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Rooted k-connections in digraphs

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Abstract

The problem of computing a minimum cost subgraph $D' = (V, A')$ of a directed graph $D = (V, A)$ so that D' contains k edge-disjoint paths from a specified root $r \in V$ to every other node in V is known to be nicely solvable since it can be formulated as a matroid intersection problem. A corresponding problem when openly disjoint paths are requested rather than edge-disjoint was solved in [12] with the help of submodular flows. Here we show that the use of submodular flows is actually avoidable and even a common generalization of the two rooted k-connection problems is a matroid intersection problem. We also provide a polyhedral description using supermodular functions on bi-sets and this approach enables us to handle more general rooted k-connection problems. For example, with the help of a submodular flow algorithm the following restricted version of the generalized Steiner-network problem is solvable in polynomial time: given a digraph $D = (V, A)$ with a root-node r, a terminal set T, and a cost function $c : A \to \mathbf{R}_+$ so that each edge of positive cost has its head in T, find a subgraph $D' = (V, A')$ of D of minimum cost so that there are k openly disjoint paths in D' from r to every node in T.

1 Introduction

Let $D = (V, A)$ be a directed graph. For two specified nodes s and t of D, a (directed) path from s to t is called an st-path. Let $\lambda(s, t; D)$ and $\kappa(s, t; D)$ denote the maximum number of edge-disjoint, respectively, openly disjoint, st paths. Two paths from s to t are called **openly disjoint** if their nodes in common are exactly s and t. In particular, k parallel edges from s to t form k openly disjoint paths.

It is well-known that $\lambda(s, t)$ can be computed via a max-flow min-cut algorithm and even more, given a nonnegative cost function on A, the cheapest set of k edge-disjoint paths from s to t can also be computed in strongly polynomial time with the help of a min-cost flow algorithm. There is a well-known and easy node-splitting technique (described, for example, in [8]) to reduce the computation of k openly disjoint st-paths to that of k edge-disjoint st-paths.

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Let r be a specified node of D called a root. We will throughout assume that no edge of D enters r. The digraph is called **rooted k-edge-connected** (resp., **rooted k-node-connected** or in short, **rooted k-connected**) if $\lambda(r, v; D) \geq k$ (resp., $\kappa(r, v; D) \geq k$) holds for every $v \in V - r$. Suppose that D is endowed with a non-negative cost-function $c : A \to \mathbf{R}_+$.

We are interested in the **rooted k-edge-connection** and the **rooted k-node-connection** problems which consist of finding a cheapest subgraph $D' = (V, A')$ of D so that

$$D' \text{ is rooted } k\text{-edge-connected from } r \tag{1}$$

and, respectively,

$$D' \text{ is rooted } k\text{-node-connected from } r. \tag{2}$$

When $k = 1$ the two problems coincide and it amounts to finding a minimum cost spanning arborescence. This was solved by Yong-Jin Chu and Tseng-Hong Liu [1] and by D.R. Fulkerson [9]. For higher k, two approaches have been known for solving (1). The first one consists of showing that there are two matroids on the edge set of D so that their common bases are exactly the minimal rooted k-edge-connected subgraphs of D. Namely, a subset F of edges is a basis of M_1 if it is the union of k edge-disjoint spanning trees, that is, M_1 is the sum of k copies of the circuit matroid of the underlying undirected graph of D, while M_2 is a partition matroid in which a set is a basis if it contains no edge entering r and contains exactly k edges entering each node distinct from r.

Having this observation, one may apply any algorithm for finding a minimum cost common base of two matroids. (The first one was developed by J. Edmonds [6]. For an efficient realization of this approach, see Gabow's work [17]). For later generalizations, we remark that this is not the only way to formulate the rooted k-edge-connection problem as matroid intersection. For example, one may replace M_1 by a matroid M_1' that arises from M_1 by declaring each edge of tail r to be a cut-element of the matroid. By a theorem of Nash-Williams [21], this is equivalent to requiring that a subset F of edges is independent in M_1' if every non-empty subset Z of $V - r$ induces at most $k(|Z| - 1)$ elements of F. Although this matroid is more free than M_1, it is still true (and will be proved in a more general context) that the minimal rooted k-edge-connected subgraphs of D are exactly the common indepent sets of $k(|V| - 1)$ elements of M_1' and M_2.

The second approach uses a more general framework. In handling edge-connectivity optimization problems, it is rather typical that a general result on covering supermodular functions by directed graphs is in the background. For the rooted k-edge-connection problem such a framework can be formulated as follows. A set-function $p : 2^V \to \mathbf{Z}$ is said to satisfy the supermodular inequality on subsets $X, Y \subseteq V$ if

$$p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y). \tag{3}$$

If this holds whenever $X \cap Y = \emptyset$, then p is called **intersecting supermodular**. If (3) is required only for subsets with $p(X) > 0$, $p(Y) > 0$, and $X \cap Y = \emptyset$, then p is **positively intersecting supermodular**.

A typical way to create a positively intersecting supermodular function is to take the 'nonnegative part' of an intersecting supermodular one with possible negative values which means replacing each negative value by zero. Example shows, however, that not every non-negative, positively intersecting supermodular function arises this way.

A digraph $D = (V, A)$ (or a function $x : A \to \mathbf{Z}_+ \cup \{\infty\}$ on its edge set) is said to **cover** p if $\varrho_D(X) \geq p(X)$ (resp., $\varrho_x(X) \geq p(X)$) for every subset $X \subseteq V$ where $\varrho_D(X)$ denotes the number of edges of D entering X while $\varrho_x(X) := \sum[x(e) : e \in A, e$ enters $X]$. The general problem consists of finding a cheapest subgraph of D covering a positively intersecting supermodular function. When p is defined to be identically k on the nonempty subsets of $V - r$ and zero elsewhere, we are back at the rooted k-edge-connection problem. To formulate the result on covering intersecting supermodular functions, let $g : A \to \mathbf{Z}_+ \cup \{\infty\}$ be a non-negative upper bound on the edges of D that covers p.

Theorem 1.1 ([10, 14]). *If p is a positively intersecting supermodular set-function, the linear system*

$$\varrho_x(Z) \geq p(Z) \text{ for every } Z \subset V,\ 0 \leq x \leq g \tag{4}$$

described for $x \in \mathbf{R}^A$ is totally dual integral (TDI). *In particular, the linear programming problem*

$$\min\{cx : x \text{ satisfies } (4)\} \tag{5}$$

has an integer-valued optimum solution and so has its linear programming dual provided c is integer-valued.

Theorem 1.2 (A. Schrijver [22]). *If p is an intersecting supermodular set-function, the polyhedron R defined by* (4) *is a submodular flow polyhedron.*

Note that the statement in Schrijver's theorem is not known to be true for the larger class of positively intersecting supermodular functions. Fortunately, each known application of the framework described in Theorem 1.1 requires intersecting supermodular functions. Beyond the fact that positively intersecting supemodular functions are formally more general than the intersecting supermodular ones, the main reason of their usage is that the proofs become technically simpler.

Schrijver's proof is a clever (though not difficult) method to formulate (4) as a submodular flow problem. Since this reduction can be carried out in polynomial time and since there are good (combinatorial) algorithms for submodular flows (for a comprehensive overview, see S. Fujishige's book [16]), the optimization problem (5) is also solvable in polynomial time.

