Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2004-19. Published by the Egrerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

A note on the degree prescribed factor problem

Jácint Szabó

December 2004

A note on the degree prescribed factor problem

Jácint Szabó*

Abstract

The degree prescribed factor problem is to decide if a graph has a subgraph satisfying given degree prescriptions at each vertex. Lovász, and later Cornuéjols, gave structural descriptions on this problem in case the prescriptions have no two consecutive gaps. We state the Edmonds-Gallai-type structure theorem of Cornuéjols which is only implicit in his paper. In these results the difficulty of checking the property of criticality is near to the original problem. By extending a result of Loebl, we prove that a degree prescription can be reduced to the edge and factor-critical graph packing problem by a 'gadget' if and only if all of its gaps have the same parity. With this gadget technique it is possible to obtain a description of the critical components. Finally, we prove two matroidal results. First, the up hulls of the distance vectors of all subgraphs form a contra-polymatroid. Second, we prove that the vertex sets coverable by subgraphs F satisfying the degree prescriptions for all $v \in V(F)$ form a matroid, in case 1 is contained in all prescriptions.

Keywords: degree prescribed factor problem

1 Introduction

The $\mathcal{H}$*-factor problem* is the following. Let G be an undirected graph and let $H_v \subseteq \mathbb{N}$ be a degree prescription for each $v \in V(G)$. For a subgraph F of G define $\delta^F(v) = \mathrm{dist}(\deg_F(v), H_v)$ where $\mathrm{dist}(I, J) = \min\{|i - j| : i \in I, j \in J\}$ for $I, J \subseteq \mathbb{N}$. Let $\delta_F = \sum\{\delta^F(v) : v \in V(G)\}$. The minimum δ_F among the subgraphs F is denoted by $\delta_{\mathcal{H}}(G)$. A subgraph F is called $\mathcal{H}$*-optimal* if $\delta_F = \delta_{\mathcal{H}}(G)$ and it is an $\mathcal{H}$*-factor* if $\delta_F = 0$, i.e. if $\deg_F(v) \in H_v$ for all $v \in V(G)$. The $\mathcal{H}$*-factor problem* is to decide if there exists an $\mathcal{H}$-factor of G, or in general, to determine the value of $\delta_{\mathcal{H}}(G)$. Throughout we assume that $H_v \neq \emptyset$.

Lovász [18, 19] gave a structural description for the $\mathcal{H}$-factor problem in case H_v has no two consecutive gaps for all $v \in V(G)$. An integer h is called a *gap* of $H \subseteq \mathbb{N}$ if

*Dept. of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, Budapest, Hungary H-1117. The author is a member of the Egerváry Research Group (EGRES). Research is supported by OTKA grant T 037547, by European MCRTN Adonet, Contract Grant No. 504438 and by the Egerváry Research Group of the Hungarian Academy of Sciences. e-mail: `jacint@elte.hu`

$h \notin H$ but H contains an element less than h and an element greater than h. Lovász [19] showed that the problem is NP-complete without this restriction. However, the issue of polynomiality remained open. Later, Cornuéjols [3] gave two polynomial algorithms. It is implicit in Cornuéjols [3] that one of these algorithms implies an Edmonds-Gallai-type structure theorem for the $\mathcal{H}$-factor problem, which is very close to the structure theorem of Lovász'. We state this result of Cornuéjols in Section 2, point out the relation with the structure theorem of Lovász, and as an application we deduce an Edmonds-Gallai-type theorem for the $(1, f)$-*odd factor problem*, i.e. when $H_v = \{1, 3, 5, \ldots, f(v)\}$ for an odd value function $f : V(G) \to \mathbb{N}$.

Section 3 extends the work of Loebl [14] by classifying those degree prescriptions which can be reduced by a gadget to the edge and factor-critical graph packing problem of Cornuéjols, Hartvigsen and Pulleyblank [4, 5]. Loebl considered reductions to the edge and triangle packing problem. However, if we allow any factor-critical graphs not just triangles then many new representable prescriptions arise. Namely, a prescription H can be represented if and only if all of its gaps have the same parity. Note that this condition implies that H has no two consecutive gaps. In some way, these prescriptions form the broadest class of the degree prescribed factor problem where parity reasons can be used. Applying this gadget technique, we then characterize the critical components.

Finally, in Section 4 we concern with matroidal properties of the $\mathcal{H}$-factor problem. We prove in three ways that the up-hulls of the vectors $\delta^F \in \mathbb{N}^{V(G)}$ with component $\delta^F(v)$ for $v \in V(G)$ form a contra-polymatroid. Another matroidal result is the following.

Definition 1.1. A subgraph F of G is called an $\mathcal{H}$-*subgraph* if $\deg_F(v) \in H_v$ for all $v \in V(F)$. Let $\mathcal{M}$ consist of those subsets of vertices of G which can be covered by $\mathcal{H}$-subgraphs.

We prove that $\mathcal{M}$ is a matroid in case $1 \in H_v$ and H_v has no two consecutive gaps for all $v \in V(G)$.

In this paper the size of a graph G denotes $|V(G)|$ and $c(G)$ denotes the number of connected components of G.

2 The Edmonds-Gallai decomposition of Cornuéjols

Cornuéjols [3] gave two polynomial algorithms for the $\mathcal{H}$-factor problem, in the case when H_v contains no two consecutive gaps for all $v \in V(G)$. One of these is an Edmonds-type alternating forest algorithm, which implies an Edmonds-Gallai-type structure theorem for the $\mathcal{H}$-factor problem. The existence of such a structure theorem is mentioned in Cornuéjols [3] but it is not stated explicitly. Lovász [19] also described a decomposition for the $\mathcal{H}$-factor problem, which is very close to that of Cornuéjols'. We briefly point out the connection of these two versions in this section. We consider in details only the formulation of Cornuéjols.

Definition 2.1. For $v \in V(G)$ let $l(v) = \min H_v$, $u(v) = \max H_v$ and $H_v^{\downarrow} = \{0, 1, \ldots, u(v)\}$.

Wlog. we assume that $0 \leq l(v) \leq u(v) \leq deg_G(v)$ for all $v \in V(G)$.

Definition 2.2. [3] A graph G with $|V(G)| \geq 2$ is called *$\mathcal{H}$-critical* if G does not have an $\mathcal{H}$-factor but for all $v \in V(G)$ there exists a subgraph K of G with the property that $\deg_K(v) + 1 \in H_v$ and $\deg_K(w) \in H_w$ for all vertices $w \neq v$. Call G *non-trivial* and define $val(G) = 1$. Moreover, G is *H-critical* if $V(G) = \{v\}$ and $l(v) \geq 1$. In this case G is said to be *trivial* and let $val(G) = l(v)$.

Definition 2.3. [3] G_{sub} is the graph what we get from G after subdividing each edge $e = uv$ with two new vertices e_u and e_v (resulting in three new edges ue_u, e_ue_v and e_vv). Let the set of these new vertices be V_E and the degree prescription on the new vertices be $\{1\}$.

It is easy to see that $\delta_{\mathcal{H}}(G) = \delta_{\mathcal{H}}(G_{sub})$. An ingenious idea of Cornuéjols is that his Edmonds-type algorithm should work on the subdivided graph G_{sub}. Thus the Edmonds-Gallai-type theorem, implicit in [3], considers the $\mathcal{H}$-factor problem on G_{sub}. We need to state this result for a slightly more general class of graphs.

Definition 2.4. A simple graph S is called *subdivided* if it is an induced subgraph of a graph of form G_{sub}. Let $S_V = V(S) \cap V(G)$ and $S_E = V(S) \cap V_E$.

