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Notes on well-balanced orientations

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Abstract

This note contains some remarks on the well-balanced orientation theorem of Nash-Williams [10]. He announced in [11] an extension of his theorem. We present a proof for a generalization of this extension. We show some new consequences of Nash-Williams' odd vertex pairing theorem. A slight generalization of a theorem of Lovász [7] will also be proved.

1 Introduction

This paper concerns undirected and directed graphs, more precisely we shall consider orientations of undirected graphs. The starting point is Robbins' theorem [13] that states that an undirected graph G has a strongly connected orientation if and only if G is 2-edge-connected. The following generalization was proved by Nash-Williams [10]: an undirected graph G has a k-arc-connected orientation if and only if G is $2k$-edge-connected (Theorem 3). This result about global edge-connectivity can be easily proved by applying Lovász' splitting off theorem [6]. Nash-Williams [10] also provided the following extension on local edge-connectivity: for any undirected graph G there exists a **well-balanced** orientation that is an orientation of G such that for every ordered pair of vertices u, v, if the maximum number of edge disjoint $(u, v)-$paths was $\lambda_G(u, v)$ in G then the maximum number of arc disjoint directed $(u, v)-$paths is at least $\lfloor \lambda_G(u, v)/2 \rfloor$ in the resulting directed graph. The well-balanced orientation may also be required to be **smooth**, that is the difference between the in-degree and the out-degree of every vertex is at most one (Theorem 4). A smooth well-balanced orientation will be called **best-balanced**. In fact, Nash-Williams proved an even stronger result in [10] the so-called odd vertex pairing theorem (Theorem 6). In [11], Nash-Williams announced an extension of his orientation theorem: for an arbitrary subgraph H of an undirected graph G there exists a best-balanced orientation of H that can be extended to a best-balanced orientation of G (Theorem 5). He mentioned that *"Given Theorem 6, the proof of Theorem 5 is not unreasonably difficult. At a certain stage in the proof Theorem 4 ... we had occasion to select an arbitrary di-Eulerian orientation Δ of the finite Eulerian graph $G + P$. ... the proof of Theorem*

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5 depends essentially on the idea of modifying this step ... by choosing Δ to be, not just any di-Eulerian orientation of $G+P$, but one which satisfies certain additional restrictions." The main contribution of the present note is to provide a simple proof for a generalization of this result, namely for an arbitrary partition of (the edge-set of) G into subgraphs $\{G_1, ..., G_k\}$, there is an orientation $\vec{G}$ of G such that $\vec{G}$ and $\vec{G}_i$ are best-balanced orientations of G and of G_i for $1 \le i \le k$.

The organization of this note is as follows. After the Introduction the necessary notation are given. Then we give some preliminary observations, we show among others that for Eulerian graphs well-balanced orientations are necessarily Eulerian. This follows easily from a result of Lovász [7]. We shall provide a slight generalization of this result of Lovász. In Section 4 we present the above mentioned theorems and then we add some new ones which will be proved in Section 5. Then we show some new applications of the odd vertex pairing theorem and Theorem 5.

