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On admissible edges

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Abstract

Let $G = (V + s, E)$ be a 2-edge-connected graph. A pair of edges rs, st is called admissible if splitting off these edges (replacing rs and st by rt) preserves the local edge connectivities between all pairs of vertices in V.

First we generalize Mader's result [2] by showing that if $d(s) \geq 4$ then there exists an edge that belongs to at least $\lfloor \frac{d(s)}{3} \rfloor$ admissible pairs. An infinite family of graphs shows that this is best possible.

Second we generalize Frank's result [1] by characterizing when an edge belongs to no admissible pairs. It provides a new proof for Mader's theorem.

1 Introduction

In this note $G = (V + s, E)$ is always a 2-edge-connected graph. The operation **splitting off** is defined as usually: two edges rs and st are replaced by rt. A pair of edges rs, st is called **admissible** if splitting off these edges preserves the local edge connectivities between all pairs of vertices in V. We say that an edge incident to s is **admissible** if it belong to an admissible pair, otherwise it is called **non-admissible**. Mader proved in [2] that if $d(s) \neq 3$ then there exists an admissible edge. Here we shall strengthen this result by showing in Theorem 3.1 that if $d(s) \geq 4$ then there exists an edge that belongs to at least $\lfloor \frac{d(s)}{3} \rfloor$ admissible pairs. The proof follows the line of [1]. We shall also present an infinite family of graphs showing that our result is best possible. Mader's theorem [2] implies that at most three edges are non-admissible. Frank showed in [1] that in fact there is at most one non-admissible edge. We shall refine this result by giving in Theorem 4.1 the structure of the graph if it contains a non-admissible edge. The proof technic developed for Theorem 4.1 provides a new proof for Mader's theorem.

2 Preliminaries

Recall that $G = (V + s, E)$ is a 2-edge-connected graph. $\boldsymbol{\Gamma(s)}$ denotes the neighbours of s. For a set $T \subset V$, $\boldsymbol{G/X}$ denotes the graph obtained from G by contracting T

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into one vertex. $\boldsymbol{d(X,Y)}$ (resp. $\boldsymbol{\overline{d}(X,Y)}$) denotes the number of edges between $X-Y$ and $Y-X$ (resp. $X\cap Y$ and $V+s-(X\cup Y)$), $d(X)=d(X,V+s-X)$. The **local edge-connectivity** between two vertices x and y is defined by $\boldsymbol{\lambda(x,y)}=\min\{d(X) : x\in X, y\notin X\}$. Let $\boldsymbol{R(X)}:=\max\{\lambda(x,y) : x\in X, y\in V-X\}$ and $\boldsymbol{h(X)}:=d(X)-R(X)$. Then, for $X,Y\subseteq V$, (1), (2), (3), (4) and at least one of (5) and (6) and hence at least one of (7) and (8) hold:

$$\begin{aligned}
h(X) &\geq 0, &(1)\\
h(X) &= h(V-X)+2d(s,X)-d(s), &(2)\\
d(X)+d(Y) &= d(X\cap Y)+d(X\cup Y)+2d(X,Y), &(3)\\
d(X)+d(Y) &= d(X-Y)+d(Y-X)+2\overline{d}(X,Y), &(4)\\
R(X)+R(Y) &\leq R(X\cap Y)+R(X\cup Y), &(5)\\
R(X)+R(Y) &\leq R(X-Y)+R(Y-X), &(6)\\
h(X)+h(Y) &\geq h(X\cap Y)+h(X\cup Y)+2d(X,Y), &(7)\\
h(X)+h(Y) &\geq h(X-Y)+h(Y-X)+2\overline{d}(X,Y). &(8)
\end{aligned}$$

A set X is called **tight** (resp. **dangerous**) if $h(X)=0$ (resp. $h(X)\leq 1$).

The following three claims can be found in [1].

