Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2004-09. Published by the Egrerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

A short proof on the local detachment theorem

Zoltán Szigeti

May 2004

A short proof on the local detachment theorem

Zoltán Szigeti*

Abstract

A simplified and shortened proof is presented for a theorem of Jordán and Szigeti [2] on detachments preserving local edge-connectivity.

1 Introduction

Let $G = (V + s, E)$ be a graph. A *degree specification* for s is a sequence $f(s) = (d_1, ..., d_p)$ of positive integers with $\sum_{j=1}^{p} d_j = d_G(s)$. An $f(s)$-*detachment* of G at s is the graph G' obtained from G by replacing s by a set $s_1, ..., s_p$ of independent vertices and distributing the edges incident to s among them in such a way that $d_{G'}(s_i) = d_i$ $(1 \leq i \leq p)$. Note that all the other ends of the edges in G remain the same. For a requirement function $r : V \times V \to Z_+$, we say that G is $\mathbf{r}$-*edge-connected* if $\lambda_G(u, v) \geq r(u, v)$ $\forall u, v \in V$, where $\lambda_G(u, v)$ is the *local edge-connectivity* between u and v in G, that is the size of a minimum edge cut separating u and v in G. The following theorem characterizes graphs having an $\mathbf{r}$-edge-connected $f(s)$-detachment.

Theorem 1.1 (Jordán, Szigeti [2]). *Let r be a requirement function for $G = (V + s, E)$ with $r(u, v) \geq 2$ $\forall u, v \in V$. Let $f(s) = (d_1, ..., d_p)$ be a degree specification for s with $d_i \geq 2$ $\forall i$. Let $\varphi = \sum_1^p \lfloor \frac{d_i}{2} \rfloor$. Then there exists an $\mathbf{r}$-edge-connected $f(s)$-detachment of G at s if and only if*

$$G \quad \textit{is} \quad \mathbf{r}\textit{-edge-connected,} \tag{1}$$

$$G - s \quad \textit{is} \quad (\mathbf{r}-\boldsymbol{\varphi})\textit{-edge-connected.} \tag{2}$$

The aim of this paper is to provide a short proof for Theorem 1.1. We mention that Theorem 1.1 is a common generalization of Mader's theorem [4] on splitting off preserving local edge-connectivities between vertices in V $(f(s) = (2, d_G(s)-2), r(u, v) = \lambda_G(u, v)$ $\forall u, v \in V)$ and Fleiner's theorem [1] on k-edge-connected detachments $(r(u, v) = k$ $\forall u, v \in V)$. This paper does not provide a new proof for Mader's theorem because it applies it. For a new proof on a generalization of Mader's threorem the reader is referred to [5].

*Equipe Combinatoire, Université Paris 6, 75252 Paris, Cedex 05, France. This work was done while the author was visiting the Egerváry Research Group (EGRES), Department of Operations Research, Eötvös University, Budapest.

2 Definitions, preliminary results

Recall that $G = (V + s, E)$ is **r**-edge-connected if $\lambda_G(u,v) \geq r(u,v) \quad \forall u,v \in V$. Note that it is equivalent to $h^r_G(X) \geq 0 \ \forall X \subseteq V$, where $h^r_G(X) := d_G(X) - R(X)$ and $R(X) := \max\{r(u,v) : u \in X, v \in V - X\}$. The following basic property "skew-submodularity" of the function h (see in [3]) will be usefull. $d_G(X,Y)$ denotes the number of edges between $X - Y$ and $Y - X$, $\overline{d}_G(X,Y) = d_G(X \cap Y, V + s - (X \cup Y))$.

For any two subsets $X, Y \subseteq V$ at least one of the following inequalities holds:

$$h^r_G(X) + h^r_G(Y) \geq h^r_G(X \cap Y) + h^r_G(X \cup Y) + 2d_G(X,Y), \quad (3)$$
$$h^r_G(X) + h^r_G(Y) \geq h^r_G(X - Y) + h^r_G(Y - X) + 2\overline{d}_G(X,Y). \quad (4)$$

If $X \cup Y = V$ then (4) always holds (with equality).

