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Two-connected orientations of Eulerian graphs

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Abstract

A graph $G = (V, E)$ is said to be weakly four-connected if G is 4-edge-connected and $G - x$ is 2-edge-connected for every $x \in V$. We prove that every weakly four-connected Eulerian graph has a 2-connected Eulerian orientation. This verifies a special case of a conjecture of A. Frank.

1 Introduction

A directed graph D is an *orientation* of an undirected graph G if the underlying graph of D is G. Robbins [8] proved that a graph G has a strongly connected orientation if and only if G is 2-edge-connected. A deep result of Nash-Williams [7] from 1960 implies the extension to higher edge-connectivity: a graph G has a k-edge-connected orientation if and only if G is $2k$-edge-connected. See also [6, Problem 6.54] for a direct proof and [9, Chapter 61] for more results on graph orientations.

The vertex-connected version is still unsolved, even for $k = 2$. It is also open whether sufficiently highly vertex-connected graphs have k-vertex-connected orientations, which was conjectured by Thomassen [10]. A stronger conjecture, due to Frank [2], states that a graph G has a k-vertex-connected orientation if and only if $G - X$ is $2(k - |X|)$-edge-connected for every subset X of vertices of G. Note that this is a necessary condition. Thus, if true, the latter conjecture would yield the desired characterisation. The only partial result known is due to Gerards [3], who verified Frank's conjecture in the special case when $k = 2$ and the graph is 4-regular.

In this paper we also consider the case $k = 2$ and show that Frank's conjecture holds for Eulerian graphs. This fact will be used in a forthcoming paper [5] to verify Thomassen's conjecture for $k = 2$. Although most orientation theorems with edge-connectivity conditions are straightforward for Eulerian graphs (by taking an Eulerian orientation), this is not the case for vertex-connectivity. For example the graph of Figure 1 has a 2-vertex-connected orientation, but the Eulerian orientation obtained by reversing all arcs of the directed three-cycle at the bottom is not 2-vertex-connected.

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We shall use some recent results on edge splittings and removable cycles in weakly four-connected graphs (these will be summarised in Sections 2 and 5, respectively), as well as new results on merging and hooking up in 2-connected digraphs (described in Sections 3 and 4). The proof of our main result is in Section 6. Algorithmic remarks will follow in Section 7.

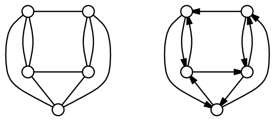

Figure 1: A weakly four-connected graph G and its 2-connected orientation.

1.1 Definitions and notation

Graphs and digraphs (directed graphs) in this paper may contain parallel edges and loops. Let $G = (V, E)$ be a graph with vertex set V and edge set E. For two disjoint subsets X, Y of V let $d(X, Y)$ denote the number of edges between X and Y. For a subset X we use $d(X) := d(X, V - X)$ to denote the *degree* of X. For a singleton x we simply write $d(x)$. G is *Eulerian* if $d(v)$ is even for all $v \in V$. The *multiplicity* of an edge $uv \in E$ is $d(u, v)$. For some $v \in V$ we use $N(v)$ to denote the set of vertices adjacent to v (that is, the set of *neighbours* of v). We say that G is *k-edge-connected* if $d(X) \geq k$ for every $\emptyset \neq X \subset V$. (We use $\subset$ to denote proper inclusion and $\subseteq$ to mean $\subset$ or $=$.)

Let $D = (V, A)$ be a digraph with vertex set V and arc set A. For two disjoint subsets X, Y of V let $d^+(X, Y)$ denote the number of arcs with tail in X and head in Y. For a subset X we use $d^+(X) := d^+(X, V - X)$ to denote the *out-degree* of X. Similarly, the *in-degree* of X is $d^-(X) := d^+(V - X, X)$. D is *Eulerian* if $d^+(v) = d^-(v)$ for all $v \in V$. We let $d(X, Y) = d^+(X, Y) + d^+(Y, X)$. For some $v \in V$ let $N^+(v) := \{u \in V : vu \in A\}$ denote the *out-neighbours* of v. The set $N^-(v)$ of *in-neighbours* is defined analogously. We say that D is *strongly connected* if $d^+(X) \geq 1$ for every $\emptyset \neq X \subset V$. D is *k-vertex-connected* (*k-connected*, for short) if $|V| \geq k + 1$ and $D - X$ is strongly connected for all $X \subset V$ with $|X| \leq k - 1$.

We call a graph $G = (V, E)$ *weakly four-connected* if $|V| \geq 3$, G is 4-edge-connected, and $G - x$ is 2-edge-connected for all $x \in V$. Let X, Y, Z be pairwise disjoint subsets of V with $V = X \cup Y \cup Z$, where only Z may be empty. We say that (X, Y, Z) is a *mixed cut* in G of *size* $d(X, Y)$. If $Z = \{z\}$ for some $z \in V$, we write (X, Y, z). Note that G is weakly four-connected if and only if $2|Z| + d(X, Y) \geq 4$ for all mixed cuts (X, Y, Z) in G. Let (X, Y, Z) be a mixed cut. It is *non-trivial* if $|X|, |Y| \geq 2$. Otherwise it is *trivial.* If $d(X, Y) = 2$ then we call it *tight.* If there is an edge $uv \in E$ with $u \in X$,$v \in Y$, such that the multiplicity of uv equals 3 and $d(X, Y) = 3$, then we call it *narrow.*

We use $G - X$ (G/X) to denote the graph obtained from graph G by deleting (contracting, respectively) a set X of edges or vertices. Adding a set X of edges or vertices is denoted by $G+X$. For $X \subseteq V$ the subgraph induced by X is denoted by $G[X]$. The notation for digraphs is similar. We apply standard notation to indicate the graph we are referring to, when it is not clear from the context, e.g. $d_{G'}(X)$ or $V(D')$ denotes the degree of X in G' and the vertex set of D', respectively. We close this section with a simple lemma.

