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Rigid realizations of graphs on small grids

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Abstract

A framework (G,p) is a graph $G = (V,E)$ and a mapping $p : V \to \mathbb{R}^2$. We prove that if (G,p) is an infinitesimally rigid framework then there is an infinitesimally rigid framework (G,q) for which the points $q(v)$, $v \in V(G)$, are distinct points of the $k \times k$ grid, where $k = \lceil \sqrt{|V|-1} \rceil + 9$. We also show that such a framework on G can be constructed in $O(|V|^3)$ time.

1 Introduction

A *bar-and-joint framework*, or *framework* for short, in d-space is a graph $G = (V,E)$ and a mapping $p : V \to \mathbb{R}^d$. It is denoted by (G,p) and is also called a *realization* of G in $\mathbb{R}^d$. A framework is *non-degenerate* if the points $p(v)$, $v \in V$, are pairwise distinct. Otherwise it is *degenerate*. The *rigidity matrix* of the framework is the matrix $R(G,p)$ of size $|E| \times d|V|$, where, for each edge $v_iv_j \in E$, in the row corresponding to v_iv_j, the entries in the d columns corresponding to vertex i (j) contain the d coordinates of $(p(v_i) - p(v_j))$ $((p(v_j) - p(v_i))$, respectively), and the remaining entries are zeros. The rigidity matrix of (G,p) defines the *rigidity matroid* of (G,p) on the ground set E by independence of rows of the rigidity matrix.

Lemma 1.1. *[8, Lemma 11.1.3] Let (G,p) be a framework in $\mathbb{R}^d$. Then* $\operatorname{rank} R(G,p) \leq S(n,d)$*, where $n = |V(G)|$ and*

$$S(n,d) = \begin{cases} nd - \binom{d+1}{2} & \text{if } n \geq d+1 \\ \binom{n}{2} & \text{if } n \leq d+1. \end{cases}$$

We say that a framework (G,p) in $\mathbb{R}^d$ is *infinitesimaly rigid* if $\operatorname{rank} R(G,p) = S(n,d)$. A *framework* (G,p) is *generic* if the coordinates of the points $p(v)$, $v \in V$, are algebraically independent over the rationals. Any two generic frameworks (G,p) and (G,p') have the same rigidity matroid. We call this the d-dimensional *rigidity*

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matroid $\mathcal{R}_d(G) = (E, r_d)$ of the graph G. We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$. We say that G is *M-independent* in $\mathbb{R}^d$ if E is independent in $\mathcal{R}_d(G)$. We say that a graph $G = (V, E)$ is *generically rigid* (or simply *rigid*) in $\mathbb{R}^d$ if $r_d(G) = S(n, d)$. See [2, 8, 7, 9] for more details on the rigidity of frameworks and graphs.

It follows that a graph G has an infinitesimally rigid realization if and only if G is rigid. In this paper we consider the problem of finding infinitesimally rigid (non-degenerate) realizations (G, p) of rigid graphs G for which the coordinates of the points $p(v)$, $v \in V(G)$, are integers between 1 and k, for some small k.

The existence of such a realization (which may be degenerate and where small means $O(n)$) follows from a lemma of Schwartz [5]. It implies that rigidity is in NP and it also leads to an efficient randomized algorithm for testing rigidity, for any d. It will also follow from the next 'moving' lemma, which is a kind of deterministic and algorithmic version of the above mentioned lemma of Schwartz, formulated for polynomials obtained from the rigidity matrix. This lemma will be used in the proof of our main result: we shall prove that for $d = 2$ a grid of size $k = O(n^{\frac{1}{2}})$ suffices, even if we require that the points $p(v)$ are pairwise distinct. Furthermore, such a realization can be found in $O(n^3)$ time.

Let (G, p) be a framework in $\mathbb{R}^d$. Suppose that we create a new framework on G by replacing the l-th coordinate of vertex u by some real number z and leaving all other coordinates of all vertices unchanged. Then we say that the resulting framework (G, p') is obtained from (G, p) *by moving u along axis l to z*. The *degree* of vertex u in G is denoted by $d_G(u)$.