For $X \subset V$, the *cut* $\delta_G(X)$ is the set of edges leaving X. For $T \subset \delta_G(s)$, the *T-split* of G is the $(|T|, d_G(s) - |T|)$-detachment G' of G at s where $\delta_{G'}(s_1) = T$. Let $e(T,X) := |\delta_G(X) \cap T|$.

3 The proof

Proof. Necessity: Let $G' := (V + \{s_1, ..., s_p\}, E)$ be an **r**-edge-connected $f(s)$-detachment of G at s. Since the identification of $\{s_1, ..., s_p\}$ does not destroy **r**-edge-connectivity in V, (1) is satisfied. Applying for every vertex s_i $1 \leq i \leq p$ that the deletion of s_i can decrease the local edge-connectivities in V by at most $\lfloor \frac{d_i}{2} \rfloor$ it follows that (2) is satisfied. □

Sufficiency: Wlog. $p \geq 2$ and $\varphi \geq 2$. As we already mentioned, (1) and (2) can be reformulated as

$$h^r_G(X) \geq 0 \quad \forall X \subseteq V, \quad (5)$$
$$h^{r-\varphi}_{G-s}(X) \geq 0 \quad \forall X \subseteq V. \quad (6)$$

We shall use induction on $z(G) := |V| + d_G(s)$. Note that

$$h^{r-\varphi}_{G-s}(X) = h^r_G(X) - d_G(s,X) + \varphi \quad \forall X \subseteq V. \quad (7)$$

Lemma 3.1. *We may assume that*

$$\textit{every set } X \textit{ with } h^r_G(X) = 0 \textit{ is a singleton.} \quad (8)$$

Proof. Suppose there exists a set Q with $h^r_G(Q) = 0$ and $|Q| > 1$. Then let $\hat{G} := (\hat{V}, \hat{E})$ be obtained from G by contracting Q into a vertex q and let $\hat{r}(u,v) := r(u,v)$ if $u,v \in \hat{V} - q$, and $\max\{r(w,x) : w \in Q\}$ if $q \in \{u,v\}$ where $x = \{u,v\} - q$. It can be verified easily that $\hat{R}(\hat{X}) = R(X) \ \forall \hat{X} \subseteq \hat{V}$, so (5) and (6) are satisfied for $\hat{G}$ and $\hat{r}$. Since $|Q| > 1$, $z(\hat{G}) < z(G)$ and hence, by induction, $\hat{G}$ has an $\hat{\boldsymbol{r}}$-edge-connected $f(s)$-detachment $\hat{G}'$. We show that the graph G' obtained from $\hat{G}'$ by "blowing up" Q

is $\boldsymbol{r}$-edge-connected and we are done. Let $X' \subseteq V'$. Using that $h^r_{G'}(Q) = h^r_G(Q) = 0$, the skew-submodularity of $h^r_{G'}$ and the fact that if X' and Q are not intersecting then $h^r_{G'}(X') \geq 0$ (because if $X' \subset Q$ then $h^r_{G'}(X') = h^r_G(X) \geq 0$ by (5) and if $Q \subseteq X'$ or $Q \cap X' = \emptyset$ then $h^r_{G'}(X') = h^{\hat{r}}_{\hat{G}'}(\hat{X}') \geq 0$ since $\hat{G}'$ is $\hat{\boldsymbol{r}}$-edge-connected) we get that $h^r_{G'}(X') \geq 0$ as we wanted. □

Lemma 3.2. *There exists $T \subset \delta_G(s)$ with $|T| = 3$ if $f(s) = (3, 3, ..., 3)$ and $|T| = 2$ otherwise such that the graph G' obtained from G by the T-split satisfy*

$$G' \quad \text{is} \quad \mathbf{r'}\text{-edge-connected in } V', \tag{9}$$
$$G' - s \quad \text{is} \quad (\mathbf{r'} - (\boldsymbol{\varphi} - \mathbf{1}))\text{-edge-connected in } V', \tag{10}$$

where $\mathbf{r'}(u, v) := r(u, v)$ if $u, v \in V$ and 2 otherwise and $V' = V \cup s_1$.

