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On the existence of k edge-disjoint 2-connected spanning subgraphs

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Abstract

We prove that every $6k$-connected graph contains k edge-disjoint 2-connected spanning subgraphs. By using this result we can settle special cases of two conjectures, due to Kriesell and Thomassen, respectively: we show that every 12-connected graph G has a spanning tree T for which $G-E(T)$ is 2-connected, and that every 18-connected graph has a 2-connected orientation.

1 Introduction

In this note we consider undirected graphs without multiple edges or loops. It is well-known that every $2k$-edge-connected graph contains k edge-disjoint connected spanning subgraphs. We shall prove the following similar result on 2-connected spanning subgraphs of highly connected graphs.

Theorem 1.1. *Every $6k$-connected graph contains k edge-disjoint 2-connected spanning subgraphs.*

Theorem 1.1 will follow from a stronger result on the existence of k edge-disjoint 'rigid' subgraphs that we shall prove in Section 3 by using matroidal methods. In the rest of this section we discuss some corollaries.

The following conjecture, which has been open for $k \geq 2$, is due to Kriesell.

Conjecture 1.2. *[5] For every integer k there exists a (smallest) $f(k)$ such that every $f(k)$-connected graph G contains a spanning tree T for which $G-E(T)$ is k-connected.*

Theorem 1.1 implies $f(2) \leq 12$.

Theorem 1.3. *Every 12-connected graph G has a spanning tree T for which $G-E(T)$ is 2-connected.*

Thomassen posed the following conjecture, which has also been open for $k \geq 2$.

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Conjecture 1.4. *[8] For every integer k there exists a (smallest) $g(k)$ such that every $g(k)$-connected graph has a k-connected orientation.*

Eulerian graphs with 2-connected orientations have been characterised in [1].

Theorem 1.5. *[1] Let $G = (V, E)$ be an Eulerian graph with $|V| \geq 3$. Then G has a 2-connected orientation if and only if G is 4-edge-connected and $G - v$ is 2-edge-connected for all $v \in V$.*

Theorems 1.1 and 1.5 imply $g(2) \leq 18$ as follows.

Theorem 1.6. *Every 18-connected graph has a 2-connected orientation.*

Proof: Let $G = (V, E)$ be 18-connected. By Theorem 1.5 it suffices to prove that G has a 4-edge-connected Eulerian spanning subgraph H for which $H - v$ is 2-edge-connected for all $v \in V$. It follows from Theorem 1.1 that G has three edge-disjoint 2-connected spanning subgraphs $H_i = (V, E_i)$, $1 \leq i \leq 3$. It is easy to see that $H' = (V, E_1 \cup E_2)$ is 4-edge-connected and $H' - v$ is 2-edge-connected for all $v \in V$. It is well-known (and easy to show) that, since H_3 is a connected spanning subgraph, there is a set $F \subset E_3$ for which $H' + F$ is Eulerian. Thus $H = (V, E_1 \cup E_3 \cup F)$ is the required spanning subgraph of G. •

We note that a conjecture of Frank [3] would imply $g(k) = 2k$ for $k \geq 1$.

2 Preliminaries

Let $G = (V, E)$ be a graph. For sets $X \subseteq V$ and $F \subseteq E$ let $E_F(X)$ and $i_F(X)$ denote the set and the number of edges of F induced by X, respectively. If $F = E$ then we may omit the subscript F. We say that a set $S \subseteq E$ is *sparse* if

$$i_S(X) \leq 2|X| - 3 \text{ for all } X \subseteq V \text{ with } |X| \geq 2. \tag{1}$$

Let $\mathcal{S}$ be the collection of sparse sets in G. It follows from the theory of submodular functions (and can also be verified directly [4, Section 2]) that $\mathcal{S}$ corresponds to the independent sets of a matroid on ground-set E. Due to its importance in the theory of 'graph rigidity', this matroid, denoted by $\mathcal{R}(G) = (E, \mathcal{S})$, is called the *rigidity matroid* of G. For the rank function r of $\mathcal{R}(G)$ we have the following formula.