Lemma 1.1. *Let $G=(V,E)$ be a weakly four-connected Eulerian graph, let $d(u,v)=3$ for some pair $u,v \in V$, and suppose that $G-\{uv,uv\}$ is not weakly four-connected. Then there is a narrow mixed cut (X,Y,z) in G with $u \in X$ and $v \in Y$.*

Proof: If $|V|=3$ then the lemma is easy to verify, so we shall suppose that $|V| \geq 4$. Let e,f,g denote the three parallel edges between u and v and let $G'=G-\{e,f\}$. Since G' is not weakly four-connected, either there is a set $\emptyset \neq X \subset V$ with $d_{G'}(X) \leq 3$ or there is a pair z,Y with $z \in V$, $\emptyset \neq Y \subset V-z$ and $d_{G'-z}(Y) \leq 1$. In the former case we must have $d_G(X)=4$ and $|X \cap \{u,v\}|=1$, since G is weakly four-connected and Eulerian. This implies that $G-u$ or $G-v$ is not 2-edge-connected, a contradiction.

In the latter case let $W=V-Y-z$. Since G is weakly four-connected and $d(u,v)=3$, we must have $d_{G-z}(Y,W)=3$, and e,f,g must connect Y to W. Thus (Y,W,z) is a narrow mixed cut in G. •

2 Splittable vertices of degree four

By *splitting off* a pair su,sv of edges from a vertex s in a graph we mean the operation of deleting the edges su,sv and adding (a new copy of) the edge uv. If $d(s)$ is even, we may consider a *complete splitting at* s, which is a sequence of $d(s)/2$ splittings at s.

Let $G=(V,E)$ be a weakly four-connected graph with $|V| \geq 4$ and let $s \in V$ be a designated vertex with $d(s)=4$. We say that a complete splitting at s is *admissible* if the graph on vertex set $V-s$, obtained by splitting off s, is also weakly four-connected. We call s *admissible* if there is an admissible complete splitting at s. Note that the admissibility of a complete splitting does not depend on the order of splittings.

The proof of the next lemma is easy, using the definition of weak four-connectivity. It also follows from [4, Proposition 2.1, Lemma 2.2, Lemma 2.3].

Lemma 2.1. *Let $G=(V,E)$ be weakly four-connected with $|V| \geq 4$ and let $\{sx,sy,sa,sb\}$ be the set of edges incident to some $s \in V$ with $d(s)=4$. A complete splitting at s on pairs sx,sy and sa,sb is non-admissible if and only if one of the following holds:*
(a) there is a set $\emptyset \neq X \subseteq V-\{s,a,b\}$ with $d(X)=4$ and $x,y \in X$,
(b) there is a pair (X,w), where $w \in V-s$ and $\emptyset \neq X \subseteq V-\{s,w,a,b\}$, for which $d_{G-w}(X) \leq 3$ and $x,y \in X$,
(c) there is a set $\emptyset \neq X \subseteq V-\{s,y,a,b\}$, for which $d_{G-y}(X)=2$ and $x \in X$ (or there is a set $\emptyset \neq X \subseteq V-\{s,x,a,b\}$, for which $d_{G-x}(X)=2$ and $y \in X$).

We shall also need the following results.

Lemma 2.2. *[4, Lemma 3.2] Let $G = (V, E)$ be weakly four-connected with $|V| \geq 4$ and let $s \in V$ with $d(s) = 4$. If $|N(s)| \leq 3$ then s is admissible.*

A subset $\emptyset \neq X \subset V$ is called a *mixed fragment* if $d_{G-z}(X) = 2$ for some $z \in V - X$. Note that X is a mixed fragment if and only if (X, Y, z) is a tight mixed cut for some $z \in V$ and $Y = V - X - z$.

Theorem 2.3. *[4, Theorem 6.3] Let $G = (V, E)$ be weakly four-connected with $|V| \geq 4$ and suppose that V as well as every mixed fragment of G contains a vertex of degree four. Then G has an admissible vertex s with $d(s) = 4$.*

3 Reducible configurations

At some point in the proof of our main result we shall orient a weakly four-connected Eulerian graph by first orienting two smaller graphs, obtained by contracting the 'sides' of an appropriate mixed cut, and then merging these oriented graphs to obtain a 2-connected Eulerian orientation of G. This section contains the proofs of the lemmas which make this step work.

