Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2005-05. Published by the Egrerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

A note on $[k, l]$-sparse graphs

Zsolt Fekete and László Szegő

April 2005

A note on $[k, l]$-sparse graphs

Zsolt Fekete⋆ and László Szegő⋆⋆

Abstract

In this note we provide a Henneberg-type constructive characterization theorem of $[k, l]$-sparse graphs, that is, the graphs for which the number of induced edges in any subset X of nodes is at most $k|X| - l$. We consider the case $0 \leq l \leq k$.

1 Introduction

Constructive characterization theorems play an important role in the theory of combinatorial rigidity. Constructions serve as useful inductive tools for proving theorems. Tay characterized rigidity of several structures using construction theorems: bar-and-body structures [11], $(n - 2, 2)$-frameworks and identified body-and-hinge structures [9]. A deep result of Tibor Jordán and Bill Jackson [5] was a constructive characterization theorem concerning a connectivity and sparsity conditions (see also [1]) and this was the key of the characterization of global rigidity in the plane.

We will consider graphs satisfying sparsity condition arose in Whiteley's investigation of rigidity on surfaces [13]. We give a constructive characterization theorem of these graphs.

In this paper we consider undirected graphs and we allow parallel edges and loops. Let $G = (V, E)$ be a graph. If $u, v \in V$ and $e \in E$, then $e = uv$ denotes that edge e has end-nodes u and v (there may be other edges parallel to e).

For a subset $X \subseteq V$, $\gamma_G(X)$ denotes the number of induced edges in X, i.e. $\gamma_G(X) := |\{e \in E : e = uv \text{ where } u, v \in X\}|$. If $v \in V$, then $\gamma_G(v) := \gamma_G(\{v\})$ is the number of loops on v. If $X, Y \subseteq V$, then $d_G(X, Y) := |\{e \in E : e = uv \text{ where } u \in X - Y, v \in Y - X\}|$. $d_G(X) := d_G(X, V - X)$. For a node $v \in V$, $d_G(v)$ will denote the degree of

⋆Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, Budapest, Hungary, H-1117 and Communication Networks Laboratory, Pázmány sétány 1/A, Budapest, Hungary, H-1117. e-mail: `fezso@cs.elte.hu`. The author is a member of the Egerváry Research Group (EGRES). Supported by the Hungarian National Foundation for Scientific Research, OTKA T037547 and TS049788 ,by European MCRTN Adonet, Contract Grant No. 504438.

⋆⋆Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, Budapest, Hungary, H-1117 and Department of Combinatorics & Optimization, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1. e-mail: `eszel@cs.elte.hu`. The author is a member of the Egerváry Research Group (EGRES). Supported by the Hungarian National Foundation for Scientific Research, OTKA T037547.

v, that is, $d_G(v) := d_G(\{v\}, V - \{v\}) + 2\gamma_G(v)$ (note that a loop contributes 2 to the degree). We omit the index G if it is clear from the context.

Let l, k be integers and $l \leq k$. We say that a graph $G = (V, E)$ is $[k, l]$-*sparse in* Z ($\emptyset \neq Z \subseteq V$) if $\gamma(X) \leq k|X| - l$ holds for every $\emptyset \neq X \subseteq Z$. If $k + 1 \leq l \leq 2k - 1$, then we say that a graph $G = (V, E)$ is $[k, l]$-*sparse in* Z ($\emptyset \neq Z \subseteq V$) if G is loopless and $\gamma(X) \leq k|X| - l$ holds for every $X \subseteq Z, |X| \geq 2$. We say that a graph $G = (V, E)$ is $[k, l]$-*sparse* if $|E| = k|V| - l$ and G is $[k, l]$-sparse in V. Remark that if $l < k$, then there can be (at most $k - l$) loops incident to any node in a $[k, l]$-sparse graph.

Nash-Williams [7] proved the following theorem concerning coverings by trees.

Theorem 1.1 (Nash-Williams). *A graph $G = (V, E)$ is the union of k edge-disjoint forests if and only if G is $[k, k]$-sparse in V.*

A consequence of this theorem that a graph is $[k, k]$-sparse if and only if its edge-set is a disjoint union of k spanning trees. An undirected graph is called k-*tree-connected* if it contains k edge-disjoint spanning trees. Remark that a graph is minimally k-tree-connected if and only if it is $[k, k]$-sparse.