Theorem 2.5. (Cornuéjols) *Let S be a subdivided graph. Let $D \subseteq V(S)$ consist of vertices v with the property that there exists an $\mathcal{H}$-optimal subgraph F of S such that $\deg_F(v) \in H_v^{\downarrow} \backslash H_v$. Let A be the set of neighbors of D in S and let $C = V(S) - (D \cup A)$. Let $val(D)$ denote the sum of the values of the H-critical components of $S[D]$. Then*

1. *the components of $S[D]$ are $\mathcal{H}$-critical,*
2. $\delta_{\mathcal{H}}(S) = val(D) - u(A)$,
3. $\sum\{val(K) : K$ *is a component of* $S[D]$ *adjacent to* $A'\} \geq u(A') + 1$ *for all* $\emptyset \neq A' \subseteq A$,
4. *for all $\mathcal{H}$-optimal subgraphs F of S there is no edge of F between A and C and $F[C]$ is an $\mathcal{H}$-factor of $S[C]$.*

For sake of completeness, we briefly describe without proof the relation of Thm. 2.5 with the decomposition formulated by Lovász. This is defined on the original graph G. It consists of four vertex sets.

$$C_L = \{v \in V(G) : \deg_F(v) \in H_v \text{ for all } \mathcal{H}\text{-optimal subgraphs } F \text{ of } G\},$$

$$A_L = \{v \in V(G) \setminus C_L : \deg_F(v) \geq u(v) \text{ for all } \mathcal{H}\text{-optimal subgraphs } F \text{ of } G\},$$

$$B_L = \{v \in V(G) \setminus C_L : \deg_F(v) \leq l(v) \text{ for all } \mathcal{H}\text{-optimal subgraphs } F \text{ of } G\},$$

and finally

$$D_L = V(G) - (A_L \cup B_L \cup C_L).$$

Let $V(G_{sub}) = D \dot\cup A \dot\cup C$ be the decomposition of Thm. 2.5. The relation is as follows. $A_L = V(G) \cap A$, $C_L = V(G) \cap C$,

$$B_L = \{v \in V(G) \colon \{v\} \text{ is a trivial component of } G_{sub}[D]\} \text{ and}$$

$$D_L = \{v \in V(G) \colon v \text{ is contained in a non-trivial component of } G_{sub}[D]\}.$$

Note that if a vertex $v \in A \cap V_E$ is adjacent only to non-trivial components of $G_{sub}[D]$ then it is adjacent to exactly two such components by Thm. 2.5, *3.* On the other hand, it can be proved that if $e = xy$ is a cut edge of a component of $G[D_L]$ then exactly one of e_x and e_y belongs to A. So the maximal 2-edge connected subgraphs of the components of $G[D_L]$ correspond to non-trivial components of $G_{sub}[D]$. We do not go into details. Observe that with the help of the subdivision of the edges of G it is possible to encode the two sets B_L and D_L in one set D.

Another characterization of the decomposition of Thm. 2.5 is that

$$C = \{v \in V(G_{sub}) \colon \deg_F(v) \in H_v \text{ for all } \mathcal{H}\text{-optimal subgraphs } F\},$$

$$A = \{v \in V(G_{sub}) \setminus C \colon \deg_F(v) \geq u(v) \text{ for all } \mathcal{H}\text{-optimal subgraphs } F\}, \text{ and}$$
$$D = V(G_{sub}) - (C \cup A).$$

Thm. 2.5 immediately implies a Berge-type theorem for the $\mathcal{H}$-factor problem in subdivided graphs S. Note that $\geq$ is trivial.

Theorem 2.6. (Cornuéjols) [3]

$$\delta_H(S) = \max_{A \subseteq V(S)} val(S - A) - u(A).$$

Here $val(S - A)$ denotes the sum of the values of the $\mathcal{H}$-critical components of $S - A$.

As an application of Thm. 2.5 we deduce an Edmonds-Gallai-type structure theorem for the $(1, f)$-*odd factor problem.* This was introduced by Amahashi [1] who gave a Tutte-type existence theorem for the $\mathcal{H}$-factor problem in case $H_v = \{1, 3, 5, \ldots, 2k + 1\}$ for some $k \in \mathbb{N}$ for all $v \in V(G)$. Let $f : V(G) \to \mathbb{N}$ be a function with odd values. For the case when $H_v = \{1, 3, 5, \ldots, f(v)\}$, a Tutte-type theorem was proved by Cui and Kano [6], and a Berge-type minimax formula by Kano and Katona [12]. These are generalized by the next theorem.

Theorem 2.7. *Let $H_v = \{1, 3, 5, \ldots, f(v)\}$ for all $v \in V(G)$. Let $D_f \subseteq V(G)$ consist of those vertices v for which there exists an $\mathcal{H}$-optimal subgraph F of G with $\deg_F(v) \in \{0, 2, 4, \ldots, f(v) - 1\}$. Let A_f be the set of neighbors of D_f in G and let $C_f = V(G) - (D_f \cup A_f)$. Then*

1. *the components of $G[D_f]$ have odd size,*
2. *$\delta_{\mathcal{H}}(G) = c(D_f) - f(A_f)$,*
3. *$|\{K : K$ is a component of $G[D_f]$ adjacent to $A'\}| \geq f(A') + 1$ for all $\emptyset \neq A' \subseteq A_f$,*

4. for all $\mathcal{H}$-optimal subgraphs F of G there is no edge of F between A_f and C_f and $F[C_f]$ is an $\mathcal{H}$-factor of $G[C_f]$.

Proof. Let $V(G_{sub}) = D \dot\cup A \dot\cup C$ be the Edmonds-Gallai decomposition of G_{sub} by Thm. 2.5. An $\mathcal{H}$-optimal subgraph F of G with $\deg_F(v) \in \{0, 2, 4, \dots, f(v)-1\}$ gives an $\mathcal{H}$-optimal subgraph F_{sub} of G_{sub} with $\deg_{F_{sub}}(v) \in H_v^{\downarrow} \setminus H_v$. So $D_f \subseteq D \cap V(G)$. On the other hand, if there exists an $\mathcal{H}$-optimal subgraph F_{sub} of G_{sub} with $\deg_{F_{sub}}(v) \in H_v^{\downarrow} \setminus H_v$ for some $v \in V(G)$ then we can choose it to have $\deg_{F_{sub}}(w) = 1$ for all $w \in V_E$. So in fact, $D_f = D \cap V(G)$. By parity reasons, the $\mathcal{H}$-critical components of $G_{sub}[D]$ have odd size. A vertex in $V_E \cap A$ is adjacent to two components of $G_{sub}[D]$ by Thm. 2.5, *3.* Thus the components of $G_{sub}[D \cup (V_E \cap A)]$ have odd size, too. It is clear that the components of $G[D_f]$ are obtained from the components of $G_{sub}[D \cup (V_E \cap A)]$ by contracting each edge of type xe_x. This proves *1.* Properties *2.*, *3.* and *4.* follow from the related properties of Thm. 2.5. □

Observe that if $f \equiv 1$ then the components of $G[D_f]$ are factor-critical by the classical Edmonds-Gallai theorem [7, 9, 10]. However, for general f, these components are only of odd size.

3 The representable prescriptions

Unfortunately, checking $\mathcal{H}$-criticality in Theorems 2.5 and 2.6 is almost as difficult as the original problem. This applies to the related notion of 'criticality' in Lovász' structure theorem as well. Actually, we can check $\mathcal{H}$-criticality by any algorithm of Cornuéjols [3]. However, beside an algorithmic proof, we want a nice description of the $\mathcal{H}$-critical components. Under this we mean checking '$\mathcal{F}$-criticality' in the edge and factor-critical graph packing problem of Cornuéjols, Hartvigsen and Pulleyblank [4, 5]. This task is well-solved and amounts to checking factor-criticality and the existence of perfect matchings. The possibility of producing such characterizations is available for those prescriptions which can be represented by *gadgets*, which are auxiliary graphs reducing the behavior of a prescription to the edge and factor-critical graph packing problem. Using gadgets offers the theoretical possibility of describing the H-critical graphs, and hence of deriving exact Tutte-, Berge- and Edmonds-Gallai-type theorems in the representable cases. In Thm. 3.5 we prove that a prescription can be represented if and only if *all of its gaps have the same parity.* These prescriptions form the broadest class of the degree prescribed factor problem where parity reasons occur, as it will be justified by Thm. 3.14. It also covers the antifactor problem of Lovász [17], when $H_v = [0, \deg_G(v)] \setminus \{g(v)\}$ for some $g : V(G) \to \mathbb{N}$.

