = p(V) = 0$. Then a set $T \subseteq V$ arises as the set of odd out-degree vertices in an orientation of H covering p if and only if*

$$b(V) \leq \sum_{j=1}^{t} b(U_j) - |\{j : b(U_j) \not\equiv |T \cap U_j| \bmod 2\}| \tag{9}$$

holds for every partition $U_1, U_2, \ldots, U_t$ of V.

Let us give an example showing that polymatroids without non-trivial compatible double circuits are not closed under contraction. Let $V = \{v_1, v_2, v_3, v_4\}$, $\mathcal{E} = \{v_1v_i,\ v_iv_i : i \in \{2, 3, 4\}\}$, $p(\{v_1\}) = 1$ and $p(U) = 0$ for all other sets $U \subseteq V$. Then by Remark 3.4, b is non-decreasing and non-negative, thus by Lemma 3.6, $\widehat{b}$ has no non-trivial compatible double circuits. However, the polymatroid obtained from $\widehat{b}$ by contracting an element from the preimage of v_1 in the prematroid of $\widehat{b}$ has the non-trivial compatible double circuit $(1, 2, 2, 2)$, with the trivial principal partition.

4 A planar rigidity problem

If $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^2$ is an embedding into the Euclidean plane then (G, p) is said to be a **framework**. We think of the edges of G as rigid bars with flexible joins at the vertices. An **infinitesimal motion** means an assignment of a speed $x(v) \in \mathbb{R}^2$ to each vertex $v \in V$ such that the bar lengths are preserved, that is $(p(u) - p(v)) \perp (x(u) - x(v))$. The framework (G, p) is called **rigid** if all infinitesimal motions of (G, p) correspond to isometries of $\mathbb{R}^2$. The question of finding a vertex set $Z \subseteq V$ of minimum size such that $(G + K_Z, p)$ is rigid was solved by Lovász in his seminal paper [9] about matroid parity. We say that $G = (V, E)$ is **generic rigid** if all frameworks (G, p) with *algebraically independent* coordinates p are rigid. The problem of finding a vertex set $Z \subseteq V$ of minimum size such that $G + K_Z$ is generic rigid is left open by [9], and it was solved recently by Fekete [2]. For more on the 2-dimensional rigidity see Laman [8] and Lovász and Yemini [12].

The setup of [2] puts the problem into a bit more general setting. Let $G = (V, E)$ be a graph, and for $l \in \{2, 3\}$ let $M_{2,l}$ be the matroid on ground set E such that $F \subseteq E$ is independent in $M_{2,l}$ if and only if $|F[X]| \leq 2|X| - l$ for all $X \subseteq V$, $|X| \geq 2$. It can be proved that M is really a matroid. For clarity, $M_{2,2}$ is two times the cycle matroid of G, and so G has two edge-disjoint spanning trees if and only if $r_{2,2}(E) = 2|V| - 2$. As $M_{2,3}$ is the rigidity matroid of G, the graph G is generic rigid if and only if $r_{2,3}(E) = 2|V| - 3$. For $Z \subseteq V$ let $K_Z = (Z, E_Z)$ be the graph with vertex set Z having $4 - l$ parallel edges between any two vertices of Z. Our goal is to find a set $Z \subseteq V$ of minimum size such that $E + E_Z$ has rank $2|V| - l$ in $M_{2,l}$.

We assume that E is independent in $M_{2,l}$, since if E is replaced by one of its bases then the solution set does not change. Fekete [2] proved the following lemma. For $X \subseteq V$ let $e(X)$ denote the number of edges having at least one end vertex in X.

Lemma 4.1 ([2]). *Let $l \in \{2,3\}$. Assume that E is independent in $M_{2,l}$ and that $r_{2,l}(E) < 2|V| - l$. Let $Z \subseteq V$. Then $r(E + E_Z) = 2|V| - l$ if and only if $e(Y) \geq 2|Y|$ for every $Y \subseteq V - Z$.*

Therefore, the goal is to find a set $Z \subseteq V$ of minimum size such that $e(Y) \geq 2|Y|$ for every $Y \subseteq V - Z$. Let $f : 2^V \to \mathbb{Z}_+$ be the polymatroid function with $f(X) = \min_{Y \subseteq X} 2|Y| + e(X - Y)$ for $X \subseteq V$. That is, f is obtained from the polymatroid function $X \mapsto e(X)$ by deleting with the vector $(2, 2, \ldots, 2)$. Hence for $l = 2$ the value $|V| - \nu(f)$ is just the minimum size of a set Z whose contraction results in a graph with two edge-disjoint spanning trees, and for $l = 3$ it is the minimum size of a set Z such that $G + K_Z$ is generic rigid. In [9] the computation of $\nu(f)$ is reduced to the matching problem of graphs, yielding a partition type characterization. This characterization follows from the previous results of this paper, too. First, by Lemma 3.6 with the choice $p = 0$, the polymatroid function $X \mapsto e(X)$ has no non-trivial compatible double circuits. As f is obtained from $X \mapsto e(X)$ by deletion, Claim 2.4 yields that nor f has. Thus,

$$\nu(f) = \min \sum_{j=1}^{t} \left\lfloor \frac{f(U_j)}{2} \right\rfloor,$$

where the minimum is taken over all partitions $U_1, U_2, \ldots, U_t$ of V. By the definition of f, we get the following.

Theorem 4.2 (Fekete, [2]). *Let $l \in \{2,3\}$. Assume that E is independent in $M_{2,l}$ and that $r_{2,l}(E) < 2|V| - l$. Then the minimum size of a set $Z \subseteq V$ such that $r(E + E_Z) = 2|V| - l$ is $|V| - \nu(f)$, where*

$$\nu(f) = \min \left| V - \bigcup_{j=1}^{t} U_j \right| + \sum_{j=1}^{t} \left\lfloor \frac{e(U_j)}{2} \right\rfloor,$$

where the minimum is taken over all subpartitions $U_1, U_2, \ldots, U_t$ of V.

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