Theorem 2.1. *[6, Theorem 1],[4, Corollary 2.5] Let $F \subseteq E$. Then*

$$r(F) = \min\{\sum_{i=1}^{t}(2|X_i| - 3)\},$$

where the minimum is taken over all collections of subsets $\{X_1, X_2, \ldots, X_t\}$ of V such that $\{E_F(X_1), E_F(X_2), \ldots, E_F(X_t)\}$ partitions F.

We say that a set $S \subseteq E$ is *rigid* if S is sparse and $r(S) = 2|V| - 3$, that is, if S is independent and has maximum possible rank in $\mathcal{R}(G)$. It follows easily from (1) that if $|V| \geq 3$ then every rigid set induces a 2-connected spanning subgraph in G. A graph $G = (V, E)$ is called *rigid* if the rank of $\mathcal{R}(G)$ is equal to $2|V| - 3$, that is, if E has a rigid subset. The following result is due to Lovász and Yemini [6].

Theorem 2.2. *[6] Every* 6*-connected graph is rigid.*

3 Edge-disjoint rigid subgraphs

In this section we extend Theorem 2.2 and show that every sufficiently highly connected graph contains k edge-disjoint rigid spanning subgraphs.

Theorem 3.1. *Every* $6k$*-connected graph contains* k *edge-disjoint rigid spanning subgraphs.*

Proof: For a contradiction suppose that for some integer k there exists a $6k$-connected graph without k edge-disjoint rigid spanning subgraphs, and suppose that $G = (V, E)$ is such a counterexample with the minimum number $n = |V|$ of vertices, and subject to this, with the maximum number of edges. By Theorem 2.2 we must have $k \geq 2$.

Let $\mathcal{M}(G)$ be the matroid on ground-set E, obtained by taking the matroid union of k copies of $\mathcal{R}(G)$. Let r^* be the rank function of $\mathcal{M}(G)$. Since G is a counterexample, $r^*(E) < 2kn - 3k$ must hold. By using Edmonds' formula for the rank function of matroid unions (see [2] or [7, Chapter 42] for details), this implies that there is a set $F \subseteq E$ with $kr(F) + |E - F| < 2kn - 3k$, where r is the rank function of $\mathcal{R}(G)$. By inserting the equality for $r(F)$ from Theorem 2.1 we obtain that

$$k(\sum_{1}^{t} 2|X_i| - 3) + |E - F| < 2kn - 3k \tag{2}$$

holds for some collection of subsets $\mathcal{X} = \{X_1, X_2, \ldots, X_t\}$ of V such that $\{E_F(X_1), E_F(X_2), \ldots, E_F(X_t)\}$ partitions F. Since adding new edges to G (and to F) between vertices of some X_i does not increase the left hand side of (2) and preserves $6k$-connectivity, the maximal choice of $|E|$ implies that X_i induces a complete subgraph in G for $1 \leq i \leq t$. Furthermore, since moving and edge, induced by some X_i, from $E - F$ to F does not increase the left hand side, we may assume that every edge induced by X_i belongs to F, for $1 \leq i \leq t$.

For a vertex $v \in V$ let $g(v)$ denote the number of edges in $E - F$ incident to v.

Claim 3.2. *If* $g(v) = 0$ *then* v *belongs to at least two sets of* $\mathcal{X}$*.*

Proof: Suppose that $g(v) = 0$. Then for every edge vu there is an $X_i \in \mathcal{X}$ with $uv \in E(X_i)$. For a contradiction suppose also that v belongs to exactly one member, say X_1, of $\mathcal{X}$. Then X_1 contains all neighbours of v. This implies that v cannot be adjacent to all other vertices of G, for otherwise $X_1 = V$ would follow, contradicting (2).

Consider $G - v$ and F', where F' is the edge set obtained from F by deleting all edges incident to v. Let $X_1' = X_1 - v$ and let $X_i' = X_i$ for $2 \leq i \leq t$. Now (2) implies that $k(\sum_1^t 2|X_i'| - 3) + |E(G - v) - F'| < 2k(n-1) - 3k$, and hence $G - v$ has no k edge-disjoint rigid spanning subgraphs. Thus, by the minimal choice of G, $G - v$ is not $6k$-connected. Since G is not complete (v is not adjacent to every other vertex), this implies that v is in a minimum vertex separator of G. This contradicts the fact that the neigbours of v induce a complete subgraph. •