2 Notation, definitions

A directed graph will be denoted by $\vec{G} = (V, A)$ and an undirected graph by $G = (V, E)$. For a directed graph $\vec{G}$, a set $X \subseteq V$ and $u, v \in V$, let $\delta_{\vec{G}}(X) := |\{uv \in A : u \in X,\ v \notin X\}|$, $\varrho_{\vec{G}}(X) := \delta_{\vec{G}}(V - X)$, $f_{\vec{G}}(X) := \varrho_{\vec{G}}(X) - \delta_{\vec{G}}(X)$, $T^+_{\vec{G}} := \{v \in V : \varrho_{\vec{G}}(v) > \delta_{\vec{G}}(v)\}$, $T^-_{\vec{G}} := \{v \in V : \varrho_{\vec{G}}(v) < \delta_{\vec{G}}(v)\}$, $\lambda_{\vec{G}}(u, v) := \min\{\delta_{\vec{G}}(Y) : u \in Y,\ v \notin Y\}$, and $\overleftarrow{G} := (V, \{vu : uv \in A\})$. For an undirected graph G, a set $X \subseteq V$ and $u, v \in V$, let $\Delta_G(X) := \{uv \in E : u \in X,\ v \notin X\}$, $d_G(X) := |\Delta_G(X)|$, $\lambda_G(u, v) := \min\{d_G(X) : u \in X,\ v \notin X\}$, $R_G(X) := \max\{\lambda_G(x, y) : x \in X,\ y \notin X\}$, $\hat{R}_G(X) := 2\lfloor R_G(X)/2 \rfloor$, $b_G(X) := d_G(X) - \hat{R}_G(X)$ and $T_G := \{v \in V : d_G(v) \text{ is odd}\}$. Observe that $\forall X \subseteq V$, (considering $R_G(X) = d_G(X) = b_G(X) = \varrho_{\vec{G}}(X) = \delta_{\vec{G}}(X) = 0$ if $X = \emptyset$ or $X = V$)

$$f_{\vec{G}}(X) = \sum_{v \in X} f_{\vec{G}}(v), \tag{1}$$
$$0 \le b_G(X) \le d_G(X). \tag{2}$$

Let $G = (V, E)$ be an undirected graph. G is **connected** if for every pair $u, v \in V$ of vertices there is a (u, v)-path in G. G is called **k-edge-connected** if $G - F$ is connected for $\forall F \subseteq E$ with $|F| \le k - 1$. In this paper, if it is not explicitly stated, graphs may be disconnected, and we use the notion "Eulerian graph" for a possibly disconnected graph with all degree even. Let $D = (V, A)$ be a directed graph. D is **strongly connected** if for every ordered pair $(u, v) \in V \times V$ of vertices there is a directed (u, v)-path in D. D is called **k-arc-connected** if $G - F$ is strongly connected for $\forall F \subseteq A$ with $|F| \le k - 1$. An orientation $\vec{G}$ of G is called **well-balanced** if $\vec{G}$ satisfies (3) and it is called **smooth** if $\vec{G}$ satisfies (4). A smooth well-balanced orientation is called **best-balanced**. Note that if $\vec{G}$ is best-balanced then so is $\overleftarrow{G}$.

$$\lambda_{\vec{G}}(x, y) \ge \lfloor \lambda_G(x, y)/2 \rfloor \quad \forall\ (x, y) \in V \times V, \tag{3}$$
$$|f_{\vec{G}}(v)| \le 1 \quad \forall\ v \in V. \tag{4}$$

A **pairing** M of G is a new graph on vertex set T_G in which each vertex has degree one. Let M be a pairing of G. An orientation $\vec{M}$ of M that satisfies (5) is called **good**. M is **well-orientable** if there *exists* a good orientation of M, M is **strong** if *every* orientation of M is good and M is **feasible** if (6) is satisfied. Clearly an oriented pairing $\vec{M}$ is good iff $\overleftarrow{M}$ is good. We say that the orientations $\vec{M}$ and $\vec{G}$ are **compatible** if (7) is satisfied or equivalently if $\vec{G} + \overleftarrow{M}$ is Eulerian.

$$\begin{aligned} f_{\vec{M}}(X) &\leq b_G(X) \quad \forall X \subseteq V, & (5)\\ d_M(X) &\leq b_G(X) \quad \forall X \subseteq V, & (6)\\ f_{\vec{G}}(X) &= f_{\vec{M}}(X) \quad \forall X \subseteq V. & (7) \end{aligned}$$