Proof.

Claim 3.3. *(9) and (10) are equivalent to*

$$h^r_G(X) \geq 2e(T, X) - |T| \quad \forall X \subset V, \tag{11}$$
$$e(T, C) \geq 1 \quad \forall C \in \mathcal{C}, \tag{12}$$

where $\mathcal{C}$ is defined as the minimal sets X with $h^{r-\varphi}_{G-s}(X) = 0$.

Proof. (9) is satisfied if and only if $0 \leq h^{r'}_{G'}(X')$ wlog. $\forall s_1 \in X'$ which is, by $h^{r'}_{G'}(X') = h^r_G(X) - e(T, X) + (|T| - e(T, X))$ with $X = X' - s_1$, equivalent to (11). (10) is satisfied if and only if $0 \leq h^{r'-\varphi'}_{G'-s}(X)$ wlog. $\forall s_1 \notin X'$ which is, by $h^{r'-\varphi'}_{G'-s}(X) = h^{r-\varphi}_{G-s}(X) + e(T, X) - 1$, equivalent to (12). □

Claim 3.4. *The following are true for $\mathcal{C}$:*

$$\text{the sets in } \mathcal{C} \text{ are pairwise disjoint,} \tag{13}$$
$$d_G(s, C) \geq \varphi \text{ for each } C \in \mathcal{C}, \tag{14}$$
$$|\mathcal{C}| \in \{0, 2, 3\}, \tag{15}$$
$$\text{if } |\mathcal{C}| = 3 \text{ then } f(s) = (3, 3, ..., 3) \text{ and } h^r_G(C) = 0 \quad \forall C \in \mathcal{C}. \tag{16}$$

Proof. By the submodularity of $h^{r-\varphi}_{G-s}(X)$, the minimality of the sets in $\mathcal{C}$ and (6), (13) follows. (7) and (5) imply (14). By (14), (13) and $d_i \geq 2$, $|\mathcal{C}|\varphi \leq \sum_{C \in \mathcal{C}} d_G(s, C) \leq d_G(s) = \sum_{i=1}^p d_i \leq 3 \sum_{i=1}^p \lfloor \frac{d_i}{2} \rfloor = 3\varphi$ that is $|\mathcal{C}| \leq 3$. Moreover, if $X \in \mathcal{C}$, then there exists $Y \subseteq V - X$ with $Y \in \mathcal{C}$ implying (15). It also follows that if $|\mathcal{C}| = 3$ then each $d_i = 3$, that is $f(s) = (3, 3, ..., 3)$ and for every $C \in \mathcal{C}$, $d_G(s, C) = \varphi$, so by (7), $h^r_G(C) = 0$. □

By (15), either $|\mathcal{C}| = 3$ or $|\mathcal{C}| \in \{0, 2\}$. If $\boldsymbol{|\mathcal{C}| = 3}$**,** then, by (16), $f(s) = (3, 3, ..., 3)$. By (14), there exists $T \subset \delta_G(s)$ with $|T| = 3$ that satisfies (12). T also satisfies (11). Indeed, by (16), (8) and (14), $d_G(s, X) \geq \varphi\ e(T, X)$. So, by (7), (6) and $\varphi \geq 2$, $h^r_G(X) \geq d_G(s, X) - \varphi \geq \varphi(e(T, X) - 1) \geq 2(e(T, X) - 1) \geq 2e(T, X) - |T|$. From now on $\boldsymbol{|\mathcal{C}| \in \{0, 2\}}$**.**