To see how the framework in Theorem 1.1 includes Problem (1), consider the special case when $g \equiv 1$ and p is defined by $p(Z) = k$ for every subset $\emptyset \subset Z \subseteq V - r$ and $p(Z) = 0$ otherwise. In this case there is a one-to-one correspondence between the $0-1$-valued solutions of (4) and the rooted k-edge-connected subgraphs of D. Therefore a submodular flow algorithm may be used to solve (1).

Naturally, this second approach is more complex than the first one relying on matroid intersection but it has the advantage that more general rooted k-edge-connection problems can also be handled with its help. For example, one may be interested in rooted k-edge-connected subgraphs for which, in addition, the indegree of every node (distinct from r) meets a prescribed lower bound. Let me mention two further extensions.

(A) We are given a digraph $D = (V, A_0 \cup A)$ with a root-node r and a terminal set $T \subseteq V - r$ so that T contains the head of every edge in A and so that there are k edge-disjoint paths from r to every node $t \in T$. There is also a cost function $c : A \to \mathbf{R}$. The problem is to find a minimum cost subset F of A so that there are k edge-disjoint paths from r to every node $t \in T$. Note that this problem specializes to the rooted k-edge-connection problem when $T = V - r$ and $A_0 = \emptyset$, while it becomes NP-complete even for $k = 1$ if $A_0 = \emptyset$ and the assumption on the head of edges in A (to be in T) is dropped. (This is the directed Steiner tree problem). The problem is indeed a special case of (5) when the groundset chosen to be T and function p is defined by $p(X) := \max\{k - \varrho_0(X \cup Y) : Y \subseteq V - (T + r)\}$ when $X \subseteq T$ is nonempty and $p(\emptyset) = 0$ since this p is easily proved to be intersecting supermodular. (This application was explicitely mentioned in [10] only for the special case $T = V - r$.)

(B) Suppose that $F_1, \dots, F_k$ are edge-disjoint arborescences of root r. An equivalent version of Edmonds' disjoint branchig theorem [5] asserts that *these arborescences can be extended into k edge-disjoint spanning arborescences by using edges from a given edge set A if and only $\varrho_A(X) \geq p(X)$ holds for every nonempty subset $X \subseteq V$ where $p(X)$ denotes the number of arborescences F_i for which their node set $V(F_i)$ is disjoint from X.* Since this function p is intersecting supermodular, the problem of finding a requested arborescence-extensions with minimum total cost is a special case of (5).

The minimum cost rooted k-connected subgraph problem is not covered by the above model (4). Frank and Tardos [12] described a rather complicated way to reduce it to submodular flows and in this sense a polynomial algorithm is available. Since the the k openly disjoint rt-paths problem can so easily be reduced, via the node-splitting technique, to that of k edge-disjoint rt-paths, it has been tempting to avoid the difficult reduction of [12] by invoking node-splitting. The natural direct approach, however, fails since node-splitting gives rise to new nodes of in-degree 1 and therefore the resulting digraph certainly will not contain k edge-disjoint paths from r to *every* other node (when $k \geq 2$).

The goal of this paper is double. On one hand, it will be shown that the rooted k-node-connection problem, like its k-edge-connection counter-part, can also be reduced to matroid intersection, and in fact we prove this for a common generalization of the two versions. In other words, the use of submodular flows is avoidable. On the other hand, by introducing a simpler and more natural approach than the one in [12], a TDI description of the rooted k-connected subgraphs will be provided. By extending Schrijver's theorem 1.2, we show that the polyhedron in question is also a submodular flow polyhedron. Again, this second framework, though more complicated than matroid intersections, will have the advantage of being suitable for handling more general problems. For example, this way one is able to find a cheapest subgraph of a digraph in which there are k edge-disjoint (resp., openly disjoint) paths from r to every node in a specified terminal set $T \subseteq V - r$ provided that the head of every edge with positive cost is in T, a requirement satisfied automatically when $T = V - r$. In fact, these solutions will be formulated in a general framework that includes both the rooted k-edge-connection and the rooted k-node-connection problems.

2 Preliminaries on node-connectivity and bi-set functions

A key to handling node-connectivity type optimization problems in digraphs is to consider supermodular functions defined on ordered pairs of subsets rather than using supermodular set-functions. In [15] disjoint pairs of subsets (X_T, X_H) were used to solve the directed node-conectivity augmentation problem where X_T and X_H were called the tail and the head of the pair. In the present work it is more convenient to work with pairs of subsets where one member of the pair includes the other. Given a ground-set V, by a **bi-set** $X = (X_O, X_I)$ we mean a pair of subsets X_O, X_I of V for which $\emptyset \subseteq X_I \subseteq X_O \subseteq V$. X_O is the **outer** member of X while X_I is the **inner** member. Note that there is a simple one-to-one correspondence between ordered pairs (X_T, X_H) of disjoint sets and bi-sets (X_O, X_I), namely, $X_O = V - X_T$, $X_I = X_H$, and hence every theorem concerning bi-sets may be formulated in terms of pairs of disjoint sets and vice versa.

Let $\mathcal{P}_2 = \mathcal{P}_2(V)$ denote the set of all bi-sets of V. A bi-set X with $X_I = \emptyset$ or with $X_O = V$ is called **trivial**. When $X_I = \emptyset$ the bi-set is **void**. A set of bi-sets is called **inner-disjoint** if the inner sets are pairwise disjoint. A function on $\mathcal{P}_2(V)$ will be called a bi-set function on V. We will assume throughout that the bi-set functions in question are integer-valuded and that their value on non-void bi-sets is always zero. A family of bi-sets is called **laminar** if it has no two properly intersecting members. A family $\mathcal{F}$ of bi-sets is **intersecting** if both the union and the intersection of any two intersecting members of $\mathcal{F}$ belong to $\mathcal{F}$. A laminar family is obviously intersecting.

The intersection $\cap$ and the union $\cup$ of bi-sets is defined in a staightforward manner: for $X, Y \in \mathcal{P}_2$ let $X \cap Y := (X_O \cap Y_O, X_I \cap Y_I)$, $X \cup Y := (X_O \cup Y_O, X_I \cup Y_I)$. We write $X \subseteq Y$ if $X_O \subseteq Y_O, X_I \subseteq Y_I$. This determines a partial order on $\mathcal{P}_2$. Accordingly, when $X \subseteq Y$ or $Y \subseteq X$, we call X and Y **comparable**. A family of pairwise comparable bi-sets is called a **chain**. Two bi-sets are **intersecting** if $X_I \cap Y_I \neq \emptyset$ and **properly intersecting** if, in addition, they are not comparable.

A directed edge $a = uv$ **enters** or **covers** a bi-set $X = (X_O, X_I)$ if a enters both X_O and X_I. Edge a **leaves** X if it leaves both X_O and X_I. For a directed graph $D = (V, A)$, $\varrho(X) := \varrho_D(X) := \varrho_A(X)$ denotes the number of edges entering (covering) X while $\delta(X) := \delta_D(X) := \delta_A(X)$ denotes the number of edges leaving X. For a vector $z : A \to \mathbf{R}$, let $\varrho_z(X) := \sum[z(a) : a \in A, a \text{ covers } X]$. $\delta_z(X)$ is defined analogously. For a bi-set function p, a digraph $D = (V, A)$ is said to cover p if $\varrho_D(X) \geq p(X)$ for every $X \in \mathcal{P}_2(V)$. A function $z : A \to \mathbf{R}$ **covers** p if $\varrho_z(X) \geq p(X)$ for every $X \in \mathcal{P}_2(V)$. We say that an edge $e = uv$ is **induced** by a bi-set $X = (X_O, X_I)$, if the head v of e is in X_I while its tail u is in X_O. Let $I_D(X)$ denote the set of edges induced by X and $i_D(X) := |I_D(X)|$.