3 Equivalent forms

Claim 1. *For an orientation $\vec{G}$ of an undirected graph G, the following are equivalent:*

$$\begin{aligned} \vec{G} &\quad is \quad well-balanced, & (8)\\ \delta_{\vec{G}}(X) &\geq \lfloor R_G(X)/2 \rfloor \quad \forall\, X \subseteq V, & (9)\\ f_{\vec{G}}(X) &\leq b_G(X) \qquad \forall\, X \subseteq V. & (10) \end{aligned}$$

Proof. (8) $\Longleftrightarrow$ (9) : For $X \subset V$, let $x \in X,\ y \in V - X$ with $\lambda_G(x,y) = R_G(X)$. Then, by (3), $\delta_{\vec{G}}(X) \geq \lambda_{\vec{G}}(x,y) \geq \lfloor \lambda_G(x,y)/2 \rfloor = \lfloor R_G(X)/2 \rfloor$. For $x, y \in V$, let $X \ni x,\ V - X \ni y$ with $\delta_{\vec{G}}(X) = \lambda_{\vec{G}}(x,y)$. Then, by (9), $\lambda_{\vec{G}}(x,y) = \delta_{\vec{G}}(X) \geq \lfloor R_G(X)/2 \rfloor \geq \lfloor \lambda_G(x,y)/2 \rfloor$.
(9) $\Longleftrightarrow$ (10) : $b_G(X) - f_{\vec{G}}(X) = (d_G(X) - \hat{R}_G(X)) - (\varrho_{\vec{G}}(X) - \delta_{\vec{G}}(X)) = 2(\delta_{\vec{G}}(X) - \hat{R}_G(X)/2)$. □

Claim 2. *A pairing M of G is strong if and only if M is feasible.*

Proof. If M is feasible, then every orientation $\vec{M}$ of M is good because, by (6), $f_{\vec{M}}(X) = \varrho_{\vec{M}}(X) - \delta_{\vec{M}}(X) \leq d_M(X) \leq b_G(X) \quad \forall\, X \subseteq V$. If M is strong, then for $X \subseteq V$, let $\vec{M}$ be an orientation of M with $\delta_{\vec{M}}(X) = 0$. Then (6) is satisfied because, by (5), $d_M(X) = f_{\vec{M}}(X) \leq b_G(X)$. □

Claim 3. *Let G be an undirected graph.*

(a) If $\vec{M}$ is a well-oriented pairing of G, then G has an orientation $\vec{G}$ compatible to $\vec{M}$.

(b) For a pairing M of G, let the orientations $\vec{M}$ and $\vec{G}$ be compatible. Then $\vec{G}$ is smooth. Moreover, $\vec{G}$ is well-balanced if and only if $\vec{M}$ is good.

(c) If $\vec{G}$ is a smooth orientation of G, then G has an oriented pairing $\vec{M}$ compatible to $\vec{G}$.

(d) If T_G is partitioned into $T^+ \cup T^-$ such that $|T^+| = |T^-|$ and $|X \cap T^+| - |X \cap T^-| \leq b_G(X)\ \forall X \subseteq V$, then there exists a well-oriented pairing $\vec{M}$ satisfying $T^+ = T^+_{\vec{M}}$.

(e) G has a best-balanced orientation if and only if G has a well-orientable pairing.

(f) For a strong pairing M of G, for every Eulerian orientation $\vec{G}+\vec{M}$ of $G+M$, $\vec{G}$ is best-balanced.

Proof. (a) Note that $G+M$ is Eulerian and $\overleftarrow{M}$ is also good. By (5) and (2), $f_{\overleftarrow{M}}(X) \leq b_G(X) \leq d_G(X) \quad \forall X \subseteq V$. Then, by the theorem of Ford and Fulkerson [1], $\overleftarrow{M}$ can be extended to an Eulerian orientation $\vec{G}+\overleftarrow{M}$ of $G+M$. Then, as we observed before, $\vec{M}$ and $\vec{G}$ are compatible.

(b) By (7), $|f_{\vec{G}}(v)| = |f_{\vec{M}}(v)| \leq 1 \quad \forall v \in V$, so $\vec{G}$ is smooth. By (7), (10) and (5) are equivalent.