Lemma 1.2. *Let $G = (V, E)$ be a graph and let (G, p) be an infinitesimally rigid realization of G in $\mathbb{R}^d$. Let $v \in V$ be a designated vertex, let l be an integer with $1 \leq l \leq d$ and let $z_1, z_2, ..., z_r$ be distinct real numbers with $r \geq d_G(v) + 1$. Then there is an integer m, $1 \leq m \leq r$, for which the framework obtained from (G, p) by moving v along axis l to z_m is infinitesimally rigid.*

Proof: Since (G, p) is infinitesimally rigid, we have $rank R(G, p) = S(n, d)$. Thus there is a non-singular square submatrix T of $R(G, p)$ of size $S(n, d)$. It follows from the definition of $R(G, p)$ that $p(v)^l$, the l-th coordinate of $p(v)$, appears in at most $d_G(v)$ rows of T. Thus by replacing all the entries $p(v)^l$ of T by a variable x, the determinant of T becomes a polynomial $T(x)$ of degree at most $d_G(v)$. Since $T(p(v)^l) \neq 0$ and $r \geq d_G(v) + 1$, there exists an integer m, $1 \leq m \leq r$, for which $T(z_m) \neq 0$. So the rank of the rigidity matrix remains unchanged by moving v along axis l to z_m. This completes the proof. •

Let $\mathbb{Z}_k^d \subset \mathbb{R}^d$ denote the grid points $\{(x^1, x^2, ..., x^d) : x^i \in \mathbb{Z}, 1 \leq x^i \leq k, 1 \leq i \leq d\}$. Let us say that a point $x \in \mathbb{R}^d$ is *covered* by some framework (H, q) if there is a vertex $v \in V(H)$ with $q(v) = x$. Otherwise x is *uncovered*. Given an infinitesimally rigid framework (G, p), we can use Lemma 1.2 to move any vertex $v \in V$ along any axis l to some integer between 1 and $2|V(G)| - 1$ so that the modified framework remains rigid and the new position of v was uncovered by (G, p). Thus we have:

Corollary 1.3. *Let $G = (V, E)$ be rigid in $\mathbb{R}^d$. Then there is an infinitesimally rigid non-degenerate framework (G, p) for which $p(v) \in \mathbb{Z}_{2|V|-1}^d$ for all $v \in V$.*

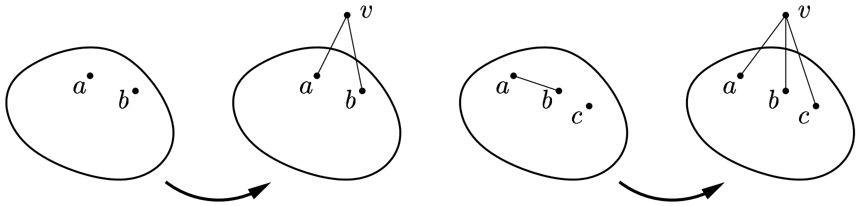

Figure 1: The operations 0-extension and 1-extension

2 Operations on graphs and frameworks in two dimensions

In the rest of the paper we shall suppose that $d = 2$. We need some further notation. For $X \subseteq V$, let $E_G(X)$ denote the set, and $i_G(X)$ the number, of edges in $G[X]$, that is, in the subgraph induced by X in G. For some $v \in V$ let $N_G(v)$ denote the set of vertices adjacent to v in G. We use $E(X)$, $i(X)$, or $N(v)$ when the graph G is clear from the context.

The following theorem, due to Laman, gives a combinatorial characterisation for rigidity in two dimensions. We say that $G = (V, E)$ is *minimally rigid* if G is rigid but $G - e$ is not rigid for all $e \in E(G)$. If G is rigid, the edge sets of the minimally rigid spanning subgraphs correspond to the bases of the rigidity matroid of G.

Theorem 2.1. *[4] A graph $G = (V, E)$ is minimally rigid if and only if $|E| = 2|V| - 3$ and*

$$i(X) \leq 2|X| - 3 \text{ for all } X \subset V \text{ with } |X| \geq 2. \tag{1}$$

Note that Theorem 2.1 leads to efficient algorithms for testing rigidity and, more generally, computing the rank in $\mathcal{R}_2(G)$. It remains an open problem to find good characterizations and algorithms for rigidity in $\mathbb{R}^d$ when $d \geq 3$.

To find the required infinitesimally rigid realizations of rigid graphs it is sufficient to consider their minimally rigid spanning subgraphs. Now we recall the basic reduction and extension operations of minimally rigid graphs.

Let v be a vertex in a minimally rigid graph G with $d_G(v) = 3$. The operation *splitting off (at vertex v)* means deleting v (and the edges incident to v) and adding a new edge connecting two non-adjacent vertices of $N(v)$. Note that v can be split off in at most three different ways. A splitting at v is *admissible* if the resulting graph is also minimally rigid.