A nonnegative integer-valued function $p : \mathcal{P}_2 \to \mathbf{Z}_+$ is said to satisfy the **supermodular inequality** on $X, Y \in \mathcal{P}_2$ if

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y). \tag{6}$$

If the reverse inequality holds, we speak of the **submodular** inequality. p is said to be **fully supermodular** or supermodular if it satisfies the supermodular inequality for every pair of bi-sets X, Y. If (6) holds for intersecting (crossing) pairs, we speak of **interecting** (**crossing**) **supermodular** functions. Analogous notions can be introduced for submodular functions. Sometimes (6) is required for those intersecting (crossing) pairs for which $p(X) > 0$ and $p(Y) > 0$. In this case p is called **positively intersecting** (resp., **positively crossing**) **supermodular**.

Proposition 2.1. *The indegree function ϱ_D on $\mathcal{P}_2$ is submodular while i_D is supermodular.*

Note that a bi-set $X = (X_O, X_I)$ with $X_O = X_I$ may (and will often) be identified with the subset X_O and hence results on bi-sets and bi-set functions may be specialized to those on sets and set-functions, respectively. Also, any set function h can be extended to bi-sets $X = (X_O, X_I)$ by taking $h(X) = h(X_O)$ if $X_O = X_I$ and $h(X) = 0$ otherwise. The set-function h is positively intersecting supermodular if and only if the bi-set function is positively intersecting supermodular. The notation for bi-set functions ϱ and i is in harmony with that of set-functions ϱ and i since $\varrho(X_O, X_O) = \varrho(X_O)$ and $i(X_O, X_O) = i(X_O)$.

Let $D = (V, F)$ be a digraph and $g : V - r \to \mathbf{Z}_+$ a function. A set of edge-disjoint rt-paths is said to be g**-bounded** if each node $v \in V - \{r, t\}$ is used by at most $g(v)$ of these paths. We stress that g-boundedness automatically means that the paths are edge-disjoint. Let $\lambda_g(r, t; D)$ denote the maximum number of g-bounded rt-paths. Note that for large g (say, $g \equiv |F|$) $\lambda_g(r, t; D)$ is the maximum number of edge-disjoint rt-paths, while for $g \equiv 1$, $\lambda_g(r, t; D)$ is the maximum number of openly disjoint rt-paths.

We will need the bi-set function μ_g defined by

$$\mu_g(X) := \sum [g(v) : v \in X_O - X_I] \ \ (= \mu_g(X_O) - \mu_g(X_I)). \tag{7}$$

It is easily seen that for bi-sets X and Y

$$\mu_g(X) + \mu_g(Y) = \mu_g(X \cap Y) + \mu_g(X \cup Y). \tag{8}$$

Proposition 2.2 (Variation of Menger's theorem). *In a digraph $D = (V, F)$ there are k g-bounded rt-paths if and only if*

$$\varrho_F(X) \geq k - \mu_g(X) \text{ holds for every bi-set } X = (X_O, X_I) \text{ with } t \in X_I \text{ and } X_O \subseteq V - r. \tag{9}$$

Proof. Suppose that there are k g-bounded rt-paths. Among these paths at most $\mu_g(X)$ use a node from $X_O - X_I$, hence at least $k - \mu_g(X)$ of them must use an edge entering bi-set X and the necessity of (9) follows.

Conversely, suppose that (9) holds. We may assume that there is no edge entering r and no edge leaving t. Define a new digraph $D' := (V' \cup V'', F' \cup E' \cup E'')$, as follows. V' and V'' are disjoint copies of V. For each edge $uv \in F$ let $u'v''$ be a member of F'. For each node $v \in V - \{r, t\}$ put $g(v)$ parallel edges from v'' to v' and k parallel edges from v' to v''. The edges form v'' to v' form E', the edges from v' to v'' form E''.

By this construction, if D' includes k edge-disjoint $r't''$-paths, then these paths correspond to k g-bounded rt-paths in D. If no such paths exist in D', then, by the directed edge-version of Menger's theorem, there is a subset X' of nodes of D' so that $\varrho_{D'}(X) < k$ and $t'' \in X' \subseteq V' \cup V'' - r'$. Let $X_I := \{v \in V : v'' \in X'\}$ and let $X_O := \{v \in V : v' \in X'\}$. Due to the edges in E'', $v'' \in X'$ implies $v' \in X'$ and hence $t \in X_I \subseteq X_O \subseteq V - r$. By the construction, we get $k > \varrho_{D'}(X') = \varrho_D(X) + \mu_g(X)$ contradicting (9). •

Note that for $g \equiv k$, (9) is automatically satisfied for bi-sets X with $t \in X_I \subset X_O \subseteq V - s$) and hence (9) is equivalent to requiring that $\varrho(X) \geq k$ holds for every *subset* X with $t \in X \subseteq V - r$.

The proposition immediately implies the following slight extension. Let $D = (V, F)$ be a digraph with a specified root-node r and terminal set $T \subseteq V - r$. Let $g : V - r \to \{1, 2, \ldots, k\}$ be a function. We say that D is (k, g)**-connected from** r **to** T if

$$\lambda_g(r, t; D) \geq k \text{ holds for every } t \in T. \tag{10}$$

In the special case when $T = V - r$, we call D **rooted** (k, g)**-connected**.

Proposition 2.3. *A digraph $D = (V, F)$ is (k, g)-connected from r to T if and only if*

$$\varrho_F(X) \geq k - \mu_g(X) \text{ holds for every bi-set } X = (X_O, X_I) \text{ with } X_I \cap T \neq \emptyset \text{ and } X_O \subseteq V - r. \tag{11}$$

A digraph is called a (k, g)**-foliage** (of root r) if it is rooted (k, g)-connected but deleting any edge destroys this property.

Proposition 2.4. *Suppose that $D = (V, F)$ is (k, g)-connected from r to T but removing any edge of D destroys this property. Then the indegree of every node in T is exactly k. In particular, in a (k, g)-foliage the indegree of every node distinct from r is k.*

Proof. Suppose indirectly that $\varrho(z) > k$ for some $z \in T$. Choose k g-bounded rz-paths $P_1, \ldots, P_k$. Then there is an edge $e = uz$ not used by these paths. We claim that there are k g-bounded rt-paths in $D' := D - e$ for every $t \in T$ and this will contradict the minimality assumption on D. If these paths do not exist for some $t \in T$, then, by Proposition 2.3 there is a bi-set X violating (11) in D'. Since X does not violate (11) in D, it follows that e must enter X and hence $t \in X_I$. But then the existence of paths $P_1, \ldots, P_k$ show that X cannot violate (11) in D' either, a contradiction. •

3 Rooted k-connections via matroid intersection

Let $D = (V, A)$ be a digraph with a root-node r, a non-negative cost-function $c : A \to \mathbf{R}_+$, and a bounding function $g : V - r \to \{1, 2, \ldots, k\}$ (but in this section no terminal set T is considered). For later purposes we denote by $D^* = (U, A^*)$ the digraph obtained from D by deleting r, that is, $U := V - r$ and A^* is the set of edges induced by U.