Frank in [2] by observing that a combination of a theorem of Mader and a theorem of Tutte gives rise to the following characterization. (For a direct proof, see Tay [10]).

Theorem 1.2. *An undirected graph $G = (V, E)$ is k-tree-connected if and only if G can be built from a single node by the following three operations:*

1. *add a new edge,*
2. *add a new node z and k new edges ending at z,*
3. *pinch i ($1 \leq i \leq k - 1$) existing edges with a new node z, and add $k - i$ new edges connecting z with existing nodes.*

Two variants of the notion of k-tree-connectivity were considered by Frank and Szegő in [3]. One of them is the following: a loopless graph G (with at least 2 nodes) is called *nearly k-tree-connected* if G is not k-tree-connected but adding any new edge to G results in a k-tree-connected graph. It is easy to see that a graph is nearly k-tree-connected if and only if it is $[k, k + 1]$-sparse.

Let K_2^{k-1} denote the graph on two nodes with $k - 1$ parallel edges. Based on the work of Henneberg [4] and Laman [6], Tay and Whiteley gave a proof of the following theorem in the special case of $k = 2$ in [12].

Theorem 1.3 (Frank and Szegő). *An undirected graph $G = (V, E)$ is nearly k-tree-connected if and only if G can be built from K_2^{k-1} by applying the following operations:*

1. *add a new node z and k new edges ending at z so that no k parallel edges can arise,*
2. *choose a subset F of i existing edges ($1 \leq i \leq k - 1$), pinch the elements of F with a new node z, and add $k - i$ new edges connecting z with other nodes so that there are no k parallel edges in the resulting graph.*

In [11] Tay proved for inductive reasons that a node of degree at most $2k-1$ either can be “split off”, or “reduced” to obtain a smaller nearly k-tree-connected graph. Theorem 1.3 says that there always exists a node which can be “split off”. The following proposition follows easily from the definition of $[k,l]$-sparse graphs.

Proposition 1.4. *Let $k+1 \le l \le \frac{3k}{2}$. If an undirected graph $G=(V,E)$ can be built up from a single node by applying the following operations, then it is $[k,l]$-sparse.*

(P1) *add a new node z and at most k new edges ending at z so that no $2k-l+1$ parallel edges can arise.*

(P2) *Choose a subset F of i existing edges ($1 \le i \le k-1$), pinch the elements of F with a new node z, and add $k-i$ new edges connecting z with other nodes so that there are no $2k-l+1$ parallel edges in the resulting graph.*

Inspiring by Theorem 1.3 we would conjecture that the reverse of the proposition above is also true for all k and l satisfying $k+1 \le l \le \frac{3k}{2}$. But as it was shown in [8], this is not true if $k+\frac{k+2}{3} \le l$, still we think the following holds.

Conjecture 1.5. *Let $k+1 \le l < k+\frac{k+2}{3}$. An undirected graph $G=(V,E)$ is $[k,l]$-sparse if and only if G can be built from a single node by applying the operations* (P1) *and* (P2)*.*

In this paper we consider a class of graphs that related to packing of trees and pseudotrees. A pseudotree is a set of edges which is connected and contains exactly one cycle. Now we show how they are related to $[k,l]$-sparse graphs, where $0 \le l \le k$.

The maximal edge-sets on vertex set V not containing a cycle form the base-set of a matroid which is called the *cycle* matroid. The maximal edge-sets B on vertex set V containing at most one cycle in every connected components of (V,B) form the base-set of a matroid which is called the *bicycle* matroid. (We note that loops and parallel edges are allowed and a loop is a cycle of length one and two parallel edges form a cycle of length two.)

It is easy to check the following.

Claim 1.6. *An edge-set F on vertex-set V is independent in the cycle matroid if and only if (V,F) is $[1,1]$-sparse.*

An edge-set F on vertex-set V is independent in the bicycle matroid if and only if (V,F) is $[1,0]$-sparse.

Whitely [13] proved the following characterization.