Claim 2.1. *$\{su,sv\}$ is admissible if and only if no dangerous set contains u,v.* □

Claim 2.2. *Let $t\in\Gamma(s)$ of minimum degree. If $t\in M$, $h(M)\leq 1$ and $|\Gamma(s)\cap M|\geq 2$, then $R(M-t)\geq R(M)$.* □

Claim 2.3. *For a tight set T, $\{su,sv\}$ is admissible in G if and only if it is admissible in G/T.* □

Claim 2.4. *If M is a dangerous set, then (a) $d(s,M)\leq\frac{d(s)+1}{2}$ (where equality holds only if $V-M$ is tight) and (b) $G[M]$ is connected.*

Proof. (a) By (1) and (2). (b) If $\emptyset\neq X\subset M$, then $-1\geq h(M)-2\geq h(X)+h(M-X)-2d(X,M-X)\geq -2d(X,M-X)$ that is there is at least one edge between X and $M-X$. □

Lemma 2.5. *Let $st\in E$ and $\mathcal{M}$ be a minimal collection of dangerous sets in V such that $t\in M_i$ for all $M_i\in\mathcal{M}$ and $d(s,\bigcup\mathcal{M})>\frac{d(s)+1}{2}$. Suppose that $|\mathcal{M}|\geq 3$ and*

$$\textit{every tight set is a singleton.} \qquad (9)$$

Then for $M_i,M_j\in\mathcal{M}$, (a) (8) does not apply, (b) $M_i\cap M_j$ is tight, so by (9), $M_i\cap M_j=t$.

Proof. (a) $1\geq h(M_i), 1\geq h(M_j)$ thus if (8) applied, then $h(M_i-M_j)=0$ (so by (9), $M_i-M_j=r_i$) and $\overline{d}(M_i,M_j)=1$. Let $M_k\in\mathcal{M}-M_i-M_j$. Then, by Claim 2.4(b), $1\leq d(M_i\cap M_j, M_k-M_i\cap M_j)\leq\overline{d}(M_i,M_j)-d(M_i\cap M_j,s)\leq 1-1=0$, contradiction. (b) By Claim 2.5(a), (7) applies for M_i and M_j. Then, since $1\geq h(M_i), 1\geq h(M_j)$, and by the minimality of $\mathcal{M}$, $h(M_i\cup M_j)\geq 2$, we have $h(M_i\cap M_j)=0$. □

3 A $\lfloor\frac{d(s)}{3}\rfloor$-admissible edge

Theorem 3.1. *If $d(s) \geq 4$, then there is an edge sr belonging to at least $\lfloor\frac{d(s)}{3}\rfloor$ admissible pairs.*

Proof. Induction on $|V|$. By Claim 2.3 we may assume that (9) is satisfied. Let t be a minimum degree neighbour of s. Suppose t belongs to less than $\lfloor\frac{d(s)}{3}\rfloor$ admissible pairs. Then, by Claim 2.1, there is a minimal collection $\mathcal{M}$ of dangerous sets in V such that $t \in M_i$ for all $M_i \in \mathcal{M}$ and $(*)$ $d(s, \bigcup \mathcal{M}) \geq d(s) - \lfloor\frac{d(s)}{3}\rfloor + 1 = \lceil\frac{2d(s)}{3}\rceil + 1$. By Claim 2.4(a), $|\mathcal{M}| \geq 2$. Let $M_1, M_2 \in \mathcal{M}$.

Claim 3.2. $\mathcal{M} = \{M_1, M_2\}$.

Proof. By Claim 2.2, $R(M_1 - t) \geq R(M_1)$ and $R(M_2 - t) \geq R(M_2)$. Suppose $|\mathcal{M}| \geq 3$. Then, by Lemma 2.5(b), $M_1 \cap M_2 = t$, thus M_1 and M_2 satisfy (8), a contradiction by Lemma 2.5(a). □

Claim 3.3. *(a) $M_1 - M_2 = r_1$, $M_2 - M_1 = r_2$, (b) $d(M_1 \cap M_2, s) = 1$, $d(s, r_1) + d(s, r_2) \geq \lceil\frac{2d(s)}{3}\rceil$.*

Proof. By (2) and (1), $h(M_1 \cup M_2) \geq 2d(s, M_1 \cup M_2) - d(s) \geq 2(\lceil\frac{2d(s)}{3}\rceil + 1) - d(s) \geq 3$, so (7) does not apply and hence (8) applies for M_1 and M_2. Then $h(M_1 - M_2) = 0 = h(M_2 - M_1)$, so by (9), $M_1 - M_2 = r_1$ and $M_2 - M_1 = r_2$; and $\overline{d}(M_1, M_2) = 1$, so $d(M_1 \cap M_2, s) = 1$. By Claim 3.2 and $(*)$, $d(s, r_1) + d(s, r_2) = d(s, M_1 \cup M_2) - d(s, M_1 \cap M_2) \geq \lceil\frac{2d(s)}{3}\rceil + 1 - 1 \geq \lceil\frac{2d(s)}{3}\rceil$. □