We define *the edge and factor-critical graph packing problem* and cite the related Edmonds-Gallai-type structure theorem.

Definition 3.1. Let G be an undirected graph and let $\mathcal{F}$ consist of factor-critical subgraphs of G. A subgraph Q of G is called an *$\mathcal{F}$-packing* if each connected component of Q is either isomorphic to K_2 or is contained in $\mathcal{F}$. Q is *maximum* if it covers a maximum number of vertices and Q is an *$\mathcal{F}$-factor* if it covers all vertices of G.

Moreover, let $d_{\mathcal{F}}$ be the number of vertices of G missed by a maximum $\mathcal{F}$-packing. G is *$\mathcal{F}$-critical* if it has no $\mathcal{F}$-factor, but $G - v$ has one for each $v \in V(G)$.

Note that when $\mathcal{F} = \emptyset$ we get the classical matching problem. Cornuéjols, Hartvigsen and Pulleyblank [4, 5] showed that the $\mathcal{F}$-packing problem is polynomial and they proved an Edmonds-Gallai-type theorem for the $\mathcal{F}$-packing problem. We cite this result.

Theorem 3.2. (Cornuéjols, Hartvigsen) [4] *Let $D_{\mathcal{F}} \subseteq V(G)$ consist of those vertices which can be missed by a maximum $\mathcal{F}$-packing of G. Let $A_{\mathcal{F}}$ be the set of neighbors of $D_{\mathcal{F}}$ in G and let $C_{\mathcal{F}} = V(G) - (D_{\mathcal{F}} \cup A_{\mathcal{F}})$. Then*

1. *the components of $G[D_{\mathcal{F}}]$ are $\mathcal{F}$-critical,*
2. *$d_{\mathcal{F}}(G) = c(D_{\mathcal{F}}) - |A_{\mathcal{F}}|$,*
3. *$|\{K : K$ is a component of $G[D_{\mathcal{F}}]$ adjacent to $A'\}| \geq |A'| + 1$ for all $\emptyset \neq A' \subseteq A_{\mathcal{F}}$,*
4. *for all maximum $\mathcal{F}$-packings F of G there is no edge of F between $A_{\mathcal{F}}$ and $C_{\mathcal{F}}$ and $F[C_{\mathcal{F}}]$ is an $\mathcal{F}$-factor of $G[C_{\mathcal{F}}]$.*

We make use of the edge and factor-critical graph packing problem in gadgets as follows.

Definition 3.3. $(T, U, \mathcal{F})$ is said to be a *gadget representing* the degree prescription $H \subseteq \mathbb{N}$ if T is a graph, $U \subseteq V(T)$ and $\mathcal{F}$ is a set of factor-critical subgraphs of T with the property that $h \in H$ if and only if there exists an h-element set $U' \subseteq U$ such that $T - U'$ has an $\mathcal{F}$-factor.

If H_v has a representing gadget for all $v \in V(G)$ then G has an $\mathcal{H}$-factor if and only if G_{aux} has an $\mathcal{F}$-factor, where G_{aux} is the auxiliary graph what we get when replacing in G_{sub} each vertex $v \in V(G)$ by a gadget representing H_v. Even $\delta_{\mathcal{H}}(G) = d_{\mathcal{F}}(G_{aux})$ holds by Thm. 3.9. The exact definition of this auxiliary graph is as follows. Recall that a subdivided graph is just an induced subgraph of a graph of form G_{sub}.

Definition 3.4. Let S be a subdivided graph. Suppose $(T_v, U_v, \mathcal{F}_v)$ represents H_v for $v \in V_S$. Let S_{aux} be the graph with vertex set

$$V(S_{aux}) = S_E \cup \bigcup_{v \in S_V} V(T_v)$$

and edge set

$$E(S_{aux}) = \{e_x e_y : e_x, e_y \in S_E\} \cup \{e_x u : u \in U_x,\ x \in S_V,\ e_x \in S_E\} \cup \bigcup_{v \in S_V} E(T_v).$$

Moreover, let $\mathcal{F} = \bigcup_{v \in S_V} \mathcal{F}_v$. $(G_{sub})_{aux}$ is denoted simply by G_{aux}.

It is well known that a parity interval $\{p,\, p+2, \ldots, p+2r\}$ can be represented by a gadget with $\mathcal{F}=\emptyset$, see Fig. 2. With the help of these gadgets Cornuéjols [3] was able to produce a non-Edmonds-type algorithm for the $\mathcal{H}$-factor problem using the local augmenting property of jump systems. By answering a question of Pulleyblank, Loebl [14] then proved that a prescription H can be represented by a gadget $(T, U, \mathcal{F})$ such that $\mathcal{F}$ *contains only triangles* if and only if H is a parity interval or

$$H = I \cap \{p,\, p+2,\, p+3, \ldots,\, p+2r-2,\, p+2r\},\ r \geq 1$$

where I is an interval. However, Thm. 3.5 shows that many new representable prescriptions arise if we allow any factor-critical graphs in $\mathcal{F}$ not just triangles. These new representable prescriptions slightly simplify the above algorithm of Cornuéjols. Namely, instead of $2|V(G)|+1$, it is enough only 1 – a bit more involved – search for a local augmentation. This is because $\{p, p+1, p+3, \ldots, p+2r-1, p+2r\} \cap I$ can be represented, where I is an interval.

Note that Thm. 3.5 implies that a representable degree prescription has no two consecutive gaps.

Theorem 3.5. *A degree prescription can be represented by a gadget if and only if all of its gaps have the same parity.*

Proof. Necessity. Suppose $(T, U, \mathcal{F})$ is a gadget representing the degree prescription H. Let $p, q \in H,\ p < q$. We prove that

i. $\{p+1,\, p+2,\, q-1\} \cap H \neq \emptyset$ and

ii. $\{p+1,\, q-1\} \cap H \neq \emptyset$ if $p \not\equiv q \mod 2$.

Now *i.* implies that there are no two consecutive gaps in H, and hence *ii.* gives that all gaps have the same parity. Let Q_p, Q_q be $\mathcal{F}$-factors of $T-U_p$, $T-U_q$ resp., with U_p, $U_q \subseteq U$ and $|U_p| = p$, $|U_q| = q$. Let $V_p = V(Q_p) = V(T) - U_p$ and define V_q similarly. Choose Q_p, Q_q with $V_p \cap V_q$ maximal. $|V_p| > |V_q|$ so let $v \in V_p \setminus V_q$. Let P be a longest alternating path starting at v with edges alternately being K_2 components of Q_p and Q_q. Note that P cannot end in $V_q \setminus V_p$ because of the maximality of $V_p \cap V_q$. So three possibilities can occur.

1. If P ends in a factor-critical component of Q_p then we can modify Q_p to an $\mathcal{F}$-factor of $T - U_p - v$.
2. If P ends in a factor-critical component of Q_q then we can modify Q_q to an $\mathcal{F}$-factor of $T - U_q + v$.
3. If P ends in $u \in V_p \setminus V_q$ then we can modify Q_p to an $\mathcal{F}$-factor of $T - U_p - u - v$.