Theorem 1.7 (Whiteley). *If $G=(V,E)$ is a graph and $0 \le l \le k$, then the following are equivalent.*

1. *G is $[k,l]$-sparse,*

2. *E is the disjoint union of l bases of the cycle matroid and $(k-l)$ bases of the bicycle matroid.*

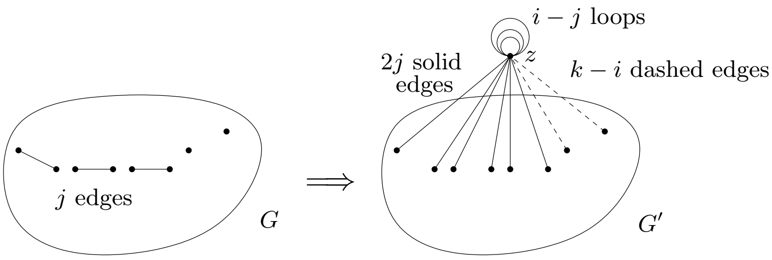

Figure 1: G' is obtained from G by operation $K(k,i,j)$.

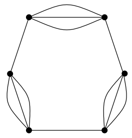

Figure 2: A loopless $[2,0]$-sparse graph, which cannot be obtained by a sequence of loopless $[2,0]$-sparse graphs.

We will prove a Henneberg-type construction of $[k,l]$-sparse graphs for $0 \leq l \leq k$. We will use the following operations.

Definition 1.8. Let $0 \leq j \leq i \leq k$. $K(k,i,j)$ will denote the following operation. Choose j edges of G, subdivide each of them by a new node and identify these nodes to a new node z. Put $i-j$ loops on z and link it with other nodes by $k-i$ new edges. (This operation results in a graph with k more edges than the original graph and the new node z has degree $(k+i)$. See Figure 1.)

The graph on one node with l loops will be denoted by P_l. Our main result is the following theorem.

Theorem 1.9. *Let $G=(V,E)$ be a graph and $1 \leq l \leq k$. Then G is a $[k,l]$-sparse graph if and only if G can be obtained from P_{k-l} by operations $K(k,i,j)$ where $0 \leq j \leq i \leq k-1, i-j \leq k-l$.*

Let $G=(V,E)$ be a graph and $l=0$. Then G is a $[k,l]$-sparse graph if and only if G can be constructed from P_{k-l} by operations $K(k,i,j)$ where $0 \leq j \leq i \leq k, i-j \leq k-l$.

Notice that operations $K(k,i,j)$ for $j \leq i = k$ must be allowed in case $l=0$ while the construction works without them in the other cases. We remark that loopless $[k,l]$-sparse graphs cannot be obtained by operations above via a sequence of loopless $[k,l]$-sparse graphs. (See Figure 2 for an example.)

In Section 3 we give some further operations which may lead to other characterizations for negative values of l and for $l > k$ or alternative characterizations to the existing ones.

2 Proof of Theorem 1.9

The if part of Theorem 1.9 is the following.

Lemma 2.1. *Let $0 \leq l \leq k$. If graph G is obtained from P_{k-l} by operations $K(k,i,j)$ where $0 \leq j \leq i \leq k,\ i-j \leq k-l$, then it is a $[k,l]$-sparse graph.*

Proof. This can be seen directly from the definition. □

We will need the following claim, which is a consequence of equality $\sum_{v\in V} d(v) = 2|E| = 2(k|V|-l)$ and the inequality $d(v) = |E| - \gamma(V-v) + \gamma(v) \geq k|V| - l - (k|V-v| - l) = k$.

Claim 2.2. *Let $G = (V,E)$ be a $[k,l]$-sparse graph.*

1. *If $0 < l \leq k$ and $|V| \geq 2$, then $\exists v \in V$ such that $k \leq d(v) \leq 2k-1$.*
2. *If $l = 0$ and $|V| \geq 2$, then $\exists v \in V$ such that $k \leq d(v) \leq 2k$.*
3. *If $l < 0$ and $|V| \geq 2|l|+1$, then $\exists v \in V$ such that $k \leq d(v) \leq 2k$.*

Let $e = sv, f = sw \in E, v \neq s, w \neq s$. *Splitting off* the pair of edges e and f means the following: delete e and f and add a new edge $g = vw$, i.e. we get by splitting off edges e, f graph $G^{ef} = (V, E - e - f + g)$. We say that the new edge g is a *split edge.*

The following will give the only if part of Theorem 1.9.