3.1 Contracting mixed cuts

Lemma 3.1. *Let $G = (V, E)$ be weakly four-connected a let $\emptyset \neq X \subset V$ be a set of vertices with $|V - X| \geq 2$. Then G/X is weakly four-connected if and only if $G - X$ is 2-edge-connected.*

Proof: Let x be the vertex of G/X obtained by contracting X. First suppose that G/X is weakly four-connected. Since $G - X = G/X - x$, it follows that $G - X$ is 2-edge-connected. To prove the other direction suppose that $G - X$ is 2-edge-connected. First observe that the edge-connectivity of a graph cannot be decreased by contracting a set of vertices. This shows that G/X is 4-edge-connected and $G/X - z$ is 2-edge-connected for all $z \in V(G/X) - x$. Now $G/X - x$ is also 2-edge-connected, since $G - X$ is 2-edge-connected. Since $|V - X| \geq 2$, we have $|V(G/X)| \geq 3$. Thus G/X is weakly four-connected. •

Lemma 3.2. *Let $G = (V, E)$ be weakly four-connected. If (X, Y, z) is a tight or narrow mixed cut in G then G/X is weakly four-connected.*

Proof: Clearly, X is a proper subset of V with $|V - X| \geq 2$. Thus, by Lemma 3.1, it is sufficient to verify that $G - X$ is 2-edge-connected. For a contradiction suppose that there exists a set $\emptyset \neq W \subset V - X$ with $d_{G-X}(W) \leq 1$. By replacing W with $V - X - W$, if necessary, we may assume that $W \subseteq Y$. Since G is weakly four-connected, $d_G(W) \geq 4$, and hence $d_G(W, X) \geq 3$ holds. If the mixed cut is tight, this gives $3 \leq d_G(W, X) \leq d_G(Y, X) = 2$, a contradiction. If the mixed cut is narrow, this implies $u \in W$ and $v \in X$ for the common end-vertices u, v of the edges from Y to X. But then $d_{G-v}(W) \leq 1$, contradicting the fact that G is weakly four-connected. •

A *special mixed cut* in a graph $G = (V, E)$, denoted by $(X + v, Y + u, z)$, is a mixed cut in G for which $|X| \geq 2$, $d(v) = 4$, $d(v, u) = 2$, $d(X + v, Y + u) = 3$ and $d(X, Y + u) = 1$ hold.

Lemma 3.3. *Let $G = (V, E)$ be weakly four-connected and let $(X + v, Y + u, z)$ be a special mixed cut. Suppose that there is no non-trivial tight mixed cut in G. Then (i) G/X and (ii) $G/(Y + u)$ are both weakly four-connected.*

Proof: First we prove that G/X is weakly four-connected. Clearly, X is a proper subset of V with $|V - X| \geq 2$. Thus, by Lemma 3.1, it is sufficient to verify that $G - X$ is 2-edge-connected. For a contradiction suppose that there exists a set $\emptyset \neq W \subset V - X$ with $d_{G-X}(W) \leq 1$. Since $d(v, u) = 2$, W does not separate v and u. By replacing W by $V - X - W$, if necessary, we may assume that $W \cap \{u, v\} = \emptyset$. If $z \notin W$ then $d_G(X, W) \leq 1$, since the mixed cut is special. Hence $d_G(W) \leq d_{G-X}(W) + 1 \leq 2$, which contradicts the fact that G is weakly four-connected. Thus $z \in W$. Let $W' = V - X - W$. Since $d_G(W') \geq 4$, $d_{G-X}(W') \leq 1$, and since the mixed cut is special, we can deduce that $d_G(X, W) = d_G(X, z)$. Now either $W - z \neq \emptyset$, in which case $d_{G-z}(W - z) \leq 1$ follows, or $W = \{z\}$, in which case $(X + z, Y + u, v)$ is a non-trivial tight mixed cut in G. Each of these conclusions contradicts the hypotheses of the lemma, which completes the proof of (i).

As above, it follows from Lemma 3.1 that to prove (ii) it is sufficient to verify that $G - (Y + u)$ is 2-edge-connected. For a contradiction suppose that there exists a set $\emptyset \neq W \subset V - (Y + u)$ with $d_{G-(Y+u)}(W) \leq 1$. We may assume that $z \notin W$. If $v \notin W$, then $d_G(W) \leq d_{G-(Y+u)}(W) + 1 \leq 2$, which contradicts the fact that G is weakly four-connected. Thus $v \in W$. Let $W' = V - (Y + u) - W$. Since $d_G(W) \geq 4$ and $d_{G-(Y+u)}(W) \leq 1$, and since the mixed cut is special, we have $d_G(Y + u, W') = d_G(Y + u, z)$. So either $W' \cap X \neq \emptyset$, in which case $d_{G-z}(W' \cap X) \leq 1$ follows, or $W' = \{z\}$, in which case $(X + v, Y + z, u)$ is a non-trivial tight mixed cut in G. Each of these conclusions contradicts the hypotheses of the lemma, which completes the proof of (ii). •

3.2 Merging digraphs

Let $G = (V, E)$ be a graph and (X, Y, Z) be a mixed cut in G. Let $G_x = G/X$ and $G_y = G/Y$ denote the graphs obtained from G by contracting the two sides of the mixed cut, and let D_x and D_y be orientations of G_x and G_y, respectively. We say that D_x and D_y are *compatible* if $D_x[Z] = D_y[Z]$ and for all edges e between X and Y the orientation of (the edge corresponding to) e in D_x and D_y is the same. If D_x and D_y are compatible, we can obtain an orientation of G in a natural way: let the edges in $G[X \cup Z]$ be oriented as in D_y, let the edges in $G[Z \cup Y]$ be oriented as in D_x, and let the edges between X and Y be oriented as in D_x and D_y. We call this operation *merging* D_x and D_y (along (X, Y, Z)). We shall use this operation only for $1 \leq |Z| \leq 2$.