Hence *i.* is proved. Also *ii.* is proved if there exists $v \in V_p \setminus V_q$ for which possibility 1. or 2. occurs. Suppose otherwise. The paths of type 3. pair the elements of $V_p \setminus V_q$ implying that $|V_p \setminus V_q|$ is even and is clearly at least 2. Let P be such an alternating path with end vertices u, $v \in V_p \setminus V_q$ and let P_p (resp. P_q) consist of the K_2 components of P belonging to Q_p (resp. Q_q). The oddness of $q - p$ implies that $|V_q \setminus V_p|$ is odd. For each $w \in V_q \setminus V_p$ let R_w be a longest alternating path starting at w with edges alternately being K_2 components of Q_q and Q_p. Observe that R_w and P are disjoint. As above, R_w either ends in a factor-critical component or it ends in $V_q \setminus V_p$. Since $|V_q \setminus V_p|$ is odd, for at least one vertex $w \in V_q \setminus V_p$, either

1. R_w ends in a factor-critical component of Q_p in which case we can modify $Q_p - P_p + P_q$ to an $\mathcal{F}$-factor of $T - U_p + w - u - v$, or

2. R_w ends in a factor-critical component of Q_q in which case we can modify $Q_q - P_q + P_p$ to an $\mathcal{F}$-factor of $T - U_q - w + u + v$.

Definition 3.6. Assume that all gaps of H have the same parity and that H is not an interval of length at least 2. Define *H-parity* to be 0 (resp. 1) if all even (resp. odd) integers in $[l,\, u]$ belong to H. Here $l = \min H$ and $u = \max H$.

Sufficiency. Let H be a prescription with no gaps of different parity. If H is an interval $\{p,\, p+1, \dots, p+r\}$ then it is well-known that H can be represented by a gadget T consisting of $p + r$ isolated vertices, with $U = V(T)$ and $\mathcal{F}$ consisting of r of these vertices as one vertex factor-critical subgraphs, see Fig. 2. Otherwise let $H^0 = \{h - l : h \in H\}$. If $(T, U, \mathcal{F})$ is a gadget representing H^0 then adding l isolated vertices to T which belong to U results in a gadget representing H. Hence we may assume that $l = 0$. Construct a gadget $(T, U, \mathcal{F})$ in the following way. If u has H-parity then define $n = 2u$, otherwise let $n = 2u+1$. Let $U = \{y_i : 1 \le i \le u\}$, $V(T) = U \cup \{x_1,\, x_2,\, \dots,\, x_n = x_0\}$ and let

$$E(T) = \{x_i x_{i+1} : 0 \le i \le n-1\} \cup \{x_{2i} x_{2i+2} : 0 \le i \le n/2 - 1\} \cup \{x_{2i-1} y_i : 1 \le i \le u\}.$$

If $u - r \in H$ has non H-parity then add the odd circuit $x_0, x_2, \dots, x_{2r}, x_{2r+1}, \dots, x_{n-1}$ to $\mathcal{F}$. Observe that an $\mathcal{F}$-packing of T can have at most one factor-critical component since $x_0 \in V(F)$ for all $F \in \mathcal{F}$. So it is easy to see that $(T, U, \mathcal{F})$ represents H. See an example in Fig. 1. $\square$

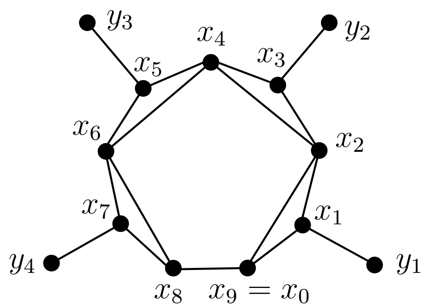

Figure 1: Gadget representing $\{0, 1, 3, 4\}$. $U = \{y_1, y_2, y_3, y_4\}$, $\mathcal{F} = \{x_0 x_2 x_4 x_6 x_8,\, x_0 \dots x_8\}$.

Of course in some special cases one can give much simpler gadgets, see Fig. 2. All but the last gadgets of Fig. 2 were already known. Note that the addition of l isolated vertices to U shifts H upwards l units.

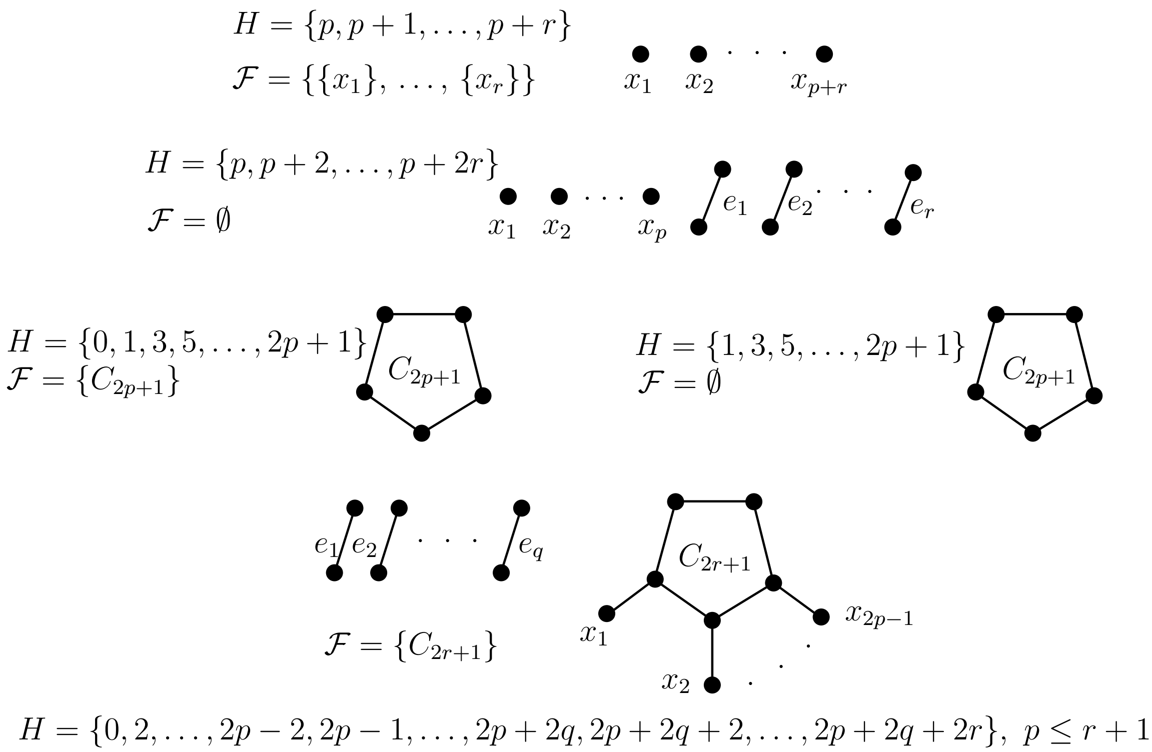

Figure 2: Some simpler gadgets. $U = V(T)$ in all cases.

We remark that the necessity part of the above proof can be directly applied to *propellers* introduced by Loebl and Poljak [15]. Thus using the more general packing of a 'closed propeller family' [15] does not yield more represented prescriptions.

Now we describe the relation of the $\mathcal{F}$-packing problem of S_{aux} and the $\mathcal{H}$-factor problem of S. Suppose that H_v can be represented by a gadget for all $v \in S_V$. Beyond the fact that G has an $\mathcal{H}$-factor if and only if G_{aux} has an $\mathcal{F}$-factor, there is a stronger relation by Thm. 3.9, namely $\delta_{\mathcal{H}}(G_{sub}) = d_{\mathcal{F}}(G_{aux})$. Recall that $\delta_H(G) = \delta_{\mathcal{H}}(G_{sub})$ holds. For the proof of Thm. 3.9 we need the following lemma.