Claim 3.4. *Let e_i be any edge connecting s and r_i for $1 \leq i \leq 2$. Then $\{e_1, e_2\}$ is admissible.*

Proof. Otherwise, by Claim 2.1, there is a dangerous set X with $r_1, r_2 \in X$, and then, by (2), (1), Claim 3.3(b) and $d(s) \geq 4$, $1 \geq h(X) \geq 2d(s, X) - d(s) \geq 2\lceil\frac{2d(s)}{3}\rceil - d(s) \geq 2$, contradiction. □

By Claim 3.3(b), wlog. $d(s, r_1) \geq \lfloor\frac{d(s)}{3}\rfloor$. Then, by Claims 3.4, e_2 belongs to at least $\lfloor\frac{d(s)}{3}\rfloor$ admissible pairs. □

Example: There exists an infinite class of graphs in which each edge incident to s belongs to exactly $\lceil\frac{d(s)}{3}\rceil$ admissible pairs. See Figure 1.

4 A non-admissible edge

Theorem 4.1. *An edge st belongs to no admissible pair if and only if $d(s)$ is odd and there exist two disjoint tight sets $C_1, C_2 \subseteq V - t$ such that $d(s, C_1) = d(s, C_2) = \frac{d(s)-1}{2}$. Moreover, if $d(s) \neq 3$, then for every $c_1 \in C_1 \cap \Gamma(s), c_2 \in C_2 \cap \Gamma(s)$, $\{sc_1, sc_2\}$ is an admissible pair.*

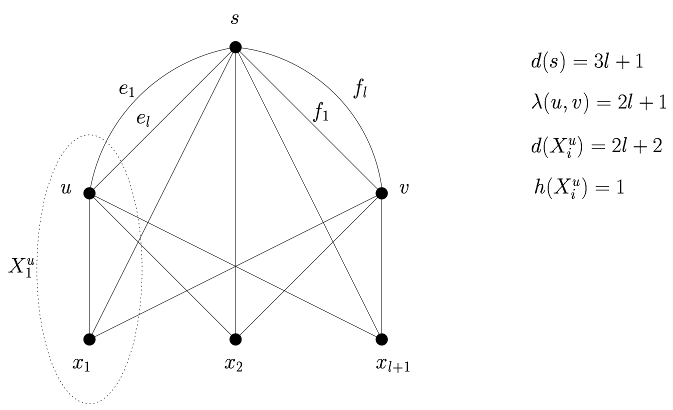

Figure 1

Proof. if: Suppose $d(s)$ is odd and there exist two disjoint tight sets $C_i \subseteq V - t$ such that $d(s, C_i) = \frac{d(s)-1}{2}$. Then, by (2), $V - C_i$ is dangerous so, by Claim 2.1, st belongs to no admissible pair.

only if: Induction on $|V|$.

Lemma 4.2. *We may assume that (9) is satisfied.*

Proof. if T was a tight set with $|T| > 1$, then let $G' := G/T$. By Claim 2.3 st belongs to no admissible pair in G' and $|V(G')| < |V|$, hence, by induction, $d(s)$ is odd and there exist two disjoint tight sets in G' $C_1, C_2 \subset V(G') - t$ such that $d(s, C_1) = d(s, C_2) = \frac{d(s)-1}{2}$ and if $d(s) \neq 3$ then for every $c_1 \in C_1 \cap \Gamma(s), c_2 \in C_2 \cap \Gamma(s)$, $\{sc_1, sc_2\}$ is an admissible pair in G'. Then, by Claim 2.3 and since C_1 and C_2 are also tight in G, we are done. □

By Claim 2.1, there is a minimal collection $\mathcal{M} \neq \emptyset$ of dangerous sets in V such that for every $r_i \in \Gamma(s) - t$ there exists $M_i \in \mathcal{M}$ containing t and r_i. $|\mathcal{M}| \geq 2$, by Claim 2.4(a) and $d(s) \geq 2$.