(c) By (4), $\{T^+_{\vec{G}}, T^-_{\vec{G}}\}$ is a partition of T_G and, by (1) and (4), $0 = f_{\vec{G}}(V) = \sum_{v\in V} f_{\vec{G}}(v) = \sum_{v\in T^+_{\vec{G}}\cup T^-_{\vec{G}}\cup(V-T_G)} f_{\vec{G}}(v) = |T^+_{\vec{G}}| - |T^-_{\vec{G}}|$. Then there exists an oriented pairing $\vec{M}$ of G from $T^-_{\vec{G}}$ to $T^+_{\vec{G}}$. Then $f_{\vec{G}}(v) = f_{\vec{M}}(v) \quad \forall v \in V$, so by (1), $\vec{M}$ and $\vec{G}$ are compatible.

(d) Define $\vec{M}$ as before and note that $f_{\vec{M}}(X) = |T^+\cap X| - |T^-\cap X| = f_{\vec{G}}(X) \; \forall X \subseteq V$.

(e) It follows from (a), (b) and (c).

(f) As M is strong, $\overleftarrow{M}$ is good and clearly $\overleftarrow{M}$ and $\vec{G}$ are compatible, so we are done by (b). □

By the following easy corollary of Theorem 1 of Lovász [7], for an orientation $\vec{G}$ of an Eulerian graph G, $\vec{G}$ is well-balanced if and only if $\vec{G}$ is Eulerian.

Claim 4. *Let $\vec{G}$ be an orientation of a graph G. Then the following are equivalent:*

(a) $\vec{G}$ is Eulerian,

(b) G is Eulerian and $\vec{G}$ is well-balanced,

(c) $\lambda_{\vec{G}}(u,v) = \lambda_{\vec{G}}(v,u) \quad \forall u,v \in V$.

Theorem 1. *Let r be a vertex of a directed graph $\vec{G}$. If $\delta_{\vec{G}}(r) > \varrho_{\vec{G}}(r)$, then there exists a vertex x such that $\lambda_{\vec{G}}(r,x) > \lambda_{\vec{G}}(x,r)$.*

The proof of Lovász [7] for Theorem 1 with a convenient modification provides a slight generalization.

Theorem 2. *Let r be a vertex of a directed graph $\vec{G}$. If $\delta_{\vec{G}}(r) > \varrho_{\vec{G}}(r)$, then there exists a vertex x such that $\lambda_{\vec{G}}(r,x) > \lambda_{\vec{G}}(x,r)$ and $\delta_{\vec{G}}(x) < \varrho_{\vec{G}}(x)$.*

Proof. A set $X \subseteq V - r$ is called $T^+_{\vec{G}}$-regular with core x if $x \in T^+_{\vec{G}} \cap X$ and $\varrho_{\vec{G}}(X) = \lambda_{\vec{G}}(r,x)$. Let $\mathcal{U} := \{U_1, ..., U_k\}$ be the set of maximal $T^+_{\vec{G}}$-regular sets. Let $V_i := U_i - \bigcup_{j\neq i} U_j$. By submodularity, the core u_i of U_i belongs to V_i (if u_i were in U_j then $U_i \cup U_j$ is a $T^+_{\vec{G}}$-regular set contradicting the maximality). Suppose $\lambda_{\vec{G}}(r,x) \leq \lambda_{\vec{G}}(x,r) \quad \forall x \in T^+_{\vec{G}}$. Then $\delta_{\vec{G}}(V_i) \geq \lambda_{\vec{G}}(u_i,r) \geq \lambda_{\vec{G}}(r,u_i) = \varrho_{\vec{G}}(U_i)$, hence $\sum_1^k \delta_{\vec{G}}(V_i) \geq$