Lemma 3.7. *Let Q be an $\mathcal{F}$-packing of a gadget $(T, U, \mathcal{F})$. If Q does not cover $V(T) - U$ then there exist vertices $x \notin V(Q) \cup U$ and y such that either $y \notin V(Q)$ ($x = y$ is allowed) and $T[V(Q) + x + y]$ has an $\mathcal{F}$-factor, or $y \in U \cap V(Q)$ and $T[V(Q) + x - y]$ has an $\mathcal{F}$-factor.*

Proof. Add to $\mathcal{F}$ the vertices of U as singleton factor-critical subgraphs, resulting in $\mathcal{F}'$. Add the vertices of $U - V(Q)$ as such subgraphs to Q resulting in the $\mathcal{F}'$-packing Q' with $V(Q') = V(Q) \cup U$. An *alternating path* is a path P starting at some vertex $x \notin V(Q')$ such that every second edge of P is a K_2 component of Q'. Our assumption that $H \neq \emptyset$ implies that T has an $\mathcal{F}'$-factor. Hence Q' is not a maximum $\mathcal{F}'$-packing. So the augmenting path theorem of Cornuéjols and Hartvigsen [4] states that

1. either there exists an alternating path ending at a vertex $y \notin V(Q')$,

2. or there exists an alternating path ending at a factor-critical component K of Q',

3. or there exists an even length alternating path P ending at a vertex $w \in W \subseteq V(T)$ such that $V(P) \cap W = \{w\}$, $T[W]$ is factor-critical having an $\mathcal{F}'$-factor Q_W and the components of Q' contained in W form a perfect matching of $T[W - w]$.

In case 2. if $K \in \mathcal{F}$ then we can modify Q to an $\mathcal{F}$-packing with vertex set $V(Q) + x$, while if $K = \{y\} \in \mathcal{F}' \setminus \mathcal{F}$ then we can modify Q to an $\mathcal{F}$-packing with vertex set $V(Q)+x+y$. This latter modification can be done in case 1. as well. In case 3. modify Q to an $\mathcal{F}'$-packing with vertex set $V(Q) + x$ using the $\mathcal{F}'$-factor Q_W. If Q_W is an $\mathcal{F}$-factor then we are done. Otherwise Q_W contains a component $\{y\} \in \mathcal{F}' \setminus \mathcal{F}$. Now $y \in U \cap V(Q)$ so replacing Q_W by a perfect matching of $T[W] - y$ gives an $\mathcal{F}$-packing with vertex set $V(Q) + x - y$. $\square$

Corollary 3.8. *If Q is an $\mathcal{F}$-packing of a gadget $(T, U, \mathcal{F})$ representing the prescription H then*

$$\mathrm{dist}(|U \setminus V(Q)|,\, H) \leq |V(T) - (U \cup V(Q))|.$$

Proof. If Q does not cover $V(T) \setminus U$ then apply the previous lemma and then use induction. $\square$

For sake of a unified treatment, for a vertex $v \in S_E$ we define T_v to be the singleton $\{v\}$, and let $U_v = \{v\}$, $\mathcal{F}_v = \emptyset$.

Theorem 3.9. $\delta_{\mathcal{H}}(S) = d_{\mathcal{F}}(S_{aux})$.

Proof. $\delta_{\mathcal{H}}(S) \geq d_{\mathcal{F}}(S_{aux})$: Let F be an $\mathcal{H}$-optimal subgraph of S. By successively deleting edges we may assume that $\deg_F(v) \in H_v^{\downarrow}$ for all $v \in V(S)$. Now F gives an $\mathcal{F}$-packing of S_{aux} in a natural way missing δ_F vertices.

$\delta_{\mathcal{H}}(S) \leq d_{\mathcal{F}}(S_{aux})$: Let Q be a maximum $\mathcal{F}$-packing of S_{aux}. For $v \in S_V$ let Q_v consist of those components of Q which are fully contained in T_v. If we contract each T_v to vertex v then $Q - \bigcup_{v \in S_V} Q_v$ gives a subgraph F of S. Now by Corollary 3.8, $\mathrm{dist}(\deg_F(v),\, H_v) \leq \mathrm{dist}(|U_v \setminus V(Q_v)|,\, H_v) + |U_v \setminus V(Q)| \leq |V(T_v) - (U_v \cup V(Q_v))| + |U_v \setminus V(Q)| = |V(T_v) \setminus V(Q)|$ for all $v \in S_V$. $\delta_F(v) \leq |V(T_v) \setminus V(Q)|$ holds for $v \in V_E$ as well. Summing up, we get that $\delta_F \leq d_{\mathcal{F}}(S_{aux})$. $\square$

Now we describe the relation of the Edmonds-Gallai decompositions $V(S) = D \,\dot\cup\, A \,\dot\cup\, C$ by Thm. 2.5 and $V(S_{aux}) = D_{\mathcal{F}} \,\dot\cup\, A_{\mathcal{F}} \,\dot\cup\, C_{\mathcal{F}}$ by Thm. 3.2. To avoid technical difficulties we assume that the gadgets $(T, U, \mathcal{F})$ used in S_{aux} are *clean.*

Definition 3.10. A gadget $(T, U, \mathcal{F})$ representing the prescription H is *clean* if it satisfies the properties below.

(a) For all vertex sets $A' \subseteq V(T)$ the number of those factor-critical components of $T - A'$ which are disjoint from U_v is at most $|A'| - 1$.

(b) $|U| = \max H$.

(c) If H is an interval of length at least 2 then $\mathcal{F}$ contains a singleton factor-critical graph.

(d) If H is not an interval of length at least 2 then for all $U' \subseteq U$, no $\mathcal{F}$-factor of $T - U'$ has more than 1 component in $\mathcal{F}$. Moreover, if it has exactly 0 (resp. 1) such component then $|U'| \in H$ has H_v-parity (resp. non H_v-parity).

In the rest of this section (a) – (d) will refer to these properties. Note that the gadgets constructed in the proof of Thm. 3.5 and that of Fig. 2 are clean. Thus each degree prescription with gaps of the same parity can be represented by clean gadgets.

Lemma 3.11. $D = \{v \in V(S) : U_v \cap D_{\mathcal{F}} \neq \emptyset\}$.

Proof. If $v \in D$ then there exists an $\mathcal{H}$-optimal subgraph F of S with $\deg_F(v) \in H_v^{\downarrow} \setminus H_v$. By deleting appropriate edges we may assume that $\deg_F(u) \in H_u^{\downarrow}$ for all vertices $u \in V(S)$ keeping the property that $\deg_F(v) \notin H_v$. By Thm. 3.9, such a subgraph F gives a maximum $\mathcal{F}$-packing F' of S_{aux} in a natural way. $U_v \not\subseteq V(F')$ hence $U_v \cap D_{\mathcal{F}} \neq \emptyset$.

On the other hand, let Q be a maximum $\mathcal{F}$-packing of S_{aux} with $U_v \not\subseteq V(Q)$. For each $w \in S_V$ do the following. If Q does not cover $V(T_w) - U_w$ then apply Lemma 3.7 to $(T_w, U_w, \mathcal{F}_w)$ and to the $\mathcal{F}_w$-packing Q_w consisting of the components of Q contained fully in $V(T_w)$. Q is maximum so there exist vertices $x \notin V(Q_w) \cup U_w$ and $y \in U_w$ such that either $y \in V(Q) \setminus V(Q_w)$ and $T_w[V(Q_w) + x + y]$ has an $\mathcal{F}_w$-factor or $y \in V(Q_w)$ and $T_w[V(Q_w) + x - y]$ has an $\mathcal{F}_w$-factor. In the first case delete from Q the K_2-component with edge $e_w y \in E(Q)$. Iterating this, we achieve that the maximum $\mathcal{F}$-packing Q covers $V(T_w) - U_w$ for all $w \in S_V$ keeping the property that $U_v \not\subseteq V(Q)$. Contracting for each $w \in S_V$ the vertex set $V(T_w)$ to w we get an $\mathcal{H}$-optimal subgraph F of S by Thm. 3.9 with $\deg_F(v) \in H_v^{\downarrow} \setminus H_v$. □

The relation of the two decompositions is as follows. Assume that $U_v \cap D_{\mathcal{F}} \neq \emptyset$ for some $v \in S_V$. Note that if $e_v \notin D_{\mathcal{F}}$ then $e_v \in A_{\mathcal{F}}$ for each adjacent vertex $e_v \in S_E$ of v.