Claim 3.3. *For each vertex $v \in V$ we have*

$$\sum_{X_i : v \in X_i} (2 - \frac{3}{|X_i|}) + \frac{g(v)}{2k} \geq 2. \tag{3}$$

Proof: Since G is $6k$-connected, each vertex has degree at least $6k$ in G. First consider a vertex v which belongs to no member of $\mathcal{X}$. Then all edges incident to v are in $E - F$, and hence $g(v) \geq 6k$ follows. Thus v satisfies (3). Next suppose that v belongs to exactly one member of $\mathcal{X}$, say X_1. As above, the inequality follows easily if $g(v) \geq 6k$. So we may suppose $g(v) \leq 6k - 1$. We also have $g(v) \geq 1$ by Claim 3.2. Since $u \in X_1$ for all edges $vu \in F$, we have $|X_1| \geq 6k - g(v) + 1$. Hence

$$2 - \frac{3}{|X_1|} + \frac{g(v)}{2k} \geq 2 - \frac{3}{6k - g(v) + 1} + \frac{g(v)}{2k} \geq 2,$$

as required.

Now let v be a vertex which belongs to at least two members of $\mathcal{X}$. Since every edge in F incident to v is induced by some member of $\mathcal{X}$ and v has degree at least $6k$ in G, we have

$$\sum_{X_i : v \in X_i} (|X_i| - 1) \geq 6k - g(v). \tag{4}$$

Suppose, without loss of generality, that v belongs to $X_1, X_2, ..., X_d$, where $d \geq 2$, and $|X_1| \geq |X_2| \geq ... \geq |X_d|$. Since $|X_i| \geq 2$ for all $1 \leq i \leq t$, each term in the sum in (3) is at least $\frac{1}{2}$. So the claim follows if $d \geq 4$. Similarly, if $d = 3$ and $g(v) \geq k$ then the left hand side of (3) is at least $3\frac{1}{2} + \frac{1}{2} = 2$, as required. If $d = 3$ and $g(v) \leq k - 1$ then $\sum_{X_i : v \in X_i} (|X_i| - 1) \geq 5k + 1$ by (4). Thus $|X_1| \geq \lceil \frac{5k+1}{3} \rceil + 1 \geq 3$, and hence $\sum_{X_i : v \in X_i} (2 - \frac{3}{|X_i|}) \geq 1 + \frac{1}{2} + \frac{1}{2} = 2$.

Finally, suppose that $d = 2$. If $g(v) \geq 2k$ then the left hand side of (3) is at least $\frac{1}{2} + \frac{1}{2} + 1 = 2$, as required. Otherwise $\sum_{X_i : v \in X_i} (|X_i| - 1) \geq 4k + 1$ by (4), and hence $|X_1| \geq \lceil \frac{4k+1}{2} \rceil + 1$. Thus, using the fact that $k \geq 2$, we obtain $|X_1| \geq 6$. Hence $\sum_{X_i : v \in X_i} (2 - \frac{3}{|X_i|}) \geq \frac{3}{2} + \frac{1}{2} = 2$. This completes the proof of the claim. •

By taking the sum of the inequalities of (3) over all vertices of G, multiplying by k and interchanging the sums, we get

$$2kn \leq k \sum_1^t |X_i|(2 - \frac{3}{|X_i|}) + \sum_v \frac{g(v)}{2} = k(\sum_1^t 2|X_i| - 3) + |E - F|,$$

which contradicts (2). This proves the theorem. •

As we noted earlier, rigid spanning subgraphs are 2-connected. Thus Theorem 1.1 follows directly from Theorem 3.1. Since there exist efficient algorithms to find a maximum size independent set in $\mathcal{M}(G)$, the required spanning subgraphs can be found in polynomial time [2, 7].

Examples for non-rigid 5-connected graphs are given in [6]. By modifying these examples one can construct $(6k-1)$-connected graphs without k edge-disjoint rigid spanning subgraphs, showing that the bound in Theorem 3.1 cannot be improved. On the other hand, the bounds in Theorems 1.3 and 1.6 can certainly be improved by more involved counting arguments. For instance, to prove Theorem 1.3 one could replace one of the two rigidity matroids by the circuit matroid of G in the definition of $\mathcal{M}(G)$. We leave the details to the interested reader.

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