Theorem 2.3. *Let $0 \leq l \leq k$. Let $G = (V+s,E)$ be a $[k,l]$-sparse graph and $d(s) = k+i, \gamma(s) = i-j$ where $0 \leq j \leq i \leq k, i-j \leq k-l$. Then we can split off j pairs of edges incident to s so that after deleting s the remaining graph is $[k,l]$-sparse.*

Let $b(X) = b_G(X) := k|X| - l - \gamma_G(X)$. We remark that a graph $G = (V,E)$ is $[k,l]$-sparse in V if and only if $b_G(Z) \geq 0$ for all $\emptyset \neq Z \subseteq V$. If $G = (V+s,E)$ is $[k,l]$-sparse in V and $e = sv, f = sw \in E$, then splitting off e and f is called *admissible* if G^{ef} is $[k,l]$-sparse in V. We will frequently use the following simple lemma.

Lemma 2.4. *Let $G = (V,E)$ be a graph and $X, Y \subseteq V$. Then the following hold.*

1. *$\gamma(X) + \gamma(Y) + d(X,Y) = \gamma(X \cap Y) + \gamma(X \cup Y)$.*
2. *$b(X) + b(Y) = b(X \cap Y) + b(X \cup Y) + d(X,Y)$.*
3. *Let $l \leq k$. If G is $[k,l]$-sparse in V, then $b(X) = b(Y) = 0, X \cap Y \neq \emptyset$ implies $b(X \cup Y) = b(X \cap Y) = 0$.*
4. *Let $l \leq 2k-1$. If G is $[k,l]$-sparse in V, then $b(X) = b(Y) = 0, |X \cap Y| \geq 2$ implies $b(X \cup Y) = b(X \cap Y) = 0$.*
5. *If $G = (V+s,E)$ is $[k,l]$-sparse in V and $e = sv, f = sw$ are edges incident to s $(v,w \in V)$, then G^{ef} is obtained by an admissible splitting off if and only if $\nexists X \subseteq V$ such that $v,w \in X$ and $b_G(X) = 0$.*

Proof. 1. Easy to check that the contribution of each edge is the same to both side.

2. A consequence of 1.

3. We know that $b(Z) \geq 0$ for all $\emptyset \neq Z \subseteq V$. $0 + 0 = b(X) + b(Y) = b(X \cap Y) + b(X \cup Y) + d(X, Y) \geq 0 + 0 + 0$, so equality holds.

4. We know that $b(Z) \geq 0$ for all $Z \subseteq V$, $|Z| \geq 2$. $0 + 0 = b(X) + b(Y) = b(X \cap Y) + b(X \cup Y) + d(X, Y) \geq 0 + 0 + 0$, so equality holds.

5. The claim follows from the fact that a graph G is $[k, l]$-sparse in V if and only if $b_G(X) \geq 0$ for all $\emptyset \neq X \subseteq V$ and

$$b_{G^{ef}}(X) = \begin{cases} b_G(X) & \text{if } v \text{ or } w \text{ is not in } X, \\ b_G(X) - 1 & \text{if } v, w \in X. \end{cases}$$

□

Proof of Theorem 2.3. Assume on the contrary that we cannot split off j pairs of edges so that the resulting graph is $[k, l]$-sparse in V. Split off as many pairs as possible. We split off say $m < j$ pairs of edges and denote the resulting graph by G'. Let $e_1 = sv_1, \ldots, e_\alpha = sv_\alpha$ be the non-loop edges incident to s in G' where $\alpha = k + i - 2(i - j) - 2m = k - i + 2j - 2m \geq 2$. By Lemma 2.4 we know that for every v_ν, v_μ $(1 \leq \nu < \mu \leq \alpha)$ there exists an $X_{\nu\mu} \subseteq V$ such that $v_\nu, v_\mu \in X_{\nu\mu}$ and $b_{G'}(X_{\nu\mu}) = 0$. Using the second statement of Lemma 2.4 we get that there exists an $X \subseteq V$ such that $v_\nu \in X$ for every ν and $b_{G'}(X) = 0$. Let $X_{G'}$ be a maximal set having these properties.

Now consider every G' which can be obtained by splitting off m pairs of edges at s in G. For each G' we have a set $X_{G'}$. Choose $G_1 := G'$ so that $|X_{G'}|$ is maximal. Let $X := X_{G_1}$.