Lemma 2.2. *[4, 6, 3] Let $G = (V, E)$ be a minimally rigid graph and let $v \in V$.*
(a) If $d(v) = 2$ then $G - v$ is minimally rigid.
(b) If $d(v) = 3$ then there is an admissible splitting at v.

We shall use the following two operations on frameworks. Both of these operations add a new vertex to the graph of the framework (by the inverse operations of deletion or splitting) and specify the position of the new vertex. The positions of the old vertices do not change.

Let $G = (V, E)$ be a graph and let (G, p) be a framework. The operation *0-extension (on distinct vertices $a, b \in V$)* adds a new vertex v to G and two edges va, vb, and determines the position $p(v)$ of v in the new framework.

Lemma 2.3. *[8, Lemma 2.1.3] Suppose that (G, p) is a rigid framework. Then the 0-extension of (G, p) on vertices a, b is rigid for all choices $p(v)$ with $p(a), p(b), p(v)$ not collinear.*

The operation *1-extension (on edge $ab \in E$ and vertex $c \in V - \{a, b\}$)* subdivides the edge ab by a new vertex v and adds a new edge vc, and determines the position $p(v)$ of v in the new framework.

Lemma 2.4. *[8, Theorem 2.2.2] Suppose that (G, p) is a rigid framework, $ab \in E(G)$, $c \in V(G) - \{a, b\}$, and the points $p(a), p(b), p(c)$ are not collinear. Then the 1-extension of (G, p) on ab and c is rigid if $p(v)$ is any point on the line of $p(a), p(b)$, distinct from $p(a), p(b)$.*

Note that in Lemmas 2.3 and 2.4 $p(v)$ may be a point already covered by (G, p).

We shall perform splittings at some vertex v only if v has a neighbour of small degree. The existence of such a vertex is guaranteed by the following lemma. Let $\delta(G)$ denote the minimum degree in graph G. It is easy to see that for a minimally rigid graph G we have $\delta(G) \in \{2, 3\}$.

Lemma 2.5. *Let $G = (V, E)$ be a minimally rigid graph with $\delta(G) = 3$. Then there is an edge $uv \in E$ with $d(v) = 3$ and $d(u) \leq 8$.*

Proof: Let $A = \{w \in V : d(w) = 3\}$ and let $B = V - A$. Since $\delta(G) = 3$, we have $A \neq \emptyset$. If there is an edge between two vertices of A then we are done. Thus we may assume that $i(A) = 0$. Since G is minimally rigid, $i(A) = 0$, and each vertex in A has degree three, we have $|E| = 2|V| - 3 = 2|A| + 2|B| - 3$ and $d(A) = 3|A|$, where $d(A)$ is the number of edges leaving A. Hence $i(B) = |E| - d(A) = 2|B| - |A| - 3$.

Let $D = \{x \in B : d_{G[B]}(x) \leq 3\}$. Clearly, each vertex $x \in D$ is connected to A by at least one edge. Since $\sum_{x \in B} d_{G[B]}(x) = 2i(B) = 4|B| - 2|A| - 6$, it follows that $|D| \geq |A|/2 + 1$. Now $d(A) = 3|A|$ implies that there is a vertex $u \in D$ which is connected to A by at most five edges. Since $d_{G[B]}(u) \leq 3$, this implies $d_G(u) \leq 8$. Thus any edge uv with $v \in A$ satisfies the requirements of the lemma. •

3 Rigid realizations on a small grid

Theorem 3.1. *Let $G = (V, E)$ be a minimally rigid graph on n vertices. Then there is an infinitesimally rigid non-degenerate framework (G, p) for which $p(v) \in \mathbb{Z}_k^2$ for all $v \in V$, where $k = \lceil \sqrt{n-1} \rceil + 9$.*

Proof: The proof is by induction on n. The theorem trivially holds for $n = 2$, so may assume that $n \geq 3$ and that the required frameworks exist for graphs on at most $n - 1$ vertices. Since G is minimally rigid, we have $\delta(G) \in \{2, 3\}$.