Lemma 3.5. *There exists $T = \{su, sv\}$ that satisfies (11) and (12).*

Proof. If $|\mathcal{C}| = 0$, then, by Mader theorem [4], there exists $T \subset \delta_G(s)$ with $|T| = 2$ that satisfies (11) and in this case (12) is automatically satisfied. If $\mathcal{C} = \{C_1, C_2\}$, then, by (14), there exists $T \subset \delta_G(s)$ with $|T| = 2$ that satisfies (12). We claim that *T satisfies (11).* Suppose $X \subset V$ violates (11). Then $e(T, X) = 2$ and $h^r_G(X) \leq 1$. Wlog. $C_1 - X \neq \emptyset$, otherwise $C_1 \cup C_2 \subset X$ so, by (7) (6) and (13) (14), $1 \geq h^r_G(X) \geq d_G(s, X) - \varphi \geq d_G(s, C_1 \cup C_2) - \varphi \geq 2\varphi - \varphi \geq 2$, contradiction. Since $C_1 \in \mathcal{C}$, $h^{r-\varphi}_{G-s}(C_1 - X) \geq 1$. Then, by (7), $h^r_G(C_1 - X) = h^{r-\varphi}_{G-s}(C_1 - X) + d_G(s, C_1 - X) - \varphi \geq 1 + d_G(s, C_1) - d_G(s, C_1 \cap X) - \varphi = h^r_G(C_1) + 1 - d_G(s, C_1 \cap X)$. Suppose (4) applies for C_1 and X. Then, by (5), $1 + h^r_G(C_1) \geq h^r_G(X) + h^r_G(C_1) \geq h^r_G(X - C_1) + h^r_G(C_1 - X) + 2\overline{d}_G(X, C_1) \geq h^r_G(C_1 - X) + 2d_G(X \cap C_1, s) \geq h^r_G(C_1) + 1 + d_G(s, C_1 \cap X) \geq h^r_G(C_1) + 2$, contradiction. So (3) applies for C_1 and X and $C_1 \cup X \neq V$. Since $\mathcal{C} = \{C_1, C_2\}$, $h^{r-\varphi}_{G-s}(C_1 \cup X) = h^{r-\varphi}_{G-s}(V - (C_1 \cup X)) \geq 1$. Then, by (7), $h^r_G(C_1 \cup X) = h^{r-\varphi}_{G-s}(C_1 \cup X) + d_G(s, C_1 \cup X) - \varphi \geq 1 + d_G(s, C_1) + d_G(s, C_2 \cap X) - \varphi \geq h^r_G(C_1) + 2$. Then, by (5), $1 + h^r_G(C_1) \geq h^r_G(X) + h^r_G(C_1) \geq h^r_G(X \cap C_1) + h^r_G(C_1 \cup X) \geq h^r_G(C_1 \cup X) \geq h^r_G(C_1) + 2$, contradiction. □

If $f(s) \neq (3, 3, ..., 3)$, then we are done. From now on $f(s) = (3, 3, ..., 3)$. Then $d_G(s) = 3\varphi$.

Lemma 3.6. *T can be extended to $T' \subset \delta_G(s)$ with $|T'| = 3$ such that T' satisfies (11).*

Proof. First suppose that $\Gamma(s) = \{u, v\}$. Since $d_G(s) = 3\varphi$ and $\varphi \geq 2$, wlog. $d_G(s, u) \geq \varphi + 1$ and hence there exists another copy e' of su. Then $T' := T \cup e'$ satisfies (11). Hence $\Gamma(s) \neq \{u, v\}$. Suppose indirect that there exists a minimal set $\mathcal{M}$ of subsets of V such that for every $z_i \in \Gamma(s) - \{u, v\}$ there exists a set $M_i \in \mathcal{M}$ violating (11) for $T' := T \cup sz_i$. Then, by the fact that T satisfies (11) and by (8), $e(T', M_i) = 3$ so $\{u, v, z_i\} \subseteq M_i$ and $h^r_G(M_i) \leq 2$. Since $\Gamma(s) \neq \{u, v\}, |\mathcal{M}| \geq 1$. By (7), (6), $h^r_G(M_i) \leq 2$ and $\varphi \geq 2$, $|\mathcal{M}| \geq 2$.