Claim 2.5. *There is a split edge $e = vw$ in G_1 such that $v, w \notin X$.*

Proof. Assume on the contrary that for every split edge $e = vw$, $v \in X$ or $w \in X$. Let $\beta := |\{e : e = vw \text{ is a split edge and } v, w \in X\}|$. $b_{G_1}(X) = 0$ implies $b_G(X) = \beta$. $b_G(X+s) = b_G(X)+k-\gamma_G(s)-d_G(s, X) = b_G(X)+k-(i-j)-(k-i+2j-(m-\beta)) = \beta + k - i + j - (k - i + 2j - m + \beta) = \beta + k - i + j - k + i - 2j + m - \beta = m - j < 0$. A contradiction. □

Let $e = vw$ be an edge given by the claim. Let $G_2 := G_1 - e + sv + sw$. We state that sv, sv_1 is an admissible splitting off in G_2. Because if $v, v_1 \in Y \subseteq V$ and $b_{G_2}(Y) = 0$, then $b_{G_1}(Y) \leq b_{G_2}(Y) = 0$ so $b_{G_1}(Y) = 0$. But $X \cap Y \neq \emptyset$ (since $v_1 \in X \cap Y$) hence $b_{G_1}(X \cup Y) = 0$ by Lemma 2.4, which contradicts the maximality of $|X_{G_1}|$.

Let $G_3 := G_2 - sv - sv_1 + vv_1$. We state that sw, sv_2 is an admissible splitting off in G_3. Assume on the contrary that $w, v_2 \in Z \subseteq V$ and $b_{G_3}(Z) = 0$.

If $v \notin Z$ or $v_1 \notin Z$, then $b_{G_1}(Z) \leq b_{G_2}(Z) = b_{G_3}(Z) = 0$ so $b_{G_1}(Z) = 0$. But $X \cap Z \neq \emptyset$ (since $v_2 \in X \cap Z$) hence $b_{G_1}(X \cup Z) = 0$, which contradicts the maximality of $|X_{G_1}|$.

If $v, v_1 \in Z$, then $b_{G_1}(Z) = b_{G_2}(Z) - 1 = b_{G_3}(Z) = 0$ so $b_{G_1}(Z) = 0$. But $X \cap Z \neq \emptyset$ (since $v_2 \in X \cap Z$) hence $b_{G_1}(X \cup Z) = 0$ but this contradicts the maximality of $|X_{G_1}|$.

We proved that sw, sv_2 is an admissible splitting off in G_3. This contradicts the maximality of m. □

Proof of Theorem 1.9. Lemma 2.1 shows the "if" part. To prove the "only if" we observe that the only $[k,l]$-sparse graph with one node is P_{k-l}. Let G be an arbitrary graph on at least two nodes. By Lemma 2.2 there exists a node s of degree at most $2k-1$ if $l>0$ or a node of degree at most $2k$ if $l=0$. Theorem 2.3 states that G is obtained from a graph G' by an operation $K(k,i,j)$. By induction we know that G' can be constructed from P_{k-l}, this implies that G can be constructed from P_{k-l} too. □

3 Partial results for other l values

In this section k,l will be integers and $k \geq 1$, but l can be negative. First we remark that Theorem 2.3 remains true without assumption $l \geq 0$ (the proof is the same). Thus for $l<0$ the following version of Theorem 1.9 follows (using 3. of Claim 2.2).

Theorem 3.1. *Let $G=(V,E)$ be a graph and $l<0<k$. Then G is a $[k,l]$-sparse graph if and only if G can be obtained from a $[k,l]$-sparse graph on at most $2|l|$ vertices by operations $K(k,i,j)$ where $0 \leq j \leq i \leq k, i-j \leq k-l$.*

In the rest of this section we give three simple operations on $[k,l]$-sparse graphs which result in smaller $[k,l]$-sparse graphs, but the inverse operations do not necessarily preserve the property in question.

For a graph $G=(V,E)$ let $X \subseteq V$, then G/X denotes the graph obtained by identifying the nodes in X into a new single node. That is, we contract X into a new node and we do not delete the loops arising.

Proposition 3.2. *Let $l \leq k$. Let $G=(V+s+t,E)$ be a $[k,l]$-sparse graph and $\gamma(\{s,t\}) \geq k$. If we delete k loops on z from $G/\{s,t\}$ where z is the new vertex obtained by contracting $\{s,t\}$, then we get a $[k,l]$-sparse graph.*

Proof. It is obvious by the definition of $[k,l]$-sparse graphs. □

Theorem 3.3. *Let $l \leq 2k-1$. If $G=(V+s+t,E)$ is a $[k,l]$-sparse graph and $\gamma(\{s,t\}) \leq k$, then we can delete k non-loop edges from $G/\{s,t\}$ incident to z (where z is the vertex which is obtained from $\{s,t\}$) such that we get a $[k,l]$-sparse graph.*

Proof. We will prove the following claim by induction on j.