Our goal is to reduce the problem of finding a cheapest rooted (k, g)-connected subgraph of D to matroid intersection. Since c is non-negative, it suffices to find a cheapest (k, g)-foliage (of root r). As a main result, we will prove that there are two matroids M_1 and M_2 on A so that their common independent sets of cardinality $k(|V| - 1)$ are exactly the (k, g)-foliages of D.

3.1 Matroids on the edge set of digraphs

Let us invoke a fundamental construction of matroids due to J. Edmonds [3]. Let b be an integer-valued, monoton increasing, intersecting submodular function on a groundset S.

Theorem 3.1 (Edmonds). *The set $\{F \subseteq S : |F \cap X| \leq b(X)$ for every subset $X \subseteq F\}$ forms the independent sets of a matroid M_b whose rank function is given by*

$$r_b(Z) = \min\{b(X) + |Z - X| : X \subseteq Z\}. \tag{12}$$

M_b is called the matroid of b. (A geometric interpretation of this matroid is as follows: the convex hull of independent sets of M_b is the intersection of the polymatroid $\{x \in \mathbf{R}^S_+ : x(Z) \leq b(Z)$ for every $Z \subseteq S\}$ defined by b with the unit $0 - 1$ cube $\{x \in \mathbf{R}^S : 0 \leq x \leq 1\}$.)

For a subset $J \subseteq A^*$ of edges of digraph D^*, let $H(J) := \{v : v$ is the head of some edge in $J\}$ and let $V(J) := \{u : u$ is the head or the tail of some edges in $J\}$. Note that $V(J)$ is a set-function on the underlying undirected edge-set and independent of the orientation of the edges.

For a function $m : V \to \mathbf{R}$ let $V_m(J) := \sum[m(v) : v \in V(J)]$ and $H_m(J) := \sum[m(v) : v \in H(J)]$. The proof of the following proposition is an easy exercise and is left to the reader.

Proposition 3.2. *If m is non-negative, then both V_m and H_m are monotone increasing submodular functions on groundset A^*.*

We need the following set-function b^* defined to be 0 on the empty set and

$$b^*(J) := \sum[k - g(v) : v \in H(J)] + \sum[g(v) : v \in V(J)] - k \text{ for } \emptyset \subset J \subseteq A^*. \tag{13}$$

By the proposition b^* is intersecting submodular. Furthermore b^* is non-negative since, for a non-empty set J, $b^*(J) = k(|H(J)| - 1) + \sum[g(v) : v \in V(J) - H(J)] \geq 0 = b^*(\emptyset)$. This and the assumption $1 \leq g(v) \leq k$ imply that b^* is monotone increasing. We abbreviate M_{b^*} by M^* and call it the **master matroid** of D^* (determined by k and g).

Proposition 3.3. *A subset $F \subseteq A^*$ is independent in M^* if and only if*

$$i_F(X) \leq k(|X_I| - 1) + \mu_g(X). \tag{14}$$

for every bi-set $X = (X_O, X_I)$ with $\emptyset \subset X_I \subseteq X_O \subseteq U$.

Proof. Let F be independent in M^* and X a bi-set whose inner set is non-empty. Consider the subset $J := I_F(X)$ of F induced by X. If this is empty, then $i_F(X) = 0 \leq k(|X_I| - 1) + \mu_g(X)$. If J is non-empty, then $|J| \leq b^*(J) = \sum[k - g(v) : v \in H(J)] + \sum[g(v) : v \in V(J)] - k \leq \sum[k - g(v) : v \in X_I] + \sum[g(v) : v \in X_O] - k = k(|X_I| - 1) + \mu_g(X)$, that is (14) holds true.

Conversely, if F is not independent in M^*, then it has a subset J for which $|J| > b^*(J)$. Let $X_I := H(J)$ and $X_O := V(J)$. Then every element of J is induced by bi-set $X = (X_O, X_I)$ and hence $i_F(X) \geq |J| > b^*(J) = \sum[k - g(v) : v \in H(J)] + \sum[g(v) : v \in V(J)] - k = k(|X_I| - 1) + \mu_g(X)$, that is, azaz (14) is violated. •

Note that in case $g \equiv k$ (14) turns to $i_F(X) \leq k(|X_I|-1)+k|X_O-X_I| = k(|X_O|-1)$. If a bi-set (X_O, X_I) violates this, then so does the bi-set (X_O, X_O), which corresponds to a subset X_O of V. Therefore in this case a subset F of edges of D^* is independent in the master matroid M^* if and only if every non-empty subset X of nodes induces at most $k(|X|-1)$ elements of F. Hence M^* depends only on the underlying undirected graph and no on the orientation of the edges. Moreover, by a theorem of Nash-Williams [21], M^* is the sum of k copies of the circuit matroid of the underlying undirected graph in which a subset of edges is independent, by definition, if it can be partitioned into k forests.

3.2 Foliages as matroid intersections

Let $A_r := A - A^*$, that is, A_r is the set of edges of D whose tail is r. Define a matroid M_1 on A to be the direct sum of the free matroid on A_r (in which, by definition, every subset is independent) and the master matroid M^* defined above on A^*.

Let M_2 denote the partition matroid on groundset A in which a subset $I \subseteq A$ is independent if $\varrho_I(v) \leq k$ for every node $v \in V - r$ (and $\varrho_I(r) = 0$).

Theorem 3.4. *A subgraph $D_B = (V, B)$ of digraph $D = (V, A)$ is a (k, g)-foliage if and only if B is a common independent set of matroids M_1 and M_2 and $|B| = k(n-1)$ where $n = |V|$.*

Proof. If D_B is a (k, g)-foliage, then Proposition 2.4 implies that $\varrho_B(v) = k$ and $\varrho_B(r) = 0$. Hence D_B has exactly $k(n-1)$ edges and B is a basis in M_2. For a bi-set $X = (X_O, X_I)$ with $\emptyset \subset X_I \subseteq X_O \subseteq U$, one has $\varrho_B(X) + \mu_g(X) \geq k$ and hence $i_B(X) = \sum[\varrho_B(v) : v \in X_I] - \varrho_B(X) \leq k|X_I| + \mu_g(X) - k$. This and Proposition 3.3 implies that B is independent in M_1.

Conversely, suppose that a $k(n-1)$-element subset $B \subseteq A$ of edges is independent in both M_1 and M_2. Then $\varrho_B(v) = k$ for every $v \in V - r$ and $\varrho_B(r) = 0$. Furthermore, for a bi-set $X = (X_O, X_I)$ with $\emptyset \subset X_I \subseteq X_O \subseteq U$, one has $i_B(X) \leq k(|X_I|-1)+\mu_g(X)$. Therefore $\varrho_B(X) + \mu_g X) = \sum[\varrho_B(v) : v \in X_I] - i_B(X) + \mu_g(X) = k|X_I| - i_B(X) +$

$\mu_g(X) \geq k|X_I| - k(|X_I| - 1) = k$ and hence Proposition 2.3 implies that $D_B = (V, B)$ is a (k, g)-foliage. •

This matroid approach enables us to handle a variation of the rooted (k, g)-connection problem in which one wants to find a cheapest rooted (k, g)-connected subgraph obeying a specified upper bound β imposed on the out-degree of root r. In this case it again suffices to restrict ourselves to consider (k, g)-foliages and the only change in Theorem 3.4 is that matroid M_1 should be changed with the direct sum of the master matroid M^* and the uniform matroid on A_r in which the subsets of cardinality at most β are the independent sets.