First suppose that $\delta(G) = 2$ and let $v \in V$ with $d_G(v) = 2$. By Lemma 2.2 $H := G - v$ is minimally rigid. By the induction hypothesis this implies that there is a rigid framework (H, q) with $q(z) \in \mathbb{Z}_k^2$ for all $z \in V(H)$. Let $N_G(v) = \{u, w\} \subseteq V(H)$ and let $L \subset \mathbb{R}^2$ be the line of $q(u), q(w)$. We claim that there is a point $(x, y) \in \mathbb{Z}_k^2$ which is not on L and which is uncovered by (H, q). To see this observe that we have at most k grid points on L and at most $|V(H)| - 2 = n - 3$ grid points covered (H, q) which are not on L. Thus

$$\begin{aligned} |\mathbb{Z}_k^2| &= k^2 \geq (\sqrt{n-1} + 9)^2 = n - 1 + 18\sqrt{n-1} + 81 \\ &= n + 18\sqrt{n-1} + 80 > \sqrt{n-1} + 10 + n - 3 \geq k + n - 3, \end{aligned} \tag{2}$$

which implies the claim. Let $p(v) = (x, y)$ and let $p(x) = q(x)$ for all $x \in V - v$. By Lemma 2.3, and by the choice of (x, y), (G, p) is the required rigid framework on G.

Next suppose that $\delta(G) = 3$. By Lemma 2.5 there is an edge uv with $d(v) = 3$ and $d(u) \leq 8$. Let $N_G(v) = \{u, w, t\}$. By Lemma 2.2 the graph $H = G - v + e$ is minimally rigid, where e is some edge connecting two non-adjacent vertices from $N_G(v)$. By the induction hypothesis this implies that there is a rigid framework (H, q) with $q(z) \in \mathbb{Z}_k^2$ for all $z \in V(H)$.

Claim 3.2. *There is an infinitesimally rigid framework (H, q') for which $q'(z) \in \mathbb{Z}_k^2$ for all $z \in V(H)$ and such that $q'(u), q'(w), q'(t)$ are not collinear.*

Proof: Suppose that $q(u), q(w), q(t)$ are collinear in (H, q). By symmetry we may assume that the line L of $q(u), q(w), q(t)$ is not vertical. Now suppose, for a contradiction, that there exist $k - 8$ columns in $\mathbb{Z}_k^2$ which contain at least $k - 9$ grid points covered by (H, q). Then

$$\begin{aligned} n - 1 &= |V(H)| \geq (k - 8)(k - 9) \\ &> (k - 9)^2 \geq (\sqrt{n-1} + 9 - 9)^2 = n - 1 \end{aligned} \tag{3}$$

follows, a contradiction. Thus at least 9 columns of $\mathbb{Z}_k^2$ contain at least 10 uncovered points with respect to (H, q). By using this fact and Lemma 1.2, and since $d_H(u) \leq d_G(u) \leq 8$, we can first move u horizontally to a a point z which belongs to a the line C of some column of $\mathbb{Z}_k^2$ containing at least 10 uncovered points, such that the resulting framework remains rigid. This temporary position of u need not be on the grid and need not be uncovered by (H, q).

Since C contains at least 10 uncovered grid points, it contains at least 9 uncovered grid points $p_1, \ldots, p_9$, such that $p(t), p(w), p_i$ are not collinear, $1 \leq i \leq 9$. By applying Lemma 1.2 again we can move u further vertically to one of these grid points p_i such that the resulting framework (H, q') remains rigid and such that $q'(u), q'(w), q'(t)$ are not on a line. •

By Claim 3.2 we may assume that $q(u), q(w), q(t)$ are not collinear. Suppose, without loss of generality, that $e = uw$, i.e. the splitting operation adds a new edge uw (we shall no longer use the fact that $d_G(u)$ is small). First we construct a rigid framework (G, p') by applying a 1-extension on (H, q) so that $p'(v)$ is a point in the intersection of the line of $q(u), q(w)$, and the line of some column of the grid, and such that $p'(v) \neq q(u), q(w)$. This is possible by Lemma 2.4, since $q(u), q(w), q(t)$ are not collinear, and $k \geq 3$. This temporary position of v need not be on the grid and need not be uncovered by (H, q).