$\sum_1^k \varrho_{\vec{G}}(U_i)$. Let $V_0 := U_0 := V - \bigcup_1^k U_i$. By an easy counting argument (as in [7]), we have $\sum_0^k \delta_{\vec{G}}(V_i) \leq \sum_0^k \varrho_{\vec{G}}(U_i)$. However, since $T_{\vec{G}}^+ \subseteq \bigcup_1^k U_i$, $\varrho_{\vec{G}}(x) - \delta_{\vec{G}}(x) \leq 0 \ \ \forall x \in U_0$ and $\varrho_{\vec{G}}(r) - \delta_{\vec{G}}(r) < 0$, thus $\varrho_{\vec{G}}(V_0) - \delta_{\vec{G}}(U_0) = \sum_{x \in U_0}(\varrho_{\vec{G}}(x) - \delta_{\vec{G}}(x)) < 0$, a contradiction. □

4 Theorems

The following four theorems are due to Nash-Williams [10], [11].

Theorem 3. *A graph G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.*

Theorem 4. *Every graph has a best-balanced orientation.*

Theorem 5. *For every subgraph H of G, there exists a best-balanced orientation of H that can be extended to a best-balanced orientation of G.*

Theorem 6. *Every graph has a feasible pairing.*

We shall show that the above "odd vertex pairing" theorem implies all the other results presented in this section. By Claim 2, Theorem 6 is equivalent to Theorem 7.

Theorem 7. *Every graph has a strong pairing.*

By Theorem 4 and Claim 3, every graph has a well-orientable pairing. In the following theorem we generalize this result.

Theorem 8. *Every pairing is well-orientable.*

The main results of this note are the following generalizations of Theorem 5 and Theorem 4.

Theorem 9. *For every partition $\{E_1, E_2, ..., E_k\}$ of $E(G)$, if $G_i = (V, E_i)$ then G has a best-balanced orientation $\vec{G}$, such that the inherited orientation of each G_i is also best-balanced.*

Theorem 10. *For every partition $\{X_1, ..., X_l\}$ of $V = V(G)$, G has an orientation $\vec{G}$ such that $\vec{G}$, $((\vec{G}/X_1)...)/X_l$ and $\vec{G}/(V - X_i)$ $(1 \leq i \leq l)$ are best-balanced orientations of the corresponding graphs.*

5 Proofs

In this section we shall apply Theorem 7 (that is equivalent to Theorem 6) to prove all the results in the previous section. We must emphasize that we do not have a new proof for Theorem 6.

Proof of Theorem 3: If graph G is $2k$-edge-connected, then, by Menger's Theorem, $\lambda_G(x,y) \geq 2k \ \ \forall x,y \in V$. Then, by Theorem 4, there exists an orientation $\vec{G}$ of G such that $\lambda_{\vec{G}}(x,y) \geq \lfloor \lambda_G(x,y)/2 \rfloor \geq k \ \ \forall (x,y) \in V \times V$, that is, by Menger's Theorem, $\vec{G}$ is k-arc-connected. □

Proof of Theorem 4: By Theorem 7 and Claim 3(e). □

Proof of Theorem 5: By Theorem 9 with $E_1 = E(H),\ E_2 = E(G) - E(H)$. □

Proof of Theorem 8: Let M_1 be an arbitrary and M_2 be a strong pairing of G. M_2 exists by Theorem 7. $M_1 \cup M_2$ is an Eulerian graph so it has an Eulerian orientation $\vec{M}_1 \cup \vec{M}_2$. Then $f_{\vec{M}_1}(v) = -f_{\vec{M}_2}(v) = f_{\overleftarrow{M_2}}(v) \ \ \forall v \in V$. Then, by (1) and using that $\overleftarrow{M}_2$ is a good orientation of M_2, $f_{\vec{M}_1}(X) = \sum_{v \in X} f_{\vec{M}_1}(v) = \sum_{v \in X} f_{\overleftarrow{M_2}}(v) = f_{\overleftarrow{M_2}}(X) \leq b_G(X) \ \ \forall X \subseteq V$, so $\vec{M}_1$ is a good orientation of M_1. □

By the above proof, if we know a feasible pairing, then for every pairing we can find the required orientation in polynomial time. Note that if we apply Theorem 5 with $H' = G$ and $G' = G + M$ we get another proof for Theorem 8.