Lemma 3.4. *Suppose that D_x and D_y are compatible 2-connected orientations of G_x and G_y, respectively. If $|Z| = 1$ and $d_G(X,Y) \le 3$ or $|Z| = 2$ and $d_G(X,Y) \le 1$ then the directed graph $D = (V,A)$, obtained by merging D_x and D_y along (X,Y,Z), is also 2-connected.*

Proof: For a contradiction suppose that $D - w$ is not strongly connected for some $w \in V$. Then $V - w$ can be partitioned into two non-empty sets S, T such that $d^+_{D-w}(S,T) = 0$ holds.

First suppose $w \in Y$. If S properly intersects $X \cup Z$ (that is, $S \cap (X \cup Z) \neq \emptyset \neq (X \cup Z) - S$) then we obtain $d^+_{D-w}(S,T) \ge d^+_{D-w}(S \cap (X \cup Z), (X \cup Z) - S) = d^+_{D_y-y}(S \cap (X \cup Z), (X \cup Z) - S) \ge 1$, by using the definition of merging and the fact that D_y is 2-connected. Similarly, if $\emptyset \neq S \subset Y$, then $d^+_{D-w}(S,T) = d^+_{D_x-w}(S, V(D_x) - w - S) \ge 1$, since D_x is 2-connected. A similar argument proves that there is at least one arc from S to T in $D - w$ if T properly intersects $X \cup Z$ or $\emptyset \neq T \subset Y$. Thus, for all possible pairs S, T, we have deduced $d^+_D(S,T) \ge 1$, a contradiction. By symmetry, we have a similar contradiction when $w \in X$.

Thus it remains to consider the case $w \in Z$. As above, the 2-connectivity of D_x and D_y, and the definition of merging imply that $d^+_{D-w}(W) \ge 1$ and $d^-_{D-w}(W) \ge 1$ whenever $W \subseteq (X \cup Z) - w$ or $W \subseteq (Y \cup Z) - w$. So we may assume that each of S, T properly intersects the sets X and Y. We can also assume, without loss of generality, that if $|Z| = 2$ then $S \cap Z = \emptyset$.

Since $d^+_{D-w}(S \cap X) \ge 1$, $d^+_{D-w}(S \cap Y) \ge 1$, and $d^+_D(S,T) \ge 1$, we must have $d_D(S \cap X, S \cap Y) \ge 2$. Similarly, if $|Z| = 1$ (and hence $T \subset X \cup Y$) then we get $d_D(T \cap X, T \cap Y) \ge 2$. Thus for $|Z| = 1$ we obtain $d_G(X,Y) \ge d_D(S \cap X, S \cap Y) + d_D(T \cap X, T \cap Y) \ge 4$. For $|Z| = 2$ we obtain $d_G(X,Y) \ge d_D(S \cap X, S \cap Y) \ge 2$. This contradicts the hypotheses of the lemma. •

Lemma 3.5. *Let (X,Y,z) be a tight or narrow mixed cut in $G = (V,E)$ and suppose that G/X and G/Y both have 2-connected Eulerian orientations. Then G also has a 2-connected Eulerian orientation.*

Proof: We shall prove that there exist 2-connected Eulerian orientations D_x and D_y of $G_x = G/X$ and $G_y = G/Y$, respectively, which are compatible. Since $|Z| = 1$, $D_x[Z] = D_y[Z]$ trivially holds for any pair of orientations.

First suppose that (X,Y,z) is tight and let e, f denote the edges between X and Y in G. Let D_x, D_y be a pair of 2-connected Eulerian orientations. By reorienting all arcs of D_y, if necessary, we may assume that the orientation of (the edge corresponding to) e is the same in D_x and D_y. Since $D_x - z$ and $D_y - z$ are both strongly connected, this implies that the orientation of f is also the same in D_x and D_y. Thus the pair is compatible.

Next suppose (X,Y,z) is narrow and let e, f, g denote the (parallel) edges between X and Y in G. Let D_x, D_y be a pair of 2-connected Eulerian orientations. Since $D_x - z$ and $D_y - z$ are both strongly connected, we may assume, by reorienting all arcs of D_y, and by relabelling e, f, g, if necessary, that the orientations of the edges

e, f, g are the same in D_x and D_y. Thus we have a compatible pair in this case as well.