- If $e_v \in D_{\mathcal{F}}$ for an adjacent vertex e_v then Thm. 3.2 together with (a) imply that $A_{\mathcal{F}} \cap U_v = \emptyset$. Hence $U_v \subseteq D_{\mathcal{F}}$ and thus $V(T_v) \subseteq D_{\mathcal{F}}$ by (a). (In case $S = G_{sub}$, such a vertex of $V(G)$ belongs to D_L in the decomposition of Lovász.)

- Otherwise $e_v \in A_{\mathcal{F}}$ for all adjacent vertices e. It is easy to prove that $X_{\mathcal{F}} \cap V(T_v) = X_{\mathcal{F}_v}$ holds ($X = D, A, C$) for the Edmonds-Gallai decomposition $V(T_v) = D_{\mathcal{F}_v} \dot\cup A_{\mathcal{F}_v} \dot\cup C_{\mathcal{F}_v}$ of Thm. 3.2. Hence $c(D_{\mathcal{F}_v}) - |A_{\mathcal{F}_v}| \leq l(v)$ by the definition of the gadgets. The reverse direction is implied by Corollary 3.8 so $c(D_{\mathcal{F}_v}) - |A_{\mathcal{F}_v}| = l(v)$ holds. (If $S = G_{sub}$ then these vertices of $V(G)$ belong to B_L in the Lovász decomposition.)

If $U_v \cap D_{\mathcal{F}} = \emptyset$ for $v \in S_V$ then we have two cases.

- If $e_v \in D_{\mathcal{F}}$ for an adjacent vertex e_v then $U_v \subseteq A_{\mathcal{F}}$. Note that $|U_v| = u(v)$ by (b). Hence $T_v - U_v$ has an $\mathcal{F}_v$-factor so $V(T_v) - U_v \subseteq C_{\mathcal{F}}$. ($A \cap V(G)$ is just A_L if $S = G_{sub}$.)

- Otherwise $e_v \notin D_{\mathcal{F}}$ for all adjacent vertices e_v so clearly $V(T_v) \subseteq C_{\mathcal{F}}$ by (a). (These vertices are in C_L.)

Using these considerations, Thm. 2.5 follows from Thm. 3.2 in case H_v can be represented for each $v \in S_V$. We also get a characterization of the $\mathcal{H}$-critical graphs. It is enough to characterize those which are non-trivial.

Lemma 3.12. *Let S be a subdivided graph. Assume that all gaps of H_v have the same parity and that $(T_v, U_v, \mathcal{F}_v)$ are clean for all $v \in S_V$. Then S is $\mathcal{H}$-critical if and only S_{aux} is $\mathcal{F}$-critical.*

Proof. Observe that S is $\mathcal{H}$-critical if and only if $D = V(S)$ by Thm. 2.5. Similarly, S_{aux} is $\mathcal{F}$-critical if and only if $D_{\mathcal{F}} = V(S_{aux})$ by Thm. 3.2. Using this observation, if S is $\mathcal{H}$-critical then (a) together with Lemma 3.11 imply that S_{aux} is $\mathcal{F}$-critical. The other direction is trivial by Lemma 3.11. □

We use a result of Cornuéjols, Hartvigsen and Pulleyblank [5].

Lemma 3.13. [5] *A graph G is $\mathcal{F}$-critical if and only if it is factor-critical and does not have a subgraph $K \in \mathcal{F}$ such that $G - K$ has a perfect matching.*

So if S is $\mathcal{H}$-critical then S_{aux} is factor-critical and hence $|V(S_{aux})|$ is odd. Observe that if H_v is an interval of length at least 2 for some $v \in S_V$ then S_{aux} cannot be $\mathcal{F}$-critical by (c). Otherwise (b) and (d) together imply that $|V(T_v)|$ has H_v-parity. Thus the sum of the H_v-parities is odd in every $\mathcal{H}$-critical graph in the representable cases. Lemma 3.13 can be translated to $\mathcal{H}$-critical graphs as follows.

Theorem 3.14. *Let S be a subdivided graph. Assume that all gaps of H_v have the same parity for all $v \in S_V$ and that $|V(S)| \geq 2$. Then S is $\mathcal{H}$-critical if and only if*

1. *H_v is not an interval of length at least 2 for all $v \in S_V$,*
2. *for each $v \in S_V$, S has a subgraph F such that $\deg_F(v) \notin H_v$ has non H_v-parity, $\deg_F(v) + 1 \in H_v$ and $\deg_F(w) \in H_w$ has H_w-parity for all $w \neq v$, and*
3. *for each $v \in V(S)$ and $d \in H_v$ with non H_v-parity it holds that S has no $\mathcal{H}$-factor F such that $\deg_F(v) = d$ and $\deg_F(w)$ has H_w-parity for all $w \neq v$.*

Proof. Construct S_{aux} using clean gadgets. We already observed that by (c), S cannot be $\mathcal{H}$-critical if H_v is an interval of length at least 2 for some $v \in S_V$. By Lemmas 3.12 and 3.13, S is $\mathcal{H}$-critical if and only if S_{aux} is factor-critical and there exists no $K \in \mathcal{F}$ such that $S_{aux} - K$ has a perfect matching. Provided that *1.* holds, the first condition is equivalent to *2.* by (a) and (d), and the second is equivalent to *3.* by (d). □

4 Matroidal results

This section contains two matroidal results on the $\mathcal{H}$-factor problem, Thm. 4.2 and Thm. 4.8. We give three proofs for the first one.

Definition 4.1. For $v \in V(G)$ let $e^v \in \mathbb{N}^{V(G)}$ be the unit vector of coordinate v. For a subgraph F of G let $\delta^F \in \mathbb{N}^{V(G)}$ be the vector with component $\delta^F(v)$ for $v \in V(G)$.

$P \subseteq \mathbb{N}^V$ is a *base polyhedron* if for all a, $b \in P$ and $v \in V$ with $a(v) > b(v)$ there exists $u \in V$ such that $a(u) < b(u)$ and $a - e^v + e^u \in P$. The up-hull of a base polyhedron (i.e. $P + \mathbb{N}^{V(G)}$) is called a *contra-polymatroid.*

Observe that $a(V)$ is constant for the elements a of a base polyhedron. Usually base polyhedrons and contra-polymatroids are integer polyhedra, defined to be exactly the convex hull of the sets defined above. We used this definition because we are interested only in the integer points of these polyhedra.