The following proofs of Theorem 9 and Theorem 10 are the main contribution of the present note.

Proof of Theorem 9: Let M_0 and M_i be strong pairings of G and of G_i for $1 \leq i \leq k$ provided by Theorem 7. Note that for $K := \sum_0^k M_i$, for every $v \in V$, $d_K(v) = \sum_0^k d_{M_i}(v) \equiv d_G(v) + \sum_1^k d_{G_i}(v) = 2d_G(v)$ is even, so K has an Eulerian orientation $\vec{K} = \sum_0^k \vec{M}_i$ that is $\sum_1^k f_{\vec{M}_i}(X) = -f_{\vec{M}_0}(X) \quad \forall X \subseteq V$. For $1 \leq i \leq k$, $\vec{M}_i$ is a good orientation of M_i so, by Claim 3(a) and (b), G_i has a best-balanced orientation $\vec{G}_i$ compatible to $\vec{M}_i$. Let $\vec{G} := \cup_1^k \vec{G}_i$. Then, by (7), $f_{\vec{G}}(X) = \sum_1^k f_{\vec{G}_i}(X) = \sum_1^k f_{\vec{M}_i}(X) = -f_{\vec{M}_0}(X) = f_{\overleftarrow{M_0}}(X) \ \ \forall X \subseteq V$ so $\vec{G}$ and $\overleftarrow{M}_0$ are compatible. $\overleftarrow{M}_0$ is a good orientation of M_0 thus, by Claim 3(b), $\vec{G}$ is a best-balanced orientation of G. □

Proof of Theorem 10: Let $G_0 := (((G/X_1)/X_2)/...)/X_l$ and $G_i := G/(V - X_i)$ $(1 \leq i \leq l)$. Let M_i be a strong pairing of G_i $(0 \leq i \leq l)$ provided by Theorem 7. It is easy to see that G has a unique pairing M whose restriction in G_i is M_i. By Theorem 8, M has a good orientation $\vec{M}$. By Claim 3(a) and (b), G has a best-balanced orientation $\vec{G}$ compatible to $\vec{M}$. $\vec{G}$ and $\vec{M}$ define the orientations $\vec{G}_i$ of G_i and $\vec{M}_i$ of M_i for $0 \leq i \leq l$. Then, by construction, $\vec{G}_i$ and $\vec{M}_i$ are compatible. Since $\vec{M}_i$ is a good orientation of M_i, $\vec{G}_i$ is a best-balanced orientation of G_i by Claim 3(b). □

For a relatively simple proof for Theorem 6 see Frank [2]. A polynomial time algorithm to find a feasible pairing can be found in [4].

6 Corollaries

We start by showing that two graphs with the same set of odd degree vertices have a common well-oriented pairing.

Corollary 1. *Let* $G_i := (V_i, E_i)$ $(i = 1, 2)$ *be undirected graphs with* $E_1 \cap E_2 = \emptyset$ *and* $T_{G_1} = T_{G_2}$. *Then there exist best-balanced orientations* $\vec{G}_i$ $(i = 1, 2)$ *such that* $T^+_{\vec{G}_1} = T^+_{\vec{G}_2}$.

Proof. Let M_i be a strong pairing of G_i $(i = 1, 2)$ provided by Theorem 7. $M_1 \cup M_2$ is an Eulerian graph so it has an Eulerian orientation $\vec{M}_1 \cup \vec{M}_2$. Then $T^+_{\vec{M}_1} = T^+_{\overleftarrow{M}_2}$. As $\vec{M}_1$ and $\overleftarrow{M}_2$ are good orientations of M_1 and M_2, so, by Claim 3(a) and (b), there exist best-balanced orientations $\vec{G}_i$ of G_i that are compatible to $\vec{M}_1$ and $\overleftarrow{M}_2$ resp. Clearly $T^+_{\vec{G}_1} = T^+_{\vec{M}_1} = T^+_{\overleftarrow{M}_2} = T^+_{\vec{G}_2}$. □

Theorem 5 implies the following result for global edge-connectivity.