Hence the directed graph D, obtained by merging D_x and D_y along (X, Y, z) is a 2-connected orientation of G by Lemma 3.4. It remains to show that D is Eulerian. Since D_x and D_y are Eulerian orientations, we have $d^+_D(w) = d^-_D(w)$ for all $w \in V - z$. But then $d^+_D(z) = d^-_D(z)$ must also hold. This completes the proof of the lemma. •

Lemma 3.6. *Let $(X + v, Y + u, z)$ be a special mixed cut in $G = (V, E)$ and suppose that G/X and $G/(Y + u)$ both have 2-connected Eulerian orientations. Then G also has a 2-connected Eulerian orientation.*

Proof: First we show that there exist 2-connected Eulerian orientations D_x and D_y of $G_x = G/X$ and $G_y = G/(Y + u)$, respectively, which are compatible (with respect to the mixed cut $(X, Y + u, Z)$ with $Z = \{v, z\}$). Let e, f denote the (parallel) edges from v to u and let g denote the edge from X to $Y + u$ in G. Let D_x and D_y be a pair of 2-connected Eulerian orientations. By reorienting all arcs of D_y, if necessary, we may assume that the orientation of g is the same in D_x and D_y. By 2-connectivity, and since $d_{G_x}(v, u) = d_{G_y}(v, y) = 2$, this implies that the orientation of the edge between v and z, if it exists, is also the same in D_x and D_y. Thus $D_x[Z] = D_y[Z]$ is also satisfied, and the orientations are compatible.

Thus the directed graph D, obtained by merging D_x and D_y along $(X, Y + u, Z)$ is a 2-connected orientation of G by Lemma 3.4. It remains to show that D is Eulerian. Since D_x and D_y are Eulerian orientations, we have $d^+_D(w) = d^-_D(w)$ for all $w \in V - Z$. Since $d_G(v) = 4$ and D is 2-connected, we also have $d^+_D(v) = d^-_D(v)$. But then $d^+_D(z) = d^-_D(z)$ holds as well. This completes the proof of the lemma. •

4 Hooking up arcs

Let $D = (V, A)$ be a directed graph and $ab, xy \in A$. By *hooking up ab, xy* on vertex v we mean the operation which adds a new vertex v to D, deletes the arcs ab, xy, and adds new arcs av, vb, xv, vy. Note that in the underlying undirected graph of the resulting digraph this corresponds to the reverse operation of a complete splitting at v. We shall also use hooking up to create 2-connected orientations. We need the following lemmas to describe the situations when hooking up preserves 2-connectivity. Note that in the next two lemmas e or f (or both) may be loops.

Lemma 4.1. *Let $D = (V, A)$ be a 2-connected digraph, let $e, f \in A$, and let D' be obtained from D by hooking up e, f on vertex v. Then either D' is 2-connected or one of the following holds:*
(i) the arcs e, f have a common tail or head (or both),
(ii) there is a bipartition of V into sets X, Y with $|X|, |Y| \geq 2$, and such that the set of arcs entering Y in D equals $\{e, f\}$.

Proof: Suppose that D' is not 2-connected. Then there is a vertex $w \in V(D')$ and a set $\emptyset \neq X \subset V(D') - w$ with $d^+_{D'-w}(X) = 0$.

First suppose $w = v$. In this case $X \subset V = V(D') - v$. Since D is 2-connected, we have $d^+_D(X) \geq 2$, and hence the set of edges entering $V - X$ in D must be equal to $\{e, f\}$. If $|X|, |V - X| \geq 2$ then (ii) follows by choosing $Y = V - X$. Otherwise we must have (i).

Next suppose $w \neq v$. If $X = \{v\}$ ($X = V(D') - \{w, v\}$) then w is the common tail (head, respectively) of e and f, and hence (i) follows. So we may assume, without loss of generality, that $v \in X$ and $|X| \geq 2$. By using the assumption $d^+_{D'-w}(X) = 0$, we obtain $d^+_{D-w}(X - v) = 0$. Thus $D - w$ is not strongly connected, a contradiction. This completes the proof of the lemma. •

Lemma 4.2. *Let $D = (V, A)$ be a 2-connected digraph, let $e, f \in A$, and let D' be obtained from D by hooking up e, f on vertex v. Suppose that the underlying graph G' of D' is weakly four-connected and at least one of e, f is a loop. Then D' is 2-connected.*

Proof: Since at least one of e, f is a loop, Lemma 4.1(ii) cannot hold. Furthermore, if Lemma 4.1(i) holds, then $d_{G'}(v, u) \geq 3$ for some $u \in V(G')$. This is impossible, since $d_{G'}(v) = 4$ and G' is weakly four-connected. This completes the proof by Lemma 4.1. •

We shall also need a similar operation, which hooks up one arc without adding new vertices.

Lemma 4.3. *Let $D = (V, A)$ be a 2-connected digraph with $|V| \geq 4$ and suppose that there is an arc $e = xy \in A$ and a vertex $v \in V$ with $N^+(v) = N^-(v) = \{x, y\}$. Then $D - xy + xv + vy$ is 2-connected.*

Proof: Let $D' = D - xy + xv + vy$. Since D is 2-connected, it is easy to see that $D' - w$ is strongly connected for all $w \neq v$. Thus it remains to prove that $D' - v$ is also strongly connected. If this is not the case, then there is a bipartition of $V - v$ into two non-empty sets X, Y for which $x \in X$, $y \in Y$, and, without loss of generality, the only arc from X to Y in D is e. Since $|V| \geq 4$, we may assume, by reorienting all arcs of D and relabelling X and Y, if necessary, that $|Y| \geq 2$. Now $x \in X$ and $N^+(v) = \{x, y\}$ imply that no arc enters $Y - y$ in $D - y$, contradicting the fact that D is 2-connected. •

5 Removable cycles

A cycle C is called *removable* in a weakly four-connected graph G if $G - E(C)$ is also weakly four-connected. We shall use the following result on removable cycles.

