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Abstract

Let $G = (V, E)$ be a graph and $x, y, z \in V$ be three designated vertices. We give a necessary and sufficient condition for the existence of a rigid two-dimensional framework (G, p), in which x, y, z are collinear. This result extends a classical result of Laman on the existence of a rigid framework on G. Our proof leads to an efficient algorithm which can test whether G satisfies the condition.

1 Introduction

A *framework* (G, p) in d-space is a graph $G = (V, E)$ and an embedding $p : V \to \mathbb{R}^d$. We say that the framework (G, p) is a *realisation* of graph G. The *rigidity matrix* of the framework is the matrix $R(G, p)$ of size $|E| \times d|V|$, where, for each edge $v_iv_j \in E$, in the row corresponding to v_iv_j, the entries in the d columns corresponding to vertex i (j) contain the d coordinates of $(p(v_i) - p(v_j))$ $((p(v_j) - p(v_i))$, respectively), and the remaining entries are zeros. See [8] for more details. The rigidity matrix of (G, p) defines the *rigidity matroid* of (G, p) on the ground set E by linear independence of rows of the rigidity matrix. The framework is *independent* if the rows of $R(G, p)$ are linearly independent. A *framework* (G, p) is *generic* if the coordinates of the points $p(v)$, $v \in V$, are algebraically independent over the rationals. Any two generic frameworks (G, p) and (G, p') have the same rigidity matroid. We call this the d-dimensional *rigidity matroid* $\mathcal{R}_d(G) = (E, r_d)$ of the graph G. We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$.

Lemma 1.1. *[8, Lemma 11.1.3] Let (G, p) be a framework in $\mathbb{R}^d$. Then rank $R(G, p) \le S(n, d)$, where $n = |V(G)|$ and*

$$S(n,d) = \begin{cases} nd - \binom{d+1}{2} & \text{if } n \ge d+1 \\ \binom{n}{2} & \text{if } n \le d+1. \end{cases}$$

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We say that a framework (G, p) in $\mathbb{R}^d$ is *infinitesimaly rigid* if $\operatorname{rank} R(G, p) = S(n, d)$. This definition is motivated by the fact that if (G, p) is infinitesimally rigid then (G, p) is *rigid* in the sense that every continuous deformation of (G, p) which preserves the edge lengths $||p(u) - p(v)||$ for all $uv \in E$, must preserve the distances $||p(w)-p(x)||$ for all $w, x \in V$, see [8]. We say that the graph G is *rigid* in $\mathbb{R}^d$ if $r_d(G) = S(n, d)$ holds. (In this case every generic framework (G, p) in $\mathbb{R}^d$ is infinitesimally rigid and hence is rigid.) We say that G is *M-independent* in $\mathbb{R}^d$ if E is independent in $\mathcal{R}_d(G)$. For $X \subseteq V$, let $i_G(X)$ denote the number of edges in $G[X]$, that is, in the subgraph induced by X in G. We use $i(X)$ when the graph G is clear from the context.

Lemma 1.1 implies the following necessary condition for G to be M-independent.

Lemma 1.2. *If $G = (V, E)$ is M-independent in $\mathbb{R}^d$ then $i(X) \leq S(|X|, d)$ for all $X \subseteq V$.*

The converse of Lemma 1.2 also holds for $d = 1, 2$. The case $d = 1$ follows from the fact that the 1-dimensional rigidity matroid of G is the same as the cycle matroid of G, see [2, Theorem 2.1.1]. The case $d = 2$ is a result of Laman [4].

Theorem 1.3. *[4] A graph $G = (V, E)$ is M-independent in $\mathbb{R}^2$ if and only if*

$$i(X) \leq 2|X| - 3 \textit{ for all } X \subset V \textit{ with } |X| \geq 2. \tag{1}$$

Note that Theorem 1.3 leads to efficient algorithms for testing independence (and computing the rank) in the rigidity matroid $\mathcal{R}_2(G)$. (For the rank function formula see Lovász and Yemini [5].) Since rank $R(G, p) = r_d(G)$ when (G, p) is generic, Theorem 1.3 gives a necessary and sufficient condition for the existence of an infinitesimally rigid realisation of G in $\mathbb{R}^2$.

Corollary 1.4. *Let $H = (V, F)$ be a graph. Then H has an infinitesimally rigid realisation (H, p) in $\mathbb{R}^2$ if and only if H has a spanning subgraph $G = (V, E)$ with $|E| = 2|V| - 3$, which satisfies (1).*

The converse of Lemma 1.2 does not hold for $d \geq 3$. Indeed, it remains an open problem to find good characterizations and algorithms for independence or, more generally, the rank function in the d-dimensional rigidity matroid of a graph when $d \geq 3$.