Corollary 2. *For a subgraph* H *of* G*,* H *has an* l*-arc-connected orientation that can be extended to a* k*-arc-connected orientation of* G *if and only if* H *and* G *are* $2l$*- and* $2k$*-edge-connected respectively.*

Remark We mention that there is a short proof for Corollary 2. Indeed, we have already seen that Theorem 6 implies easily Theorem 5. Thus all we have to do is to provide an easy proof for Theorem 6 for even global edge-connectivity. This is what follows.

Claim 5. *Let* $G := (V, E)$ *be a* $2k$*-edge-connected graph. Then there is a pairing* M *of* G *so that*

$$d_M(X) \leq d_G(X) - 2k \quad \forall X \subset V,\ X \neq \emptyset. \tag{11}$$

Proof. We prove it by induction on $|E|$.

Case 1 If there is $s \in V$ with $d(s)$ even. Then, by Lovász' splitting off theorem [6], there exists a partition of $\Delta_G(s)$ into pairs $u_i s, s v_i$ such that the graph G' obtained from G by replacing each pair $u_i s, s v_i$ by a new edge $u_i v_i$ and then by deleting the vertex s is $2k$-edge-connected. Note that $T_{G'} = T_G$ and $|E(G')| < |E|$ so by induction there is a pairing M of G' that satisfies (11) for G'. Then M is a pairing of G and, since $d_{G'}(X) \leq d_G(X)$ for $X \subset V$, clearly M satisfies (11) and we are done.

Case 2 Otherwise, $\forall s \in V,\ d(s)$ is odd. Then $T_G = V$. By a result of Mader [8], since there is no vertex v with $d(v) = 2k$, there exists $uv \in E$ such that $G' := G - uv$ is $2k$-edge-connected. Note that $T_{G'} = T_G - \{u, v\}$ and $|E(G')| < |E|$ so by induction there is a pairing M' of G' so that (11) is satisfied for G' and M'. Let $M := M' \cup uv$. Then $\forall X \subseteq V$ either $d_M(X) = d_{M'}(X)$ and $d_G(X) = d_{G'}(X)$ or $d_M(X) = d_{M'}(X) + 1$ and $d_G(X) = d_{G'}(X) + 1$ so (11) is satisfied. □

Theorem 5 easily implies the following.

Corollary 3. *If* H *is an Eulerian subgraph of* G*, then any Eulerian orientation of* H *can be extended to a best-balanced orientation of* G*.*

Proof. By Theorem 5, H has a best-balanced orientation $\vec{H}$ that an be extended to a best-balanced orientation of G. Since H is Eulerian, $\vec{H}$ is an Eulerian orientation. Observe that any other Eulerian orientation of H can be reached by reversing directed cycles, and operations of this type cannot make the best-balanced orientation of G wrong. □

Let G be a non-Eulerian graph. Then, by Theorem 1, for *every* best-balanced orientation $\vec{G}$ *there exists* a pair of vertices x and y with $\lambda_{\vec{G}}(x,y) > \lambda_{\vec{G}}(y,x)$. In the following corollary we show that for *every* pair x, y of vertices with $\lambda_G(x,y)$ odd *there exists* a best-balanced orientation $\vec{G}$ with $\lambda_{\vec{G}}(x,y) > \lambda_{\vec{G}}(y,x)$.