More generally, we can deploy a matroid M_r on the edge-set A_r and a matroid M_v on the set of edges entering v for each node $v \in U$. Call a (k, g)-foliage **matroid-restricted** if its subset of edges leaving r is independent in M_r and its subset of edges entering v is independent in M_v for each $v \in U$.

Let M'_1 be the direct sum of M^* and M_r and let M'_2 be the direct sum of the $n-1$ matroids $M_v \quad (v \in U)$.

Theorem 3.5. *A subgraph $D_B = (V, B)$ of digraph $D = (V, A)$ is a matroid-restricted (k, g)-foliage if and only if B is a common independent set of matroids M'_1 and M'_2 and $|B| = k(n-1)$ where $n = |V|$.*

Proof. Suppose first that $D_B = (V, B)$ is a matroid-restricted (k, g)-foliage. Since by Theorem 3.4 the edge set of any (k, g)-foliage is independent in M^* it follows from the definitions that B is independent in both M'_1 and M'_2. The reverse implication follows similarly from Theorem 3.4. •

Edmonds' matroid intersection theorem provides a necessary and sufficient condition for the existence of a matroid-restricted (k, g)-foliage. We formulate this for our very special initial case when the only restriction was imposed on the out-degree of r.

Theorem 3.6. *In a digraph $D = (V, A)$, there exists a rooted (k, g)-foliage in which the out-degree of root r is at most β if and only if $\sum_i [k - (\varrho_{D^*}(X_i) + \mu_g(X_i))] \leq \beta$ for every set of bi-sets $X_1, \ldots, X_q$ whose outer members are subsets of U and inner members are pairwise disjoint, where $D^* = D - r$. In particular, in a digraph there is a rooted k-edge-connected subgraph in which the out-degree of the root is at most β if and only if $\sum_i [k - \varrho_{D^*}(X_i)] \leq \beta$ holds for every set of pairwise disjoint subsets X_i of U.* •

Note that if upper bound restrictions are given on the out-degree of nodes $v \in U$ rather than on the in-degrees, then the problem becomes NP-complete even in the special case $k = 1$, $g \equiv 1$ and each upper bound is 1 since in this case the restricted (k, g)-foliages are exactly the Hamiltonian paths of initial node r.

3.3 Independence oracles for M_1 and M_2

By Theorem 3.4, a cheapest rooted (k, g)-connected subgraph of D can be computed with the help of a matroid intersection algorithm provided the independences oracles for the two matroids are available. By an independence oracle, we mean an algorithm which tells us for any input subset X whether X is independent or not. Constructing such an orale for the partition matroid M_2 is straightforward, so we consider only M_1.

We need the following orientation result of Hakimi [20].

Lemma 3.7. *For a given undirected graph $G = (U_G, E)$ and upper-bound function $g' : U_G \to \mathbf{Z}$, the graph has an orientation in which $\varrho(v) \leq g'(v)$ for every node v if and only if $g'(X) \geq i_G(X)$ holds for every subset X of nodes where $g'(X) := \sum[g'(v) : v \in X]$.*

Proof. (Sufficiency) Starting with an arbitrary orientation of G, we gradually reduce the 'error-sum' $\sum[(\varrho(v) - g'(v))^+ : v \in U_G]$ by successively reorienting certain paths. Namely, as long as there are nodes z with $\varrho(z) > g'(z)$, select anyone of them. Let Z denote the set of nodes from which z is reachable along a directed path in the given orientation. If there is an undersaturated node $u \in Z$ (that is, $\varrho(u) < g'(u)$), then by reorienting any path from u to z the error-sum becomes smaller. If no such node u exists, then Z violates the condition since $\varrho(Z) = 0$ implies $i(Z) = \sum[\varrho(v) : v \in Z] > \sum[g'(v) : v \in Z] = g'(Z)$. •

Note that the proof of the lemma gives rise to an algorithm of complexity $O(|U_G||E|^2)$ and since it can be considered as is a variation of the alternating path algorithm for flows, the bound can actually be reduced to $O(|U_G|^3)$.

Since M_1 is the direct sum of a free matroid and the master matroid M^* on A^*, it suffices to construct an independence oracle for M^*. What we actually construct is a subroutine which decides for an input independent set $F' \subseteq A^*$ of M^* and for an input element $f = sz \in A^* - F'$ whether $F := F' + f$ is independent. By repeated applications of this, one can easily decide if an arbitrary subset is independent or not in M^*.

For the digraph $D_F = (U, F)$, construct a bipartite undirected graph $G = (U', U''; E)$ as follows. To every node $v \in U$, assign a node $v' \in U'$ and a node $v'' \in U''$ which are connected by $g(v)$ parallel edges. The set of these type of edges of G is denoted by E_U. Furthermore, with every directed edge $e = uv \in F$ we associate an edge $e_G = u'v''$ of G. The set of edges of this type is denoted by E_F. Let $E := E_U \cup E_F$ and $U_G := U' \cup U''$. We use the convention that the subsets of U' and U'' corresponding to a subset $X \subseteq U$ will be denoted by X' and X'', respectively. Furthermore a subset of E_F corresponding to a subset $J \subseteq F$ will be denoted by E_J.

Define a function $g' : U_G \to \mathbf{Z}_+$ as follows. Let $g(v') := g(v)$ for every node $v \in U$, let $g'(v'') := k$ for every node $v \in U - z$, and let $g'(z'') := 0$.

Lemma 3.8. *For an independent set $F' \subseteq A^*$ of M^* and for an edge $f = sz \in A^* - F'$, the set $F := F' + sz$ is independent in M^* if and only if G has an orientation in which the indegree of each node x is at most $g'(x)$.*

Proof. Assume first that the required orientation does not exists. By Lemma 3.7 there is a subset $X' \cup Y'' \subseteq U_G$ of nodes for which $i_G(X' \cup Y'') > g'(X' \cup Y'')$. Then $i_G(X' \cup Y'') > 0$ and hence $X' \neq \emptyset$ and $Y'' \neq \emptyset$. Let $J \subseteq F$ denote the set of those edges $e = uv$ for whhich $u' \in X'$ and $v'' \in Y''$. Since $X' \cup Y''$ induces $g(X \cap Y)$ edges of type E_U we have $|J|+g(X\cap Y)=i_G(X'\cup Y'')>g'(X'\cup Y'')\geq g(X)+k(|Y|-1)$, from which

$$|J|>k|Y|-k+g(X)-g(X\cap Y). \tag{15}$$

If, indirectly, F were independent in M^*, then we had $|J|\leq b^*(J)=\sum[k-g(v):v\in H(J)]+\sum[g(v):v\in V(J)]-k\leq\sum[k-g(v):v\in Y]+\sum[g(v):v\in X\cup Y]-k=k|Y|-g(Y)+g(X\cup Y)-k=k|Y|-k+g(X)-g(X\cap Y)$ contradicting (15).