Theorem 4.2. *$C = \{\delta^F + \mathbb{N}^{V(G)} : F$ is a subgraph of $G\}$ is a contra-polymatroid.*

Proof. 1. It is enough to prove that if a, $b \in C$ and $v \in V(G)$ with $a(v) > b(v)$ then either $a - e^v \in C$ or there exists $u \in V(G)$ such that $a(u) < b(u)$ and $a - e^v + e^u \in C$. We prove this by induction on $|E(F_a)\triangle E(F_b)|$ where F_a, F_b are subgraphs such that $\delta^{F_a} \le a$, $\delta^{F_b} \le b$. If $\delta^{F_a} < a(v)$ then we are done, so suppose equality. Thus $\delta^{F_b}(v) < \delta^{F_a}(v)$ so there exists an edge $e = vu \in E(F_a)\triangle E(F_b)$ such that $\delta^{F'}(v) < \delta^{F_a}(v)$ holds with the notation $F' = F_a\triangle e$. If $\delta^{F'}(u) \le a(u)$ or $\delta^{F'}(u) = a(u) + 1 \le b(u)$ then F' shows that we are done. Otherwise $\delta^{F'}(u) > b(u)$. Now $|E(F')\triangle E(F_b)| < |E(F_a)\triangle E(F_b)|$ so the statement holds for $\delta^{F'}$ and b by our induction hypothesis. Apply it to $u \in V(G)$. □

In the next two proofs it is enough to show that $P_{\mathcal{H}}(G) := \{\delta^F : F$ is an $\mathcal{H}$-optimal subgraph of $G\}$ is a base polyhedron by Lemma 4.3.

Lemma 4.3. *For any subgraph F of G there exists an $\mathcal{H}$-optimal subgraph F_0 of G such that $\delta^{F_0} \le \delta^F$.*

Proof. The $\mathcal{H}$-optimal subgraph F_0 minimizing $E(F)\triangle E(F_0)$ will do. Otherwise $\delta^{F_0}(v) > \delta^F(v)$ for some $v \in V(G)$ so there exists an edge $e = vu \in E(F)\triangle E(F_0)$ such that $\delta^{F_0\triangle e}(v) < \delta^{F_0}(v)$ holds. But then $F_0\triangle e$ contradicts to the choice of F_0. □

We need some preliminaries. Jump systems were introduced by Bouchet and Cunningham [2]. They are closely related to the $\mathcal{H}$-factor problem, which is related to the fact that the degree sequences of all subgraphs of a graph is a jump system, see Proposition 4.5.

Definition 4.4. [2] For $a, b \in \mathbb{Z}^V$ we say that a' is a *step from a to b* if either $a' = a+e^v$ and $a(v) < b(v)$ or $a' = a - e^v$ and $a(v) > b(v)$, for some $v \in V$. $J \subseteq \mathbb{Z}^V$ is a *jump system* if for all $a, b \in J$ and a' step from a to b, either $a' \in J$ or some step from a' to b is contained in J.

If $J_i \subseteq \mathbb{Z}^{V_i}$ are jump systems for $i = 1, 2$ then let $J_1 \wedge J_2 = \{a^1 \wedge a^2 \in \mathbb{Z}^{V_1\triangle V_2} : a^1 \in J_1,\ a^2 \in J_2,\ a^1|_{V_1\cap V_2} = a^2|_{V_1\cap V_2}\}$ where $(a^1 \wedge a^2)_j = a^i_j$ if $j \in V_i$ for $i = 1, 2$. If V_1 and

V_2 are disjoint then $J_1 \times J_2 = J_1 \wedge J_2$ is called the *direct sum* of J_1, J_2. For $J \subseteq \mathbb{Z}^V$ and $c \in \{-1, 0, 1\}^V$ let J_c consist of the elements of J minimizing cost function c. J has *constant sum* if $a(V) = b(V)$ for all $a, b \in J$.

Proposition 4.5. [2] *If $J \subseteq \mathbb{Z}^V$, $J_1 \subseteq \mathbb{Z}^{V_1}$, $J_2 \subseteq \mathbb{Z}^{V_2}$ are jump systems and $c \in \{-1, 0, 1\}^V$ then $J_1 \wedge J_2$ and J_c are jump systems. A constant sum jump system is a base polyhedron. The degree sequences of all subgraphs of a graph G is a jump system, denoted by J_G.*

This proposition will be used throughout in the next two proofs.

Proof. 2. (of Thm. 4.2) Note that it is enough to prove that $P_{\mathcal{H}}(G_{sub})$ is a base polyhedron since $P_{\mathcal{H}}(G) = \{a|_{V(G)} : a \in P_{\mathcal{H}}(G_{sub}),\ a(v) = 0 \text{ for all } v \in V_E\}$. It follows from Thm. 2.5 that $a(v) = 0$ for $a \in P_{\mathcal{H}}(G_{sub})$, $v \in C$ so we can assume that $C = \emptyset$. Shrink all non-trivial components of $G_{sub}[D]$ and delete the edges induced by A resulting in the bipartite graph B. For $a \in \mathbb{N}^{V(B)}$ let $a'(v) = a(v) - u(v)$ if $v \in A$, $a'(v) = l(v) - a(v)$ if $\{v\}$ is a trivial component of $G_{sub}[D]$ and $a'(v) = a(v)$ otherwise. Let $J' = \{a' \geq 0 : a \in J_B\}$ which is clearly a jump system. For a non-trivial component K of $G_{sub}[D]$ define a jump system $J_K = \{e^K, e^v : v \in V(K)\}$ on ground set $\{K\} \cup V(K)$. Let $J_D = \times\{J_K : K \text{ is a non-trivial component of } G_{sub}[D]\}$, J_D is a jump system again. Using Thm. 2.5, *1., 2., 3.* it is not hard to see that $J' \wedge J_D = \{\delta^F : F \text{ is an } \mathcal{H}\text{-optimal subgraph of } G_{sub}\}$. Thus $J' \wedge J_D$ has constant sum so it is a base polyhedron. ☐

Proof. 3. (of Thm. 4.2) We use some results of Lovász [19]. For the definitions of A_L, B_L, C_L and D_L, see page 3.

Definition 4.6. For $v \in V(G)$ let $I_{\mathcal{H}}(v) = \{\deg_F(v) : F \text{ is an } \mathcal{H}\text{-optimal subgraph of } G\}$. $[I_{\mathcal{H}}(v)]$ denotes the minimal interval containing $I_{\mathcal{H}}(v)$.

Lemma 4.7. [19] *If $v \in D_L$ then every second element of $[I_{\mathcal{H}}(v)]$ is not contained in H_v.*

For $v \in C_L$ define $J_v = \{(i, 0) : i \in H_v\}$, for $v \in A_L$ let $J_v = \{(i, i - u(v)) : i \geq u(v)\}$, and for $v \in B_L$ let $J_v = \{(i, l(v) - i) : i \leq l(v)\}$. Finally, for $v \in D_L$ define $J_v = \{(i, 0) : i \in [I_{\mathcal{H}}(v)] \cap H_v\} \cup \{(i, 1) : i \in [I_{\mathcal{H}}(v)] \setminus H_v\}$. Observe that J_v is a jump system for all $v \in V(G)$. Let $J' = \times\{J_v : v \in V(G)\}$ and $J = J' \wedge J_G$. It is clear that if $c \in \mathbb{N}^{V(G)}$ is the constant 1 vector then $J_c = \{\delta^F : F \text{ is an } \mathcal{H}\text{-optimal subgraph of } G\}$. J_c has constant sum and thus a base polyhedron. ☐

We remark that Thm. 4.2 holds for all jump systems with ground set V, namely, if $J \subseteq \mathbb{N}^V$ is a jump system and $H_v \subseteq \mathbb{N}$ for all $v \in V$ then $\{\delta^a + \mathbb{N}^V : a \in J\}$ is a contra-polymatroid, where $\delta^a(v) = \mathrm{dist}(H_v, a(v))$.

The dual of the matroid of the next theorem is contained in $P_{\mathcal{H}}(G)$. Recall that F is an $\mathcal{H}$-subgraph if $\deg_F(v) \in H_v$ for all $v \in V(F)$.

Theorem 4.8. *Suppose $1 \in H_v$ and H_v has no two consecutive gaps for all $v \in V(G)$. Let $\mathcal{M}$ consist of those vertex sets which can be covered by $\mathcal{H}$-subgraphs. Then $\mathcal{M}$ is a matroid.*

Proof.