Theorem 5.1. *[1] Let $G = (V, E)$ be a weakly four-connected graph with maximum edge multiplicity at most two, and let $y \in V$ be a designated vertex. Suppose that $d(v) \geq 6$ for all $v \in V - y$. Then there is a removable cycle C in G.*

Lemma 5.2. *Let (X, Y, w) be a tight mixed cut in G and suppose that C is a removable cycle in G/Y with $V(C) \subseteq X + w$. Then C is a removable cycle in G.*

Proof: Since $G - E(C)$ can be obtained from $G/Y - E(C)$ and G/X by the undirected version of merging, it suffices to show that merging along a tight mixed cut preserves weak four-connectivity. The proof of this fact is similar to the proof of Lemma 3.4, and is left to the reader. •

Theorem 5.3. *Let $G = (V, E)$ be a weakly four-connected Eulerian graph with maximum edge multiplicity at most two. If $d(v) \geq 6$ for all $v \in V$ then there is a removable cycle in G. If G is Eulerian and (X, Y, w) is a tight mixed cut with $d(x) \geq 6$ for all $x \in X$ then there is a removable cycle C in G with $V(C) \subseteq X + w$.*

Proof: The first part of the proof follows directly from Theorem 5.1. To see the second part consider the graph G' obtained from G by contracting Y into vertex y, and then deleting edges between y and w to make $d_{G'}(y, w) = 2$ (and hence $d_{G'}(y) = 4$). Graph G' is weakly four-connected by Lemma 3.2 and since deleting an edge with multiplicity more than 2 preserves weak four-connectivity. It is also clear that G' is an Eulerian graph with maximum edge multiplicity equal to two.

Now apply Theorem 5.1 to G' with designated vertex y to obtain a removable cycle C in G'. Since $d_{G'}(y) = 4$, $V(C) \subseteq X + w$ must hold. Now Lemma 5.2 implies that C is a removable cycle in G, as required. •

6 Orienting weakly four-connected Eulerian graphs

Theorem 6.1. *Let $G = (V, E)$ be a weakly four-connected Eulerian graph. Then G has a 2-connected Eulerian orientation.*

Proof: Suppose that the theorem is false and let $G = (V, E)$ be a counterexample for which $|V|$ is as small as possible, and subject to this, $|E|$ is as small as possible. Since the theorem trivially holds for graphs on three vertices, we have $|V| \geq 4$.

First we use our reduction lemmas to deduce some structural properties of G. Suppose that (X, Y, z) is a non-trivial mixed cut, which is either tight or narrow. By Lemma 3.2 G/X and G/Y are both weakly four-connected graphs on a smaller set of vertices. They are Eulerian, too. Since G is a minimal counterexample, this implies that each of G/X and G/Y has a 2-connected Eulerian orientation. Now Lemma 3.5 implies that G also has a 2-connected Eulerian orientation, a contradiction. Thus

Every tight or narrow mixed cut in G is trivial. (1)

Suppose that $(X + v, Y + u, z)$ is a special mixed cut in G. By Lemma 3.3 and (1) it follows that G/X and $G/(Y + u)$ are both weakly four-connected graphs on a smaller set of vertices. They are Eulerian, too. Since G is a minimal counterexample, this implies that each of G/X and $G/(Y + u)$ has a 2-connected Eulerian orientation. Now

Lemma 3.6 implies that G also has a 2-connected Eulerian orientation, a contradiction. Thus

$$\textit{There is no special mixed cut in } G. \tag{2}$$

Now suppose G has a removable cycle C. Since $G - E(C)$ is weakly four-connected, Eulerian, and has fewer edges than G, the minimal choice of G implies that $G - E(C)$ has a 2-connected Eulerian orientation D'. By adding (a directed version of) cycle C to D' we obtain a 2-connected Eulerian orientation of G, a contradiction. Thus

$$\textit{There is no removable cycle in } G. \tag{3}$$

This implies that G contains no loops, and, since deleting two copies of an edge with multiplicity at least four preserves weak four-connectivity, it gives that the maximum edge multiplicity in G is at most three.

Suppose that G contains a narrow mixed cut (X, Y, z). By (1) we may assume that $X = \{v\}$ for some $v \in V$. Since G is weakly four-connected and Eulerian, the maximum edge multiplicity is at most three, and $d(v, Y) = 3$, we must have $d(v, z) = 3$. Let u be the unique neighbour of v in Y. It is easy to see that $G' = G - vz - vu + zu$ is weakly four-connected. Furthermore, G' is Eulerian and has fewer edges than G. Thus, by the minimal choice of G, G' has a 2-connected Eulerian orientation D'. Clearly, $N^+_{D'}(v) = N^-_{D'}(v) = \{z, u\}$, and – without loss of generality – $zu \in A(D')$. Thus Lemma 4.3 implies that $D' - zu + zv + vu$ is a 2-connected Eulerian orientation of G. This contradiction implies that

$$\textit{There is no narrow mixed cut in } G. \tag{4}$$

Suppose that $d_G(u, v) = 3$ for some pair $u, v \in V$. It follows from (3) that $G - \{uv, uv\}$ is not weakly four-connected. By Lemma 1.1 this implies that there is a narrow mixed cut in G, which contradicts (4). Thus

$$\textit{The maximum edge multiplicity in } G \textit{ is at most two.} \tag{5}$$