To see the converse, assume that F is dependent in M^*, that is, there is a bi-set $X=(X_O,X_I)$ by Proposition 3.3 for which

$$|J|=i_F(X)>k(|X_I|-1)+\mu_g(X), \tag{16}$$

where J denotes the subset of F induced by X.

As F' is independent in M^*, X must induce $f=sz$, that is, $s\in X_O$ and $z\in X_I$. Hence $g'(X_I'')=k(|X_I|-1)$. The set $X_O'\cup X_I''\subseteq U_G$ induces in G $g(X_I)$ edges of type E_U. If, indirectly, the requested orientation does exist, then $|J|+g(X_I)=i_G(X_O'\cup X_I'')\leq g'(X_O')+g'(X_I'')=g(X_O)+k(|X_I|-1)$, that is, $|J|\leq g(X_O)-g(X_I)+k(|X_I|-1)=\mu_g(X)+k(|X_I|-1)$, contradicting (16). •

We can conclude that with the help of the orientation lemma the necessary independence oracle for M^* and hence for M_1, too, is available.

4 Covering supemodular bi-set functions by digraphs

In this section we show how Theorems 1.1 and 1.2 concerning supermodular set functions can be extended to those on supermodular bi-set functions.

4.1 Total dual integrality

Proposition 4.1. *Let $\mathcal{F}$ be a laminar family of bi-sets and $D=(V,A)$ a digraph. Let M be a $0-1$ matrix the rows and columns of which correspond to the members of $\mathcal{F}$ and to the edges of D, respectively. An entry of M corresponding to $X\in\mathcal{F}$ and $e\in A$ is 1 if e enters X and zero otherwise. Then M is totally unimodular.*

Proof. Since a subfamily of a laminar family is also laminar, by the characterization of Ghouila-Houri [19], it suffices to prove that there is a uniform 2-colouration of the rows of M, that is, a function $c:\mathcal{F}\to\{-1,+1\}$ so that $|\sum[c(X):e \text{ enters } X]|\leq 1$ for each edge e of D. (In words, each edge e enters near the same number of 1-coloured and of (-1)-coloured members of $\mathcal{F}$ where near the same means that the two numbers may differ by at most one.)

We may assume that the members of $\mathcal{F}$ are distinct. Indeed, if a bi-set X occurs in at least two copies then, in order to get a uniform 2-colouration of $\mathcal{F}$, remove first two copies of X, get then inductively a uniform 2-colouration of the rest and finally colour the two removed copies of X differently.

Turning to the case when the members of $\mathcal{F}$ are distinct, define first $c(X)$ to be 1 for each maximal member X of $\mathcal{F}$. In a general step take a maximal uncoloured member X of $\mathcal{F}$. By the laminarity, there is a unique smallest coloured member Y for which $X \subset Y$. Define $c(X) := -c(Y)$. By the laminarity, the members of $\mathcal{F}$ entered by an edge e form a chain in which two consecutive members $X \subset Y$ have the property that Y is the unique smallest member of $\mathcal{F}$ that is larger than X. Hence $c(X) = -c(Y)$ and therefore the 2-colouration c is indeed uniform. •

Remark The matrix M in the theorem can be rather easily shown to be a network matrix. To see this, consider the laminar family of sets consisting of the inner sets of bi-sets in $\mathcal{F}$. It is well-known that a laminar family $\mathcal{L}$ of subsets of V can be represented by an arborescence $T = (U, F)$ in the sense that there is a mapping $\varphi : V \to U$ in such a way that there is a one-to-one correspondence between $\mathcal{L}$ and F, namely, each member X of F is $\varphi^{-1}(U(f_X))$ where f_X denotes the edge of T corrsponding to X and $U(f_X)$ denotes the subset of nodes of T reachable in T from the head of f_X. Using this representation, one can show that the edges of the arborescence corresponding to the inner sets of those members of $\mathcal{F}$ that are entered by e form a directed paths in T and hence M is indeed a network matrix.

The following result is a direct extension of Theorem 1.1 to bi-set functions. Its proof is a rather standard application of the well-known uncrossing technique.

Theorem 4.2. *Let $D = (V, A)$ be a digraph. Let $p : \mathcal{P}_2 \to \mathbf{Z}$ be a positively intersecting supermodular bi-set function and $g_A : A \to \mathbf{Z}_+ \cup \{\infty\}$ a non-negative upper bound on the edges of digraph $D = (V, A)$ that covers p. The linear system*

$$\varrho_x(Z) \geq p(Z) \text{ for every bi-set } Z \in \mathcal{P}_2,\ 0 \leq x \leq g_A \tag{17}$$

is totally dual integral. In particular, the linear programming problem

$$\min\{cx : x \text{ satisfies } (17)\} \tag{18}$$

has an integer-valued optimum solution and so has its linear programming dual provided c is integer-valued.

Proof. Let $c : A \to \mathbf{Z}$ be integer-valued so that the primal optimum is bounded (which is, in the present case, is equivalent to requiring that $g_A(e)$ is finite whenever $c(e) < 0$. Let Q denote a $0-1$ matrix in which the rows and the columns correspond to the nontrivial members of $\mathcal{P}_2$ and to the edges of D, respectively. An entry of Q corresponding to a bi-set X and edge e is 1 if e covers X and zero otherwise. In what follows, we also denote by p the $|\mathcal{P}_2|$-dimensional vector whose component corresponding the member $X \in \mathcal{P}_2$ has value $p(X)$.

Then the primal linear programming problem is $\min\{cx : 0 \leq x \leq g_A, Qx \geq p\}$, while its dual is:

$$\max\{yp - zg_A : yQ - z \leq c, y \geq 0, z \geq 0\}, \tag{19}$$

where $z(e)$ denotes the dual variable corresponding to the primal inequality $x(e) \leq g_A(e)$ ($g_A(e)$ is finite).

For a given y the optimal z is uniquely given: $z(e) = (yq_e - c(e))^+$, where q_e a denotes the column of Q corresponding to edge e. Therefore we can say that a certain y is an optimal solution to (19).

What we have to prove is that the optimun to (19) is attained at an integer vector. Let y_0 be an optimal rational solution. As long as there exist two properly intersecting bi-sets $X = (X_K, X_B)$ and $Y = (Y_K, Y_B)$ with positive $y_0(X)$ and $y_0(Y)$, revise y_0 as follows. Define $\alpha := \min\{y_0(X), y_0(Y)\}$, decrease by α both $y_0(X)$ and $y_0(Y)$, and increase by α both $y_0(X \cap Y)$ and $y_0(X \cup Y)$.

Due to the submodularity of bi-set function ϱ on $\mathcal{P}_2$, the resulting dual vector continues to be feasible. Moreover it is also dual optimal since p is assumed to be positively intersecting supermodular. Let us call such a change in the dual solution an uncrossing step.

Define the linear ordering of the partially ordered set $(\mathcal{P}_2, \subseteq)$ obtained in such a way that if the first some elements of the ordering have already been determined then the subsequent element is selected to be minimal among the not yet selected elements of $\mathcal{P}_2$. In this ordering for arbitrary $X, Y \in \mathcal{P}$ $X \cap Y$ precedes both X and Y while $X \cup Y$ follows both of them.