We claim that there exist at least 4 rows of $\mathbb{Z}_k^2$ containing at least 4 uncovered grid points with respect to (G, p'). To see this suppose, for a contradiction, that there exist $k-3$ rows of the grid which contain at least $k-3$ grid points covered by (G, p'). Then

$$\begin{aligned} n &= |V(G)| \geq (k-3)^2 \geq (\sqrt{n-1}+9-3)^2 \\ &= (\sqrt{n-1}+6)^2 = n-1+12\sqrt{n-1}+36 > n \end{aligned} \tag{4}$$

follows, a contradiction. This proves the claim. By using the claim and Lemma 1.2 we can move $p'(v)$ further vertically to some row of the grid which contains at least 4 uncovered points, preserving the rigidity of the framework. This new position is also temporary, and it need not be on the grid and need not be uncovered by (H, q). Finally, using that $d_G(v) = 3$, we can use Lemma 1.2 to move v again horizontally to some uncovered grid point in this row of the grid such that the new framework (G, p) obtained is also rigid, all points $p(v)$, $v \in V$ are distinct, and $p(v) \in \mathbb{Z}_k^2$ for all $v \in V$. This proves the theorem. •

4 Concluding remarks

The proof of Theorem 3.1 is algorithmic: the required infinitesimally rigid realization of a minimally rigid graph G can be found by first reducing G to a single edge by vertex deletions and splittings, and then building up the framework by using extensions and moving coordinates. This algorithm needs a subroutine to find an admissible splitting at a vertex of degree three. This can be done in $O(n^2)$ time, see [1] and the references therein. Within the same time bound one can find a minimally rigid spanning subgraph of a rigid graph. Thus an infinitesimally rigid realization of a rigid graph in $\mathbb{Z}_k^2$ can be found in $O(n^3)$ time, where $k = \lceil \sqrt{(n-1)} \rceil + 9$.

Since we considered non-degenerate frameworks, our bound k on the size of the grid is essentially best possible. It might be possible to specify a set S of $n+c$ points in $\mathbb{R}^2$, for some constant c, such that every rigid graph on n vertices has an infinitesimally rigid non-degenerate realization on S.

For degenerate frameworks we have the following lower bound. Let H be a minimally rigid graph on a set K of $k \geq 2$ vertices and let G be obtained from H by adding $\binom{k}{2}$ new vertices of degree two in such a way that each new vertex w is adjacent to a pair of vertices of K, and these pairs of neighbours of the new vertices are pairwise distinct. Then G has $\binom{(k+1)}{2}$ vertices, and in any rigid realization of G the vertices of K must be distinct. Thus we obtain a lower bound of $O(n^{\frac{1}{4}})$ on the grid size. It may be interesting to note that in one dimension every rigid (i.e. connected) graph has a degenerate infinitesimally rigid realization on the grid of size two.

Another direction for possible extensions is to try to find a realization (G, p) of a graph G, on a small grid, for which the rigidity matroid of (G, p) is isomorphic to the rigidity matroid of G. What can we say when G is rigid and $|E| = 2|V| - 2$?

References

[1] A. Berg, T. Jordán, Algorithms for graph rigidity and scene analysis, Proc. 11th Annual European Symposium on Algorithms (ESA) 2003, (G. Di Battista, U. Zwick, eds) Springer Lecture Notes in Computer Science 2832, pp. 78-89, 2003.

[2] J. Graver, B. Servatius, and H. Servatius, Combinatorial Rigidity, AMS Graduate Studies in Mathematics Vol. 2, 1993.

[3] B. Jackson, T. Jordán, Connected rigidity matroids and unique realizations of graphs, EGRES Technical Report 2002-12, `www.cs.elte.hu/egres/`, *J. Combin. Theory Ser. B*, to appear.

[4] G. Laman, On graphs and rigidity of plane skeletal structures, *J. Engineering Math.* 4 (1970), 331-340.

[5] J.T. Schwartz, Fast probabilistic algorithms for verification of polynomial identities, *J. ACM* 27, 701-717, 1980.

[6] T.S. Tay and W. Whiteley, Generating isostatic frameworks, *Structural Topology* 11, 1985, pp. 21-69.

[7] W. Whiteley, Matroids and rigid structures, Matroid applications, 1–53, Encyclopedia Math. Appl., 40, Cambridge Univ. Press, Cambridge, 1992.

[8] W. Whiteley, Some matroids from discrete applied geometry, in Matroid theory (J.E. Bonin, J.G. Oxley and B. Servatius eds., Seattle, WA, 1995), Contemp. Math., 197, Amer. Math. Soc., Providence, RI, 1996, 171–311.

[9] W. Whiteley, W. Whiteley, Rigidity and scene analysis, Handbook of Discrete and Computational Geometry, 2nd edition, CRC Press, 1327-1354, 2004.