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Reliable Orientations of Eulerian Graphs

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Abstract

We present a characterization of Eulerian graphs that have a k-arc-connected orientation so that the deletion of any vertex results in a $(k-1)$-arc-connected directed graph. This provides an affirmative answer for a conjecture of Frank [2]. The special case, when $k = 2$, describes Eulerian graphs admitting 2-vertex-connected orientations. This case was proved earlier by Berg and Jordán [1]. These results are specializations of the related results from [5].

This paper concerns orientations of undirected graphs. We denote an undirected graph by $G = (V, E)$ and a directed graph by $\vec{G} = (V, A)$. Multiple edges are allowed, but loops are not. For an undirected graph G, a set $X \subseteq V$ and $u, v \in V$, let $\boldsymbol{d_G(X)}= |\{xy \in E : x \in X,\ y \in V - X\}|$, $\boldsymbol{T_G}= \{v \in V : d_G(v) \text{ is odd}\}$, $\boldsymbol{\lambda_G(u, v)}= \min\{d_G(Y) : u \in Y,\ v \notin Y\}$. For a directed graph $\vec{G}$, a set $X \subseteq V$ and $u, v \in V$, let $\boldsymbol{\delta_{\vec{G}}(X)}= |\{xy \in A : x \in X,\ y \in V - X\}|$, $\boldsymbol{\varrho_{\vec{G}}(X)}= \delta_{\vec{G}}(V - X)$, $\boldsymbol{\lambda_{\vec{G}}(u, v)}= \min\{\delta_{\vec{G}}(Y) : u \in Y,\ v \notin Y\}$.

An undirected graph $G = (V, E)$ is called k**-edge-connected** if $\lambda_G(u, v) \geq k$ for all $u, v \in V$. A directed graph $D = (V, A)$ is called k**-arc-connected** if $\lambda_{\vec{G}}(u, v) \geq k$ for all $u, v \in V$, and D is called k**-vertex-connected** if $|V| > k$ and $G - X$ is 1-arc-connected for all $X \subseteq V$ with $|X| < k$. Throughout the paper we assume $k \geq 1$.

The starting point is the weak orientation theorem of Nash-Williams.

Theorem 1. *[8] A graph G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.*

A **pairing** M of G is a new graph on vertex set T_G in which each vertex has degree one. A pairing M of G is called k**-feasible** if

$$d_M(X) \leq d_G(X) - 2k \quad \text{for all } X \subset V,\ X \neq \emptyset. \tag{1}$$

The following claim is straightforward using the fact, that in any Eulerian orientation $\vec{G}$ of an Eulerian graph $G = (V, E)$, $\delta_{\vec{G}}(X) = \varrho_{\vec{G}}(X) = d_G(X)/2$ for all $X \subseteq V$.

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Claim 1. *If M is a k-feasible pairing of the $2k$-edge-connected graph G then for every Eulerian orientation $\vec{G} + \vec{M}$ of $G + M$, $\vec{G}$ is a k-arc-connected orientation of G.*

The following lemma plays a central role in this paper.

Lemma 1. *Every $2k$-edge-connected graph $G = (V, E)$ has a k-feasible pairing.*

Proof. We prove the statement by induction on $|E|$, if $E = \emptyset$ then the statement is trivial.
Case 1 There is a vertex $s \in V$ with $d(s)$ even. Then, by Lovász' splitting off theorem [6], there exists an edge-pairing $\{u_i s, s v_i\}_{i=1}^{d(s)/2}$ at s, such that replacing each non-parallel pair $u_i s, s v_i$ by a new edge $u_i v_i$ and then deleting the vertex s, the new graph G' is $2k$-edge-connected. Note that $T_{G'} = T_G$ and $|E(G')| < |E|$ so by induction there is a pairing M of G' that satisfies (1) for G'. Then M is a pairing of G and, since $d_{G'}(X) \leq d_G(X)$ for all $X \subset V - \{s\}$, clearly M satisfies (1) for G as well and we are done.
Case 2 Otherwise, $T_G = V$. By a result of Mader [7], since there is no vertex v with $d(v) = 2k$, there exists an edge $uv \in E$ such that $G' := G - uv$ is $2k$-edge-connected. Note that $T_{G'} = T_G - \{u, v\}$ and $|E(G')| < |E|$ so by induction there is a pairing M' of G' so that (1) is satisfied for G' and M'. Let $M := M' \cup \{uv\}$. Then M is a pairing of G and for all $\emptyset \neq X \subset V$ either $d_M(X) = d_{M'}(X) \leq d_{G'}(X) - 2k = d_G(X) - 2k$ or $d_M(X) = d_{M'}(X) + 1 \leq d_{G'}(X) + 1 - 2k = d_G(X) - 2k$, so (1) is satisfied for G and M and this completes the proof. □

For an Eulerian graph G, an **edge-pairing** at vertex v is an arbitrary partition of the edges incident to v into pairs. Suppose that we are given an edge-pairing at each vertex. An Eulerian orientation is called **admissible** if at each vertex every prescribed edge-pair becomes a directed path. The following statement is an easy exercise.