Next suppose there is a vertex $v \in V$ with $d(v) = 4$ and $|N(v)| = 2$. Let $N(v) = \{u, w\}$. Since G is weakly four-connected, we have $d(v, u) = d(v, w) = 2$. By Lemma 2.2 $G' = G - v + uw + uw$, obtained from G by a complete splitting at v, is weakly four-connected. It is also Eulerian. So the minimality of G implies that G' has a 2-connected Eulerian orientation D'. If $d_G(u) = 4$ then $d_{G'}(u) = 4$ and $d_{G'}(u, w) = 2$. Moreover, the two parallel edges between u and w must be oppositely oriented in D'. Now Lemma 4.1 implies that hooking up the arcs uw, wu on v in D' results in a 2-connected Eulerian digraph D. Since D is an orientation of G, this contradicts our assumption. Thus $d_G(u) \geq 6$. By symmetry, we also have $d_G(w) \geq 6$.

If $G'' = G - v$ is weakly four-connected then the minimality of G implies that G'' has a 2-connected Eulerian orientation D''. This can be extended to a 2-connected Eulerian orientation of G by adding the arcs $\{vu, uv, vw, wv\}$, a contradiction. Thus G'' is not weakly four-connected.

This implies that either there is a set $\emptyset \neq X \subset V(G'')$ with $d_{G''}(X) \leq 3$ or there is a mixed cut (X, Y, z) in G'' with $d_{G''-z}(X, Y) \leq 1$. In the former case we must have

$d_{G''}(X) = 2$ (since G'' is Eulerian) and $|X|, |V(G'') - X| \geq 2$ (since $d_G(u), d_G(w) \geq 6$ and G is loopless). It follows that $(X, V(G'') - X, v)$ is a non-trivial tight mixed cut in G, which contradicts (1). In the latter case we must have $|X|, |Y| \geq 2$ by (5). We may assume that $w \in Y$. Then either $(X + v, Y, z)$ is a non-trivial tight mixed cut or $(X + v, (Y - w) + w, z)$ is a special mixed cut in G, which contradicts (1) and (2). Thus

$$\textit{There is no vertex } v \in V \textit{ with } d(v) = 4 \textit{ and } |N(v)| = 2. \tag{6}$$

It follows from (3), (5), and Theorem 5.3 that G as well as each mixed fragment of G contains a vertex of degree four. This property and Theorem 2.3 imply that G contains an admissible vertex v with $d(v) = 4$. By (6) we have $|N(v)| \geq 3$. Since v is admissible in G, we can obtain a weakly four-connected Eulerian graph G_v by a complete splitting off at v. The minimality of G implies that G_v has a 2-connected Eulerian orientation D_v. If hooking up the arcs e, f, corresponding to the split edges, on vertex v in D_v preserves 2-connectivity, then it yields a 2-connected Eulerian orientation of G, a contradiction. Thus

$$\textit{Hooking up } e, f \textit{ in } D_v \textit{ does not preserve } 2\textit{-connectivity.} \tag{7}$$

Suppose that $|N(v)| = 4$. Then it follows from (7) and Lemma 4.1 that there is a bipartition X, Y of $V(D_v)$ with $|X|, |Y| \geq 2$, and such that the set of arcs entering Y in D_v equals $\{e, f\}$. Since D_v is Eulerian, we also have $d_{D_v}(Y, X) = 2$. Then (X, Y, v) is a non-trivial tight mixed cut in G, contradicting (1). So $|N(v)| = 3$ must hold.

Let $N(v) = \{u, x, y\}$ with $d_G(v, u) = 2$. There are two possibilities: either $G_v = G - v + uu + xy$ or $G_v = G - v + ux + uy$. If $G_v = G - v + uu + xy$ then e or f is a loop. Since G is weakly four-connected, this contradicts (7) by Lemma 4.2. Thus the admissible complete splitting at v is unique and $G_v = G - v + ux + uy$.

Next we prove that $d_G(u) \geq 6$. For a contradiction suppose that $d_G(u) = 4$. By (6), and since $d_G(v, u) = 2$, we have $|N(u)| = 3$. Let $N_G(u) = \{a, b, v\}$. If u and v have a common neighbour, say $b = x$, then $d_{G_v}(u, x) = 2$ and ux, xu are oppositely oriented arcs in D_v (since $d_{G_v}(u) = 4$ and D_v is 2-connected). Hence, by reorienting all arcs in D_v, if necessary, we may assume that the split edges correspond to $ux, yu \in A(D_v)$. This contradicts (7) by Lemma 4.1. Thus

$$a, b, x, y \textit{ are pairwise distinct.} \tag{8}$$

We claim that $G^* = G_v - u + ab + xy$, obtained by splitting off u in G_v, is weakly four-connected. Suppose not. Then at least one of the three alternatives of Lemma 2.1 holds for G_v and u.

First consider the case when there is a set $\emptyset \neq X \subseteq V(G_v) - \{u, a, b\}$ with $d_{G_v}(X) = 4$ and $x, y \in X$. Let $Y = V(G_v) - X - u$. Then $a, b \in Y$ and, since $a \neq b$ by (8), $|Y| \geq 2$ follows. This implies that $(X + v, Y, u)$ is a non-trivial tight mixed cut in G, contradicting (1).