Lemma 4.3. *If $\mathcal{M} = \{M_1, M_2\}$ then $C_1 := M_1 - M_2$ and $C_2 := M_2 - M_1$ satisfy the statement of the Theorem.*

Proof. $C_1 \cap C_2 = \emptyset$, $t \in M_1 \cap M_2$ so $C_1, C_2 \subseteq V - t$. By Claim 2.4(a), $2\frac{d(s)+1}{2} \geq d(s, M_1) + d(s, M_2) = d(s) + d(s, M_1 \cap M_2) \geq d(s) + 1$, so $d(s)$ is odd, $d(s, M_i) = \frac{d(s)+1}{2}$ and $d(s, M_1 \cap M_2) = 1$, that is $d(s, C_i) = \frac{d(s)-1}{2}$. By Claim 2.4(a), $V - M_i$ is tight so by (9), $C_j \subseteq V - M_i = r_j \subseteq M_j - M_i = C_j$, hence C_j is tight. Suppose indirect that for $c_1 \in C_1 \cap \Gamma(s)$, $c_2 \in C_2 \cap \Gamma(s)$, $\{sc_1, sc_2\}$ is not an admissible pair. Then there exists a dangerous set X containing c_1 and c_2. By $c_i = C_i$ and by Claim 2.4(a), $2\frac{d(s)-1}{2} \leq d(s, X) \leq \frac{d(s)+1}{2}$, that is $d(s) \leq 3$, contradiction. □

We suppose from now on that $|\mathcal{M}| \geq 3$. By Lemma 2.5(b), for all $M_i, M_j \in \mathcal{M}$, $M_i - M_j = M_i - t$.

Claim 4.4. *If $R(M_1) = \lambda(a, b)$, $a \in M_1, b \in V - \bigcup \mathcal{M}$, then for some $M_k \in \mathcal{M} - M_1, R(M_k - t) > d(t)$.*

Proof. $\sum_{M_j \in \mathcal{M} - M_1} d(M_j) + d(M_1) \geq d(\bigcup \mathcal{M} \cup s) + d(\bigcup \mathcal{M} - t, s) + (|\mathcal{M}| - 1)d(t) + 1 \geq d(M_1) - 1 + |\mathcal{M}| + (|\mathcal{M}| - 1)d(t) + 1 = (|\mathcal{M}| - 1)(d(t) + 1) + d(M_1) + 1$ so there exists $M_k \in \mathcal{M} - M_1$ with $d(M_k) > d(t) + 1$. Since M_k is dangerous, it follows that $R(M_k) \geq d(M_k) - 1 \geq d(t) + 1$, that is $R(M_k - t) > d(t)$. □

Claim 4.5. *There exists $M_i \in \mathcal{M}$ for which $R(M_i - t) \geq d(t)$.*

Proof. By Lemma 2.5(b), $R(t) = d(t)$ thus $Y := \{y \in V - t : d(t) = \lambda(t, y)\} \neq \emptyset$. If $y \in M_i \cap Y$ for some $M_i \in \mathcal{M}$, then $R(M_i - t) \geq \lambda(t, y) = d(t)$. Thus we suppose that $Y \subseteq V - \bigcup \mathcal{M}$. Let $y \in Y$. Then $R(M_1) = d(t) = \lambda(t, y)$ and Claim 4.4 provides the statement. □

Claim 4.6. *If $M_j \in \mathcal{M} - M_i$, then $R(M_j - t) < R(M_j) = d(t)$.*

Proof. Suppose $R(M_j - t) \geq R(M_j)$. By Claim 4.5, $R(M_i - t) \geq R(M_i)$. So (8) applies for M_i and M_j, contradicting Lemma 2.5(a). $R(M_j - t) < R(M_j)$ and $R(t) = d(t)$ implies $R(M_j) = d(t)$. □

Claim 4.7. *If $R(M_i) = \lambda(a, b)$, $a \in M_i, b \in V - M_i$, then $b \in V - \bigcup \mathcal{M}$.*

Proof. Suppose indirect $b \in M_j \in \mathcal{M}$. Then, $R(M_j - t) \geq \lambda(a, b) = R(M_i)$. By Claims 4.6 and 4.5, $R(M_j) = d(t) \leq R(M_i - t)$. Thus (8) applies for M_i and M_j, contradiction by Lemma 2.5(a). □

Claim 4.7, Claim 4.4 applied for $M_1 = M_i$ and Claim 4.6 provides a contradiction. □

References

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