Claim 3.4. *Let $0 \leq j \leq k$. We can delete j edges from G incident to s or t such that $\gamma_{G'}(X) \leq k|X|-l-j$ holds for the resulting graph G' for every $s,t \in X \subseteq V$.*

Proof. If $j=0$, then it is trivial. Suppose that G' is obtained by deleting $j-1$ edges from G incident to s or t and $\gamma_{G'}(X) \leq k|X|-l-j+1$ holds for every $s,t \in X \subseteq V$. We shall prove that we can delete one more edge.

We call a set X tight if $s,t \in X \subseteq V$ and $\gamma_{G'}(X) = k|X|-l-j+1$. If there does not exist any tight set, then we can delete any edge. If there exists a tight set, then let $P_{\min}$ be the intersection of tight sets. $P_{\min}$ is tight by Lemma 2.4. But $\gamma_{G'}(P_{\min}) = \gamma_{G'}(P_{\min} - s - t) + \gamma_{G'}(\{s,t\}) + d_{G'}(\{s,t\}, P_{\min}) = \gamma_G(P_{\min} - s - t) + \gamma_G(\{s,t\}) + d_{G'}(\{s,t\}, P_{\min}) \leq k|P_{\min}| - 2k - l + k + d_{G'}(\{s,t\}, P_{\min})$, so $k|P_{\min}| - l - j + 1 \leq k|P_{\min}| - k - l + d_{G'}(\{s,t\}, P_{\min})$, hence $d_{G'}(\{s,t\}, P_{\min}) \geq k - j + 1 \geq 1$. Thus there exists an edge between $\{s,t\}$ and $P_{\min}$. We can delete that. □

For $j = k$ we get the statement. □

At last after a lemma we give a weaker form of Theorem 2.3, which is true for more values of k and l.

Lemma 3.5. *Assume $l \leq \frac{3k}{2}$ and $G = (V, E)$ is $[k,l]$-sparse. Let $X, Y, Z \subseteq V$. If $b(X) = b(Y) = b(Z) = 0$ and $|X \cap Y| = |X \cap Z| = |Y \cap Z| = 1$, $|X \cap Y \cap Z| = 0$, then $b(X \cup Y \cup Z) = 0$ and $l = \frac{3k}{2}$.*

Proof. $0 \leq b(X \cup Y \cup Z) = k|X \cup Y \cup Z| - l - \gamma(X \cup Y \cup Z) \leq k(|X| + |Y| + |Z| - 3) - l - \gamma(X) - \gamma(Y) - \gamma(Z) = k|X| - l - \gamma(X) + k|Y| - l - \gamma(Y) + k|Y| - l - \gamma(Y) - 3k + 2l = b(X) + b(Y) + b(Z) - 3k + 2l = 2l - 3k \leq 0$. □

Theorem 3.6. *Let $l \leq \frac{3k}{2}$. Let $G = (V + s, E)$ be a $[k,l]$-sparse graph and $d(s) = k + i, \gamma(s) = i - j$ where $0 \leq j \leq i \leq k, i - j \leq l$. Then there exist a j-element edge-set F on the neighbors of s such that $(G - s) + F$ is a $[k,l]$-sparse graph.*

Proof. Let $N \subseteq V$ denote the set of the neighbors of s. If $N \subseteq X \subseteq V$, then $\gamma(s) = i - j, d(s) = k + i$ and $\gamma(X + s) \leq k|X + s| - l$ implies that $b_G(X) \geq j$. $(\gamma_G(X) = \gamma_G(X + s) - \gamma_G(s) - d_G(s, X) \leq k(|X| + 1) - l - (i - j) - (d_G(s) - 2(i - j)) = k|X| + k - l - i + j - (k + i - 2i + 2j) = k|X| - l - j.)$

We prove the following claim by induction on ν.

Claim 3.7. *For every $0 \leq \nu \leq j$ there exists a ν-element edge-set F_ν on N such that $(G - s) + F_\nu$ is $[k,l]$-sparse in V.*

Proof. If $\nu = 0$, then it is trivial. Suppose that there is a $(\nu - 1)$-element edge-set $F_{\nu-1}$, such that $\gamma_{G+F_{\nu-1}}(X) \leq k|X| - l$ for all $\emptyset \neq X \subseteq V$. Now we prove that we can add one more edge.