Several methods have been applied to obtain (partial) results for $d = 3$. One approach is to prove that certain operations ('reductions' and 'extensions') on G preserve independence and then apply these operations in inductive proofs, see [2, Section 5.3],[7],[8]. Even though the goal is to characterise the generic case, the proofs often use non-generic realisations. We believe that to verify that one version of the 'degree-five extension' operation preserves independence, it would be useful to know necessary and sufficient conditions for the existence of a framework (G, p) on G in $\mathbb{R}^3$ in which four given points are co-planar. This question (which is still open) motivated us to first study the two-dimensional analogue: given three vertices $x, y, z \in V(G)$, when do we have a infinitesimally rigid two-dimensional framework (G, p) in which x, y, z are collinear? In this paper we shall answer this question in terms of a necessary and sufficient condition and an efficient algorithm. From now on, if not stated otherwise, we shall always assume that $d = 2$.

The paper is organised as follows. Section 2 contains the basic definitions and preliminary results. In Section 3 we formulate the obstacle which precludes the existence of the required infinitesimally rigid framework and prove the main combinatorial lemma, which shows that, roughly speaking, if G has no obstacle then we can reduce G without creating an obstacle in the smaller graph. In Section 4 we use this lemma to verify our main result for M-independent graphs. This is extended to arbitrary graphs in Section 5.

2 Preliminaries

Throughout this paper we shall use $\subseteq$ and $\subset$ to denote set containment and proper set containment, respectively. Let $G = (V, E)$ be a graph. For $X, Y, Z \subset V$, let $E(X)$ be the set of edges of $G[X]$, $d(X, Y) = |E(X \cup Y) - (E(X) \cup E(Y))|$, and $d(X, Y, Z) = |E(X \cup Y \cup Z) - (E(X) \cup E(Y) \cup E(Z))|$. We define the *degree* of X by $d(X) = d(X, V - X)$. The degree of a vertex v is simply denoted by $d(v)$. Let $N(v) = \{u \in V : vu \in E\}$ denote the *neighbours* of v. We shall need the following equalities, which are easy to check by counting the contribution of an edge to each of their two sides.

Lemma 2.1. *Let G be a graph and $X, Y \subseteq V(G)$. Then*

$$i(X) + i(Y) + d(X, Y) = i(X \cup Y) + i(X \cap Y). \tag{2}$$

Lemma 2.2. *Let G be a graph and $X, Y, Z \subseteq V(G)$. Then*

$$\begin{aligned} i(X) + i(Y) + i(Z) + d(X, Y, Z) &= i(X \cup Y \cup Z) + i(X \cap Y) + i(X \cap Z) + \\ &\quad i(Y \cap Z) - i(X \cap Y \cap Z). \end{aligned}$$

We shall frequently use the fact that a graph $G = (V, E)$ is *M-independent* if and only if it satisfies (1) by Theorem 1.3. If G is M-independent and $|E| = 2|V| - 3$ then we call G *isostatic*.

Henceforth in this section, we let $G = (V, E)$ be an M-independent graph. We call a set $X \subseteq V$ *critical* if $i(X) = 2|X| - 3$ holds. A set $X \subseteq V$ is *semi-critical* if $i(X) = 2|X| - 4$.

Lemma 2.3. *[3, Lemma 2.3] Let $X, Y \subset V$ be critical sets in G with $|X \cap Y| \geq 2$. Then $X \cap Y$ and $X \cup Y$ are also critical, and $d(X, Y) = 0$.*

Lemma 2.4. *[3, Lemma 2.7] Let $X, Y, Z \subset V$ be critical sets in G with $|X \cap Y| = |X \cap Z| = |Y \cap Z| = 1$ and $X \cap Y \cap Z = \emptyset$. Then $X \cup Y \cup Z$ is critical, and $d(X, Y, Z) = 0$.*

Let v be a vertex in a graph G with $d(v) = 3$ and $N(v) = \{u, w, z\}$. The operation *splitting off* means deleting v (and the edges incident to v) and adding a new edge, say uw, connecting two vertices of $N(v)$. The resulting graph is denoted by G_v^{uw} and we say that the splitting is made *on the pair* uv, wv. (Note that v can be split off in at most three different ways.) Splitting off v on the pair uv, wv is said to be *admissible* if G_v^{uw} is M-independent. Otherwise the splitting is *non-admissible.* Vertex v is *admissible* if some split at v is admissible. The next lemma follows from the definition of admissibility and criticality.

Lemma 2.5. *Let $v \in V$ be a vertex with $N(v) = \{a, b, c\}$. The split va, vb is admissible if and only if there is no critical set X with $a, b \in X$ and $v \notin X$.*

Lemma 2.6. *Let $v \in V$ be a vertex with $N(v) = \{a, b, c\}$. The splits va, vc and vb, vc are both non-admissible if and only if there is a pair A, B of critical sets with $a \in A, b \in B$, $c \in A \cap B$ and $v \notin A \cup B$. Furthermore, for any such sets A, B we have $A \cap B = \{c\}$, $d(A, B) = 0$, and $A \cup B$ is semi-critical.*

Proof: The first part of the lemma follows from Lemma 2.5. To see the second part observe that $A \cup B$ cannot be critical, since otherwise $d(v, A \cup B) = 3$ would imply that $i(A \cup B \cup \{v\}) = 2|A \cup B \cup \{v\}| - 2$, contradicting (1). So $A \cap B = \{c\}$ follows from Lemma 2.3. By applying (2) to the critical sets A, B, we obtain $d(A, B) = 0$ and $i(A \cup B) = 2|A \cup B| - 4$. •

For a proof of the next lemma see, for example, [3, Lemma 2.8].