Lemma 4.9. *If $X, Y \in \mathcal{M}$, $X \setminus Y \neq \emptyset$ and Y is maximal in $\mathcal{M}$ then for all $x \in X \setminus Y$ there exists a vertex $y \in Y \setminus X$ such that $Y + x - y \in \mathcal{M}$.*

Proof. Assume the statement fails for X and Y and let F_X and F_Y be $\mathcal{H}$-subgraphs with $X \subseteq V(F_X)$ and $Y = V(F_Y)$. For an edge set P define the subgraph F_P with vertex set $Y + x$ and edge set $E(F_Y) \triangle P$. An edge set $P \subseteq E(F_X) \triangle E(F_Y)$ is called *augmenting* if $V(P) \subseteq Y + x$ and F_P satisfies the degree prescription at every vertex of $Y + x$ except for at most one vertex v. $\emptyset$ is augmenting so choose P to be a maximal augmenting edge set. Note that $\deg_{F_P}(v) \notin H_v$ since then F_P would contradict the maximality of Y.

Suppose that there exists an edge $e \in E(F_X) \triangle E(F_Y)$ incident with v which is not in P. If we can choose $e \in E(F_Y) \setminus E(F_X)$ then $P + e$ is augmenting, since $1 \in H_v$ and hence $\deg_{F_P}(v) - 1 \in H_v$. Otherwise all edges of $E(F_Y) \setminus E(F_X)$ incident to v are in P, so let $e = uv \in E(F_X) \setminus E(F_Y)$ not in P. Now $\deg_{F_P}(v) < \deg_{F_X}(v) \in H_v$ implying that $\deg_{F_P}(v) + 1 \in H_v$. If $u \notin X + v$ then F_{P+e} would be an H-subgraph with vertex set $Y + u + v$ due to $1 \in H_u$, a contradiction. Hence $u \in X + v$ and $P + e$ is augmenting.

If all edges of $e \in E(F_X) \triangle E(F_Y)$ incident with v are in P then $\deg_{F_X}(v) = \deg_{F_P}(v) \notin H_v$ so $v \notin X$ and $\deg_{F_P}(v) = 0$. So we are done by choosing $y = v$. $\square$

This lemma yields that the maximal elements of $\mathcal{M}$ are maximum, too. Indeed, assume the contrary, and let U, W be maximal elements of $\mathcal{M}$ with $|W| < |U|$. Choose them with $|U \cap W|$ maximum. The lemma gives that there exist $u \in U \setminus W$ and $w \in W \setminus U$ such that $W' = W + u - w \in \mathcal{M}$. W' is not maximal by our choice so let W'' be a maximal set containing W'. W is maximal so $w \notin W''$ but $|W| < |W''|$. Applying the lemma with $X = W$, $Y = W''$ and $x = w$ gives a set of $\mathcal{M}$ strictly containing W, a contradiction.

So we only have to check the base axioms which are immediately implied by Lemma 4.9. $\square$

We mention that proof *2.* of Thm. 4.2 can be modified to prove Thm. 4.8. (J' should be replaced by $\{a \in J' : a_v = 0 \ \forall v \in A\}$ and J_K by $\{e^K, e^v : K - v$ has an $\mathcal{H}$-factor$\}$. Thus the dual of $\mathcal{M}$ is a matroid.) Actually, it is easy to see that Thm. 4.8 is also true if $\{0, 1\} \cap H_v \neq \emptyset$ for all $v \in V(G)$. Otherwise $\mathcal{M}$ is not necessarily a matroid: subdivide each edge of K_4 with one vertex and let the prescription is $\{2\}$ on all vertices.

The already known special cases of Thm. 4.8 is the matching case by Edmonds and Fulkerson [8] (let $H \equiv \{1\}$), the packing by a sequential set of stars by Las Vergnas [13] ($H_v = \{1, 2, \ldots, u(v)\}$) and the $(1, f)$-odd subgraph case proved by Kano and Katona [12] ($H_v = \{1, 3, 5, \ldots, f(v)\}$).

Thm. 4.8 gives some support for the following conjecture of Loebl and Poljak [16]. If $\mathcal{K}$ is a set of graphs then a subgraph F of G is called a *$\mathcal{K}$-packing* if each connected component of F is isomorphic to a member of $\mathcal{K}$. They conjecture that if $K_2 \in \mathcal{K}$ then determining the maximum size of a $\mathcal{K}$-packing is polynomial if and only if the vertex sets coverable by $\mathcal{K}$-packings form a matroid. Note that $1 \in H_v$ stands for $K_2 \in \mathcal{K}$ and that the maximum size of an $\mathcal{H}$-subgraph is $|V(G)| - \delta_{\mathcal{H}}(G)$ in case $1 \in H_v$, hence it is polynomial.

An application of Thm. 4.8 is that the 'superstar packing' is matroidal, see [11].

The author would like to thank András Sebő for helpful discussions.

References

[1] A. Amahashi On factors with all degrees odd. *Graphs and Combin.* (1985) **1** 111–114.

[2] A. Bouchet, W. H. Cunningham Delta-matroids, jump systems, and bisubmodular polyhedra. *SIAM J. Discrete Math.* (1995) **8**, no. 1, 17–32.

[3] G. Cornuéjols General factors of graphs. *J. Combin. Theory Ser. B* (1988) **42** 285–296.

[4] G. Cornuéjols, D. Hartvigsen An extension of matching theory. *J. Combin. Theory Ser. B* (1986) **40** 285–296.

[5] G. Cornuéjols, D. Hartvigsen, W. Pulleyblank Packing subgraphs in a graph. *Oper. Res. Letter* (1981/82) **1**, no. 4, 139–143.

[6] Y. Cui, M. Kano Some results on odd factors of graphs. *J. of Graph Theory* (1988) **12**, no. 3, 327–333.

[7] J. Edmonds Paths, trees, and flowers. *Canadian J. of Math.* (1965) **17** 449–467

[8] J. Edmonds, D. R. Fulkerson Transversals and matroid partition. *J. Res. Nat. Bur. Standards Sect. B* (1965), 69B, 147–153.

[9] T. Gallai Kritische Graphen II. *A Magyar Tud. Akad. – Mat. Kut. Int. Közl.* (1963) **8** 135–139.

[10] T. Gallai Maximale Systeme unabhängiger Kanten. *A Magyar Tud. Akad. – Mat. Kut. Int. Közl.* (1964) **9** 401–413.

[11] M. Janata, J. Szabó Generalized star packing problems. *EGRES Technical Reports* 2004-17.

[12] M. Kano, G. Y. Katona Odd subgraphs and matchings. *Discrete Math.* (2002) **250**, no. 1-3, 265–272.

[13] M. Las Vergnas An extension of Tutte's 1-factor theorem. *Discrete Math.* (1978) **23**, no. 3, 241–255.

[14] M. Loebl Gadget classification. *Graphs Combin.* (1993) **9** 57–62.

[15] M. Loebl, S. Poljak On matroids induced by packing subgraphs. *J. Combin. Theory Ser. B* (1988) **44**, no. 3, 338–354.

[16] M. Loebl, S. Poljak Efficient subgraph packing. *J. Combin. Theory Ser. B* (1993) **59**, no. 1, 106–121.

[17] L. Lovász Antifactors of graphs. *Period. Math. Hungar.* (1973) **4** 121–123.

[18] L. Lovász The factorization of graphs. *Combinatorial Structures and their Applications* (Proc. Calgary Internat. Conf., Calgary, Alta., 1969) (1970) 243–246.

[19] L. Lovász The factorization of graphs. II. *Acta Math. Acad. Sci. Hungar.* (1972) **23** 223–246.