Therefore the following lemma implies that the number of uncrossing steps cannot be infinite.

Lemma 4.3. *Let $r_1, \ldots, r_n$ be a sequence of nonnegative rational numbers. As long as possibly, apply the following 4-change step. Select four distinct members for which the two middle ones are positive. Let α denote the minimum of the two middle elements. Decrease by α the value of the two middle elements and increase by α the value of the first and fourth ones. Then after a finite number of 4-change steps the procedure terminates.*

Proof. By multiplying through with the least common denominator, if necessary, we may assume that the sequence consists of integers. Since the first member never decreases, each member remains nonnegative and the total sum stays constant, after a finite number of 4-change steps the first member gets fixed and the lemma follows by induction on n. $\bullet$

We may therefore assume that the set $\mathcal{H}$ of bi-sets for which the y_0-value is positive is laminar. By Proposition 4.1 the submatrix of Q determined by the rows corresponding to the members of $\mathcal{H}$ is totally unimodular. Therefore the optimal dual solution y_0 may be chosen integer-valued, as required. $\bullet\ \bullet$

Theorem 4.2 has a certain self-refining nature. Given a subset $T \subseteq V$, we say that a bi-set function p is (**positively**) T**-intersecting supermodular** if the supermodular inequality holds for bi-sets X and Y whenever ($p(X) > 0$ and $p(Y) > 0$) $X_I \cap Y_I \cap T \neq \emptyset$.

Proposition 4.4. *For bi-set function p_1, define a bi-set function p on ground-set T, as follows. For bi-set $Z = (Z_O, Z_I)$ with $Z_I \subseteq T$, let*

$$p(Z) := \max\{p_1(Z_O, Z_I \cup K) : K \subseteq Z_O - T\} \tag{20}$$

and for every other bi-set Z let $p(Z) = 0$. If p_1 is (positively) T-intersecting supermodular, then so is p.

Proof. Let X and Y be two intersecting bi-sets (for which $p(X) > 0,\ p(Y) > 0$ in case p_1 is positively T-intersecting supermodular). There are subsets $K \subseteq X_O - T,\ L \subseteq Y_O - T$ for which $p(X) = p_1(X')$ and $p(Y) = p_1(Y')$ where $X' = (X_O, X_I \cup K)$ and $(Y') = (Y_O, Y_I \cup L)$. Since $(X_I \cup K) \cap (Y_I \cup L) \neq \emptyset,\ K \cap L \subseteq (X_O \cap Y_O) - T$ and $K \cup L \subseteq (X_O \cup Y_O) - T$, therefore $p_1(X' \cap Y') \leq p(X \cap Y)$ and $p_1(X' \cup Y') \leq p(X \cup Y)$. Hence $p(X) + p(Y) = p_1(X') + p_1(Y') \leq p_1(X' \cap Y') + p_1(X' \cup Y') \leq p(X \cap Y) + p(X \cup Y)$, as required.

4.2 Relation to submodular flows

In order to have an algorithm for the optimization problem given in Theorem 4.2 we are going to prove that the linear system (17) actually describes a submodular flow polyhedron. Since there are efficient combinatorial solving algorithms for submodular flows (for rich overviews, see [16, 23]) this way we will have for optimal coverings of intersecting supermodular bi-set functions. We remark that for the special case when p is identically 1 on a given intersecting family of bi-sets and zero otherwise Theorem 4.2 was algorithmically proved in [13] with the help of a two-phase greedy algorithm.

Let $\hat{D} = (\hat{V}, \hat{A})$ be a digraph, $\mathcal{F}$ a crossing family of subsets of $\hat{V}$, $b : \mathcal{F} \to \mathbf{Z}$ a crossing submodular function. Let $\hat{f} : \hat{A} \to \mathbf{Z} + \{-\infty\}$ and $\hat{g} : \hat{A} \to \mathbf{Z} + \{\infty\}$ two functions with $\hat{f} \leq \hat{g}$. A function $\hat{x} : \hat{A} \to \mathbf{R}$ is called a **submodular flow** or in short a **subflow** if

$$\varrho_{\hat{x}}(Z) - \delta_{\hat{x}}(Z) \leq b(Z) \text{ for every } Z \in \mathcal{F} \tag{21}$$

and $\hat{f} \leq \hat{x} \leq \hat{g}$. The set of subflows is called a submodular flow (subflow) polyhedron. It is known and easy to show that for a crossing supermodular function p on $\mathcal{F}$ the polyhedron defined by

$$\varrho_{\hat{x}}(Z) - \delta_{\hat{x}}(Z) \geq p(Z) \text{ for every } Z \in \mathcal{F} \tag{22}$$

and $\hat{f} \leq \hat{x} \leq \hat{g}$ is also a subflow polyhedron. In this sense one could speak of supermodular flows as well but we stay at the conventional term of submodular flow even if the polyhedron is defined by a supermodular function. The subflow polyhedron is called **one-way** if the in-degree ore the out-degree of every member of $\mathcal{F}$ is zero.

Theorem 4.5. *Let $D = (V, A)$ be a digraph. Let $p : \mathcal{P}_2 \to \mathbf{Z}$ be an intersecting supermodular bi-set function and $g_A : A \to \mathbf{Z}_+ \cup \{\infty\}$ a non-negative upper bound on the edges of D that covers p. The polyhedron P defined by the linear system*

$$\varrho_x(Z) \geq p(Z) \textit{ for every bi-set } Z \in \mathcal{P}_2,\ 0 \leq x \leq g_A \tag{23}$$

is a one-way submodular flow polyhedron.

Proof. It follows from the definition that the intersection of a submodular flow polyhedron with a box is also a submodular flow polyhedron so it suffices to prove the theorem for the special case when $g_A = \infty$.

Construct a digraph $\hat{D} = (U, \hat{A})$ from D as follows. For each edge $e = uv$ of D, subdivide e by a new node u_e and delete uu_e, one of the two newly arising edges. The remaining edge $u_e v$ will be denoted by $\hat{e}$. Here $U = V \cup \hat{V}_A$ where $\hat{V}_A$ denotes the set of subdividing nodes. For any subset $F \subseteq A$, the corresponding subset of edges and subset of nodes of $\hat{D}$ will be denoted by $\hat{F}$ and $\hat{V}_F$, respectively.

Define a family $\mathcal{F}$ of subsets of U and a function $\hat{p}$ on $\mathcal{F}$ as follows. For each non-void bi-set $X \in \mathcal{P}_2(V)$ with finite $p(X)$ and for each subset $F \subseteq I_D(X)$, let $X_I \cup \hat{V}_F$ be a member of $\mathcal{F}$ and let $\hat{p}(X_I \cup \hat{V}_F) := p(X)$.