Suppose on the contrary that for every $uv \in E$, $u, v \in N$ there exists an X_{uv} such that $u, v \in X_{uv}$: $\gamma_{G+F_{\nu-1}}(X_{uv}) = k|X_{uv}| - l$, i.e. $b_{G+F_{\nu-1}}(X_{uv}) = 0$. We claim that there exist a set X, such that $N \subseteq X \subseteq V$ and $b_{G+F_{\nu-1}}(X) = 0$. If $|N| = 1$, then $X := X_{uu}$ (where $N = \{u\}$) is appropriate. If $|N| \geq 2$, then let $u, w \in N, u \neq w$ and let $X \subseteq V$ be a maximal set satisfying $X_{uw} \subseteq X$ and $b_{G+F_{\nu-1}}(X) = 0$. We claim that $N \subseteq X$. Suppose that $v \in N - X$. If $|X_{vu} \cap X| \geq 2$ or $|X_{vw} \cap X| \geq 2$, then X cannot be maximal by Lemma 2.4. If $|X_{vu} \cap X_{vw}| \geq 2$, then $b_{G+F_{\nu-1}}(X_{vu} \cup X_{vw}) = 0$ and $|(X_{vu} \cup X_{vw}) \cap X| = |\{u, w\}| = 2$ implies $b_{G+F_{\nu-1}}(X_{vu} \cup X_{vw} \cup X) = 0$, this contradicts the maximality of X.

But then we have $|X_{vu} \cap X| = |X_{vw} \cap X| = |X_{vu} \cap X_{vw}| = 1$ and by Lemma 3.5 $b_{G+F_{\nu-1}}(X_{vu} \cup X_{vw} \cup X) = 0$ contradicting the maximality of X.

Now we have $0 = b_{G+F_{\nu-1}}(X) = b_G(X) - (\nu - 1) \geq b_G(X) - (j - 1)$ contradicting the remark at the beginning of the proof, which said $b_G(X) \geq j$. □

For $l = j$ we get the statement. □

References

[1] Alex R. Berg and Tibor Jordán. Algorithms for graph rigidity and scene analysis. In *Algorithms—ESA 2003*, volume 2832 of *Lecture Notes in Comput. Sci.*, pages 78–89. Springer, Berlin, 2003.

[2] András Frank. Connectivity and network flows. In M. Grötschel R. Graham and L. Lovász, editors, *Handbook of combinatorics, Vol. 1, 2*, pages 111–177. Elsevier, Amsterdam, 1995.

[3] András Frank and László Szegő. Constructive characterizations for packing and covering with trees. *Discrete Appl. Math.*, 131(2):347–371, 2003.

[4] L. Henneberg. Die graphische statik der starren systeme. 1911. Leipzig.

[5] Bill Jackson and Tibor Jordán. Connected rigidity matroids and unique realizations of graphs. *J. Combin. Theory Ser. B.* in press.

[6] G. Laman. On graphs and rigidity of plane skeletal structures. *J. Engrg. Math.*, 4:331–340, 1970.

[7] C. St. J. A. Nash-Williams. Decomposition of finite graphs into forests. *J. London Math. Soc.*, 39:12, 1964.

[8] László Szegő. On constructive characterizations of (k, l)-sparse graphs, 2003. EGRES Technical Riport TR-2003-10, `www.cs.elte.hu/egres`.

[9] Tiong-Seng Tay. Linking $(n-2)$-dimensional panels in n-space. II. $(n-2, 2)$-frameworks and body and hinge structures. *Graphs Combin.*, 5(3):245–273, 1989.

[10] Tiong-Seng Tay. Henneberg's method for bar and body frameworks. *Structural Topology*, 17:53–58, 1991.

[11] Tiong-Seng Tay. Linking $(n-2)$-dimensional panels in n-space. I. $(k-1, k)$-graphs and $(k-1, k)$-frames. *Graphs Combin.*, 7(3):289–304, 1991.

[12] Tiong-Seng Tay and Walter Whiteley. Generating isostatic frameworks. *Structural Topology*, 11:21–69, 1985.

[13] Walter Whiteley. The union of matroids and the rigidity of frameworks. *SIAM J. Discrete Math.*, 1(2):237–255, 1988.