Proposition 1. *We are given an Eulerian graph G and an edge-pairing at every vertex. Then there exists an admissible Eulerian orientation of G.*

The aim of this paper is to prove the following result.

Theorem 2. *Suppose that $k \geq 2$. An Eulerian graph $G = (V, E)$ has a k-arc-connected orientation $\vec{G}$ such that $\vec{G} - v$ is $(k-1)$-arc-connected for all $v \in V$ if and only if G is $2k$-edge-connected and $G - v$ is $(2k-2)$-edge-connected for all $v \in V$.*

Proof. We define an edge-pairing for all $v \in V$ as follows. Take a maximum number of disjoint pairs of parallel edges incident to v. Since G is Eulerian, the other edges from v go to T_{G-v}. These edges can be naturally paired, defined by a $(k-1)$-feasible pairing M_v of $G - v$, where M_v exists by Lemma 1. By Proposition 1 there is an admissible Eulerian orientation $\vec{G}$ of G. Then $\vec{G}$ is a k-arc-connected orientation of G. Let $\vec{M}_v$ be the natural orientation of M_v (for all $v \in V$) defined by $\vec{G}$. Now $\vec{G} - v + \vec{M}_v$ is an Eulerian orientation of $G - v + M_v$, so by Claim 1, $\vec{G} - v$ is a $(k-1)$-arc-connected orientation of $G - v$ for all $v \in V$. The necessity of the conditions is straightforward. □

The statement of Theorem 2 is not necessarily true for non-Eulerian graphs, as an example, consider the graph obtained from K_4 by replacing each edge by three parallel edges.

We continue by a conjecture on vertex-connectivity orientation (see in [3]).

Conjecture 1. *Let $G = (V, E)$ be an undirected graph with $|V| > k$. Then G has a k-vertex-connected orientation if and only if for all $X \subset V$ with $|X| < k$, $G - X$ is $(2k - 2|X|)$-edge-connected.*

The following result, that is equivalent to Theorem 2, was conjectured by Frank in [2] as a specialization of Conjecture 1.

Corollary 1. *An Eulerian graph $G = (V, E)$ has an Eulerian orientation $\vec{G}$ such that $\vec{G} - v$ is k-arc-connected for all $v \in V$ if and only if $G - v$ is $2k$-edge-connected for all $v \in V$.*

For the special case of Conjecture 1 when the graph is Eulerian and $k = 2$, Berg and Jordán [1] provided a sophisticated proof. Their result below follows immediately from Theorem 2.

Corollary 2. *[1] Let $G = (V, E)$ be a 4-edge-connected Eulerian graph such that $|V| \geq 3$ and $G - v$ is 2-edge-connected for all $v \in V$. Then G has a 2-vertex-connected Eulerian orientation.*

References

[1] A. R. Berg, T. Jordán, Two-connected orientations of Eulerian Graphs, to appear in J. Graph Theory. Egres Technical Reports, TR-2004-03, (2004) `http://www.cs.elte.hu/egres/www/tr-04-03.html`

[2] Egres Open Problems, Problem 4, `http://www.cs.elte.hu/egres/problems/prob_04`

[3] A. Frank, Connectivity and network flows. Handbook of Combinatorics, Vol.1, Elsevier, Amsterdam, (1995) 111-177.

[4] Z. Király, Z. Szigeti, Notes on well-balanced orientations, Egres Technical Reports, TR-2004-10, (2004) `http://www.cs.elte.hu/egres/www/tr-04-10.html`

[5] Z. Király, Z. Szigeti, Simultaneous well-balanced orientations of graphs, JCT B, to appear.

[6] L. Lovász, Combinatorial problems and exercises, North-Holland, Amsterdam, (1979).

[7] W. Mader, Minimale n-fach kantenzusammenhängende Graphen, Math. Ann. 191, (1971) 21-28.

[8] C. St. J. A. Nash-Williams, On orientations, connectivity and odd-vertex-pairings in finite graphs, Canad. J. Math. 12, (1960) 555-567.