Corollary 4. *Let $x, y \in V(G)$ with $\lambda_G(x,y) = 2k+1$. Then G has a best-balanced orientation $\vec{G}$ such that $\lambda_{\vec{G}}(x,y) = k+1$.*

Proof. Let $G' = G + xy$ and $H' = G$. Note that $\lambda_{G'}(x,y) = 2k+2$. By applying Theorem 5 for G' and H' the corollary follows (either $\vec{G}$ or $\overleftarrow{G}$ is good). □

Theorem 4 and a theorem of Rizzi [12] imply the following result of Rizzi.

Corollary 5. *Let T be an even subset of vertices of a connected graph G. Then G has an orientation $\vec{G}$ so that $\varrho_{\vec{G}}(X) \geq k$ for every vertex set X with $|X \cap T|$ odd if and only if $d_G(X) \geq 2k$ for every vertex set X with $|X \cap T|$ odd.*

We generalize this result for parity families. A family $\mathcal{F}$ of subsets of V is called **parity family** if it satisfies the following properties: (0) $\emptyset, V \notin \mathcal{F}$, (1) if $A \in \mathcal{F}$, then $V - A \in \mathcal{F}$, (2) if $A, B \notin \mathcal{F}$ and $A \cap B = \emptyset$, then $A \cup B \notin \mathcal{F}$. It was shown in [14] that if $\mathcal{F}$ is a parity family on $V(G)$ such that $d_G(X) \geq 2k$ for every $X \in \mathcal{F}$, then for all $X \in \mathcal{F}$ there exist $x \in X,\ y \in V(G) - X$ so that $\lambda_G(x,y) \geq 2k$. By applying this fact and Theorem 4, Corollary 5 can be generalized as follows.

Corollary 6. *For a given graph G and a parity family $\mathcal{F}$ on $V(G)$, G has an orientation $\vec{G}$ so that $\varrho_{\vec{G}}(X) \geq k$ for every $X \in \mathcal{F}$ if and only if $d_G(X) \geq 2k$ for every $X \in \mathcal{F}$.*

Let us finish by a related conjecture (see in Frank [3]) and a proof for a very special case of it.

Conjecture 1. *An undirected graph G has a k-vertex-connected orientation if and only if for every vertex set $X \subseteq V$ with $|X| \leq k$, $G - X$ is $(2k - 2|X|)$-edge-connected.*

Corollary 7. *Let $G = (V,E)$ be undirected graph and $u \in V$. Then G has an orientation such that for every ordered pair of vertices $(x,y) \in (V-u) \times (V-u)$ there are k arc-disjoint $(x,y)-$paths such that at most one of them contains u if and only if G is $2k$-edge-connected and $G - u$ is $(2k-2)$-edge-connected.*

Proof. To show the non-trivial direction let G satisfy the conditions. Then, by Corollary 2, G has a k-arc-connected orientation $\vec{G}$ such that $\vec{G} - u$ is $(k-1)$-arc-connected. Let $H := (V', A')$ be the directed graph with $V' := V - u + u_1 + u_2$, $A' :=$

$A(\vec{G}-u) \cup \{xu_1 : xu \in A(\vec{G})\} \cup \{u_2y : uy \in A(\vec{G})\} \cup \{u_1u_2\}$. Then, $\forall X \notin \{\emptyset, u_1, V' - u_2, V'\}, \delta_H(X) \geq k$. By Menger's theorem [9], $\forall (x,y) \in (V' - u_1) \times (V' - u_2)$ there are k arc-disjoint $(x,y)-$paths in H that correspond to the required paths in $\vec{G}$. □

We conjecture that the following is true.

Conjecture 2. *Let G be a $2k$-edge-connected graph such that $G - u$ is $(2k-2)$-edge-connected $\forall u \in V(G)$. Then G has an orientation such that for every ordered triple of vertices $(x, y, u) \in V^3$ there are k arc-disjoint $(x,y)-$paths such that at most one of them contains u as an inner vertex.*

The interested readers may find many counter-examples for problems related to well-balanced orientations in [5].

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