Next consider the case when there is a vertex $w \in V(G_v) - u$ and a set $\emptyset \neq X \subseteq V(G_v) - \{u, w, a, b\}$ with $d_{G_v - w}(X) \leq 3$ and $x, y \in X$. Let $Y = V(G_v) - u - w - X$. Then either $(X + v, Y + u, w)$ is a non-trivial tight mixed cut in G (if $d_{G_v - w}(X) = 2$), or $(X+v, Y+u, w)$ is a special mixed cut in G (if $d_{G_v-w}(X)=3$). This contradicts (1) or (2).

Finally, suppose that there is a set $\emptyset \neq X \subseteq V(G_v)-\{u,y,a,b\}$ with $d_{G_v-y}(X)=2$ and $x \in X$. Let $Y=V(G_v)-X-u-y$. Then $a,b \in Y$, and hence $(Y+u, X+v, y)$ is a special mixed cut in G, contradicting (2). Thus G^* is weakly four-connected (and Eulerian), as claimed.

By the minimality of G, G^* has a 2-connected Eulerian orientation D^*. By renaming the neighbours of u and v, if necessary, we may assume that $ab, xy \in A(D^*)$. It is easy to verify that by hooking up ab, xy on u in D^*, and then hooking up xu, uy on v in the resulting digraph, we obtain a 2-connected Eulerian digraph D, which is an orientation of G, a contradiction. (To see that D is 2-connected, observe that it can also be obtained from D^* by subdividing ab with vertex u, then subdividing xy with vertex v, and then adding two arcs uv, vu. Since $a \neq x$ and $b \neq y$ by (8), this operation preserves 2-connectivity.) Thus

$$d(u) \geq 6. \tag{9}$$

Since $G'_v = G-v+uu+xy$ is not weakly four-connected, at least one of the three alternatives of Lemma 2.1 holds for G and v.

First suppose that there is a set $\emptyset \neq X \subseteq V-v-u$ with $d_G(X)=4$ and $x,y \in X$. Let $Y=V-X-v$. It follows from (9) and $u \in Y$ that $|Y| \geq 2$. We also have $|X| \geq 2$, since $|N(v)|=3$ implies $x \neq y$. Hence (X,Y,v) is a non-trivial tight mixed cut in G, contradicting (1).

Next consider the case when there is a vertex $w \in V-v$ and a set $\emptyset \neq X \subseteq V-\{v,w,u\}$ with $d_{G-w}(X) \leq 3$ and $x,y \in X$. Let $Y=V-X-v-w$. Since $d_G(v,u)=2$, we have $u \neq w$ and $u \in Y$. Now (9) and (5) imply that $Y-u \neq \emptyset$. Hence either $d_{G-w}(X)=2$, and then $(X+v, Y, w)$ is a non-trivial tight mixed cut in G, or $d_{G_v-w}(X)=3$ and then $(X+v, (Y-u)+u, w)$ is a special mixed cut in G. This contradicts (1) or (2).

Finally, suppose that there is a set $\emptyset \neq X \subseteq V-\{v,y,u\}$ with $d_{G-y}(X)=2$ and $x \in X$. Let $Y=V-X-v-y$. Clearly, $u \in Y$. If $|X| \geq 2$ then $(X, Y+v, y)$ is a non-trivial tight mixed cut in G, contradicting (1). If $X=\{x\}$ then $d_G(x)=4$ must hold by (5). Thus $N(x)=\{v,y,q\}$ for some $q \in Y$ and $d_G(x,y)=2$. Since $|N_G(x)|=3$, it follows from Lemma 2.2 that x is admissible in G. Thus either $G_x=G-x+yy+vq$ or $G'_x=G-x+yv+yq$ is weakly four-connected (and Eulerian), and hence, by the minimal choice of G, it has a 2-connected Eulerian orientation, denoted by D_x or D'_x, respectively.

If D_x is 2-connected then Lemma 4.2 implies that hooking up yy and vq (or qv) results in a 2-connected Eulerian orientation of G, a contradiction. So there is a 2-connected Eulerian orientation D'_x of G'_x. Since $d_{G'_x}(v,y)=2$ and $d_{G'_x}(v)=4$, we must have $vy, yv \in A(D'_x)$. Thus, by reorienting all arcs in D'_x, if necessary, we may assume that the split edges correspond to arcs yq, vy in D'_x. Now Lemma 4.1 implies that hooking up yq, vy on x in D'_x gives a 2-connected Eulerian orientation of G, a contradiction. This completes the proof of the theorem. •

7 Concluding remarks

The proof of our main result can be used to show that one can obtain a 2-connected Eulerian orientation D of a given weakly four-connected Eulerian graph G in polynomial time. The algorithm iteratively reduces the problem to smaller graphs by contractions, splittings, and deleting cycles or vertices. Then it builds up a 2-connected orientation of G from 2-connected orientations of the smaller graphs by merging, hooking up arcs, and adding directed cycles or new vertices. One can show that the number of smaller graphs is polynomial, and that each of the following objects, if they exist, can be found efficiently: (a) non-trivial tight, narrow, or special mixed cuts, (b) an admissible vertex v and an admissible complete splitting at v, (c) the removable cycles used in the proof [1]. We omit the details.

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