Claim 4.6. *$\mathcal{F}$ is an intersecting family of sets and $\hat{p}$ is intersecting supermodular.*

Proof. Suppose for bi-sets X, X' and edge sets $F \subseteq I_D(X)$, $F' \subseteq I_D(X')$ that $Y := X_I \cup \hat{V}_F$ and $Y' := X'_I \cup \hat{V}_{F'}$ are intersecting sets. Then X and X' are also intersecting. It easily follows from the definition that $I_D(X) \cap I_D(X') \subseteq I_D(X \cap X')$ and $I_D(X) \cup I_D(X') \subseteq I_D(X \cup X')$ and hence both Y and Y' are in $\mathcal{F}$. Furthermore we have $\hat{p}(Y) + \hat{p}(Y') = p(X) + p(X') \leq p(X \cap X') + p(X \cup X') = p((X_I \cap X'_I) \cup (F \cap F')) + p((X_I \cup X'_I) \cup (F \cup F')) = \hat{p}(Y \cap Y') + \hat{p}(Y \cup Y')$, as required. •

By the construction, no edge of $\hat{D}$ leaves any member of $\mathcal{F}$ and hence $\hat{P} := \{\hat{x} \in \mathbf{R}^{\hat{A}} : \hat{x} \geq 0, \hat{x}(Z) \geq \hat{p}(Z)$ for every $Z \in \mathcal{F}\}$ is a one-way subflow polyhedron. Since the edges of D and $\hat{D}$ correspond to each other we may speak of the polyhedron P' in $\mathbf{R}^A$ corresponding to $\hat{P}$.

Claim 4.7. *$P = P'$.*

Proof. Let $x \in P'$, that is, $\hat{x} \in \hat{P}$. For a non-void bi-set X with finite $p(X)$ and for $F := I_D(X)$ we have $\varrho_x(X) = \varrho_{\hat{x}}(X_I \cup V_F) \geq \hat{p}(X_I \cup V_F) = p(X)$ and hence $x \in P$, from which $P' \subseteq P$.

Conversely, let $x \in P$. For a non-void bi-set X with finite $p(X)$ and for $F \subseteq I_D(X)$ we have $\varrho_{\hat{x}}(X_I \cup V_F) \geq \varrho_x(X) \geq p(X) = \hat{p}(X_I \cup V_F)$ and hence $\hat{x} \in \hat{P}$, from which $P \subseteq P'$. •

By the two claims, the proof of the theorem is complete. • •

Corollary 4.8. *Let $D = (V, A)$ be a digraph and $g_A : A \to \mathbf{Z}_+ \cup \{\infty\}$ an integer-valued function. Let $T \subset V$ be a subset of nodes containing the head of every edge of D. Let p_1 be a positively T-intersecting supermodular bi-set function covered by g_A. Then the linear system*

$$\{0 \leq x \leq g_A, \varrho_x(X) \geq p_1(X) \textit{ for every bi-set } X\} \tag{24}$$

is totally dual integral. The polyhedron defined by (24) *is a submodular flow polyhedron.*

Proof. By proposition 4.4 the bi-set function p defined in (20) is positively intersecting supermodular. Since every edge has its head in T, a vecor $x : A \to \mathbf{R}$ covers p_1 if and only if x covers p. Furthermore, a dual solution y to (23) determines a dual solution y_1 to (24) as follows. For $X = (X_O, X_I)$ with $X_I \subseteq T$ let Y be the bi-set for which $Y_O = X_O, X_I \subseteq Y_I$ and $p(X) = p_1(Y)$. Define $y_1(Y) := y(X)$ if Y arises this way and $y_1(Y) := 0$ otherwise. Then y_1 is a dual feasible solution to (24) having the same value as y does. Therefore Theorem 4.2 implies that the system (24) is also TDI. •

4.3 Polyhedral descriptions of rooted (k, g)-connected subgraphs

Let $k \geq 1$ be an integer and $g : V \rightarrow \{1, \ldots, k\}$ a function. As an application, we exhibit how the problem of cheapest subgraphs which are (k, g)-connected from r to a terminal set T can be handled polyhedrally and algorithmically provided each edge of positive cost has its head in T.

Theorem 4.9. *Let $H = (V, A_0 \cup A)$ be a digraph with a specified root-node r and terminal set $T \subseteq V - r$ so that the head of each edge in A is in T. Suppose that H is (k, g)-connected from r to T. The convex hull of incidence vectors of the edge-sets $F \subseteq A$ for which the subgraph $(V, A_0 \cup F)$ is (k, g)-connected from r to T is described by*

$$\{x \in \mathbf{R}^A, 0 \leq x \leq 1, \varrho_x(Z) \geq p_1(Z) \text{ for every bi-set } Z\} \tag{25}$$

where p_1 is defined by

$$p_1(Z) = k - \varrho_{A_0}(Z) - \mu_g(Z) \text{ for bi-set } Z = (Z_O, Z_I) \\ \text{with } Z_I \cap T \neq \emptyset \text{ and } Z_O \subseteq V - r, \tag{26}$$

and $p_1(Z) := -\infty$ otherwise. Furthermore the linear system in (25) *is TDI and determines a submodular flow polyhedron.*

Proof. Observe that the function p_1 defined in the theorem is intersecting supermodular and hence Corollary 4.8 can be applied to $D = (V, A)$, p_1, and $g_A \equiv 1$. •

Let us formulate Theorem 4.9 in the special case when $g \equiv k$.

Corollary 4.10. *Let $D = (V, A_0 \cup A)$ be a digraph with a specified root-node r and terminal set $T \subseteq V - r$ so that the head of each edge in A is in T. Suppose that D is k-edge-connected from r to T. The convex hull of incidence vectors of the edge-sets $F \subseteq A$ for which the subgraph $(V, A_0 \cup F)$ is k-edge-connected from r to T is described by*

$$\{x \in \mathbf{R}^A, 0 \leq x \leq 1, \\ \varrho_x(X) \geq k - \varrho_{A_0}(X) \text{ for every subset } X \subseteq V - r \text{ for which } X \cap T \neq \emptyset\}. \tag{27}$$

Furthermore the linear system in (25) *is TDI and determines a submodular flow polyhedron.*

Let us formulate Theorem 4.9 in the special case when $T = V - r$ and $A_0 = \emptyset$.

Corollary 4.11. *Let $D = (V, A)$ be a rooted (k, g)-connected digraph. The convex hull of incidence vectors of the edge-sets $F \subseteq A$ for which the subgraph (V, F) is (k, g)-connected is described by*

$$x \in \mathbf{R}^A, 0 \leq x \leq 1, \\ \varrho_x(Z) \geq k - \mu_g(Z) \text{ for every non-void bi-set } Z = (Z_O, Z_I) \text{ with } Z_O \subseteq V - r. \tag{28}$$

Furthermore the linear system in (28) *is TDI and describes a submodular flow polyhedron.*

5 Conclusion

In this paper we considered the rooted (k, g)-connection problem which is a common generalization of those of finding a cheapest rooted k-edge-connected and k-node-connected subgraph of a digraph. By extending a known result on rooted k-edge-connectivity, we proved that the general version is also a matroid intersection problem and hence a weighted matroid intersection algorithm may be applied. We also showed that the independence oracle required for the matroids in question can be constructed through an easy orientation result. This matroid approach supersedes the only solution known earlier to the rooted k-node-connection problem which invoked the more complex model of submodular flows.

Moreover, we exhibited TDI descriptions for further generalizations of the rooted (k, g)-connection problem for which the algorithmic solution did invoke submodular flows. For example, the problem of finding a cheapest subgraph of a digraph in which there are k g-bounded paths from a root node to each element of a terminal set T could be handled this way provided that each edge of positive cost has its head in T. Without this latter restriction, even the special case $k = 1$ involves the NP-complete problem of directed Steiner-trees.

The key idea behind our approach was that earlier results on supermodular set functions could be extended to those on supermodular bi-set functions.

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