Claim 3.7. *If $M_i, M_j \in \mathcal{M}$, then*

$$h^r_G(M_i - M_j) = 0, \text{ (so, by (8), } M_i - M_j = z_i, \text{)} \tag{17}$$
$$\overline{d}_G(M_i, M_j) = 2, \tag{18}$$
$$d_G(z_i, M_i - z_i) \geq 1. \tag{19}$$

Proof. $2 \geq h^r_G(M_i), 2 \geq h^r_G(M_j)$, $h^r_G(M_i \cap M_j) \geq 2e(T, M_i \cap M_j) - |T| \geq 2 \times 2 - 2 = 2$ (by (11) and $\{u, v\} \subset M_i \cap M_j$), $h^r_G(M_i \cup M_j) \geq 3$ (by the minimality of $\mathcal{M}$), so (3) cannot be satisfied for M_i and M_j. Then M_i and M_j satisfy (4) implying (17) and (18). Moreover, $2 \leq h^r_G(z_i) + h^r_G(M_i \cap M_j) = h^r_G(z_i) + h^r_G(M_i - z_i) \leq h^r_G(M_i) - 2 + 2d_G(z_i, M_i - z_i) \leq 2d_G(z_i, M_i - z_i)$. □

Claim 3.8. $|\mathcal{M}| \geq 3$.

Proof. Suppose $\mathcal{M} = \{M_1, M_2\}$. Then, by (7),(17),(6),(18), $3\varphi = d_G(s) = d_G(s, z_1) + d_G(s, z_2) + d_G(s, M_1 \cap M_2) = h^r_G(z_1) - h^{r-\varphi}_{G-s}(z_1) + \varphi + h^r_G(z_2) - h^{r-\varphi}_{G-s}(z_2) + \varphi + d_G(s, M_1 \cap$

$M_2) \leq 2\varphi + 2 \leq 3\varphi$. It follows that $h^{r-\varphi}_{G-s}(z_1) = 0$, so $z_1 \in \mathcal{C}$, that is (12) is violated for T, contradiction. □

Let $M_1, M_2, M_3 \in \mathcal{M}$. Then, by (19), (17), (18), $1 \leq d_G(M_3 - z_3, z_3) = d_G(M_1 \cap M_2, z_3) \leq \overline{d}_G(M_1, M_2) - d_G(M_1 \cap M_2, s) \leq 2 - 2 = 0$, contradiction. This completes the proof of Lemma 3.6. □

Since T satisfies (12), so does T' and the proof of Lemma 3.2 is complete. □

Let G' be obtained from G by the T-split from Lemma 3.2. Let us denote the new vertex of G' of degree $|T|$ by t. Wlog. $d_1 \geq d_2 \geq ... \geq d_p$. If $d_p = |T|$ then let $f'(s) := (d_1, ..., d_{p-1})$ otherwise $(|T| = 2, d_1 \geq 4)$ let $f'(s) := (d_1 - 2, d_2, ..., d_p)$. Then $(G', f'(s))$ satisfies (9) and (10) and $z(G') < z(G)$, so by induction, G' has an **r**-edge-connected $f'(s)$-detachment G''. Then, in the former case G'', in the latter case the graph obtained from G'' by identifying s_1 and t, is an **r**-edge-connected $f(s)$-detachment of G. □

References

[1] B. Fleiner, Detachments of vertices of graphs preserving edge-connectivity, submitted to *SIAM J. Discrete Math.*

[2] T. Jordán, Z. Szigeti, Detachments preserving local edge-connectivity of graphs, SIAM Journal on Disc. Math. Vol 17, No. 1, (2003) 72-87,

[3] A. Frank, On a theorem of Mader, *Discrete Mathematics*, **101** (1992) 49-57.

[4] W. Mader, A reduction method for edge-connectivity in graphs, *Ann. Discrete Math.* **3** (1978) 145-164.

[5] Z. Szigeti, On admissible edges, manuscript, 2004