Lemma 2.7. *[4] Let $v \in V$.*
(a) If $d(v) = 2$ then $G - v$ is M-independent.
(b) If $d(v) = 3$ then v is admissible.

We shall also use some connectivity properties of the subgraphs induced by critical and semi-critical sets. We call $G = (V, E)$ *essentially* 3*-edge-connected* if G is 2-edge-connected, and every $X \subset V$ with $d(X) = 2$ satisfies $|X| = 1$ or $|V - X| = 1$. As usual, K_m denotes the complete graph on m vertices, for some $m \geq 1$.

Lemma 2.8. *[3, Lemma 2.6] Let $X \subseteq V$ be a critical set. Then either $G[X] = K_2$ or $G[X]$ is 2-connected. Furthermore, if $|X| \geq 3$, then $G[X]$ is essentially 3-edge-connected.*

Note that if $G = (V, E)$ is isostatic then V is critical. Thus it follows from Lemma 2.8 that isostatic graphs on at least three vertices are 2-connected and essentially 3-edge-connected.

Lemma 2.9. *Let $X \subseteq V$ be a semi-critical set. Then either $G[X]$ consists of two isolated vertices or $G[X]$ is connected. If $X = A \cup B$, $|A|, |B| \geq 2$, $|A \cap B| = 1$, and $d(A, B) = 0$, then A and B are both critical.*

Proof: If $|X| = 2$, then the fact that $i(X) = 2|X| - 4$ implies that $G[X]$ consists of two isolated vertices. Hence we may assume that $|X| \geq 3$.

Suppose that $G[X]$ is disconnected and let $X = A' \cup B'$ be a bipartition of X with $|A'|, |B'| \geq 1$ and $d(A', B') = 0$. By symmetry, we may assume that $|A'| \geq 2$. Then $i(X) = i(A') + i(B') \leq 2|A'| - 3 + 2|B'| - 2 = 2|X| - 5$, contradicting the fact that X is semi-critical. Thus $G[X]$ is connected.

To see the second part of the lemma consider a pair A, B with $X = A \cup B$, $|A|, |B| \geq 2$, $|A \cap B| = 1$, and $d(A, B) = 0$. Then (2) gives $2|A| - 3 + 2|B| - 3 \geq i(A) + i(B) = i(X) + i(A \cap B) - d(A, B) = 2|A \cup B| - 4 + 2|A \cap B| - 2 = 2|A| + 2|B| - 6$. Thus equality must hold everywhere, and hence A and B are both critical. •

2.1 Extensions of frameworks

We shall use the following two operations on frameworks. Let (G, p) be a framework. The operation 0-*extension (on vertices $a, b \in V$)* adds a new vertex v and two edges va, vb to G, and determines the position p_v of v in the new framework.

Lemma 2.10. *[8, Lemma 2.1.3] Suppose that (G, p) is an independent framework. Then the 0-extension of (G, p) on vertices a, b is independent for all choices p_v with p_a, p_b, p_v not collinear.*

The operation *1-extension (on edge $ab \in E$ and vertex $c \in V - \{a, b\}$)* subdivides the edge ab by a new vertex v and adds a new edge vc, and determines the position p_v of v in the new framework.

Lemma 2.11. *[8, Theorem 2.2.2] Suppose that (G, p) is an independent framework, $ab \in E(G)$, $c \in V(G) - \{a, b\}$, and the points p_a, p_b, p_c are not collinear. Then the 1-extension of (G, p) on ab and c is independent if p_v is any point on the line of p_a, p_b, distinct from p_a, p_b.*

Lemma 2.12. *Let (G, p) be an infinitesimally rigid framework and let $a \in V(G)$. Then there is an $\epsilon > 0$ such that every framework (G, p') is infinitesimally rigid, where p'_a is such that $||p'_a - p_a|| < \epsilon$, and $p'_v = p_v$ for all $v \in (V(G) - \{a\})$.*

Proof: Since (G, p) is infinitesimally rigid, $R(G, p)$ has a non-singular square submatrix A of size $2|V(G)| - 3$. Let $p_a = (x_a, y_a)$. By replacing all the entries x_a, y_a of A by variables x, y, respectively, the determinant of A becomes a polynomial $P(x, y)$. Since $P(x_a, y_a) \neq 0$, there exists an $\epsilon > 0$ for which $P(x'_a, y'_a) \neq 0$ for all $p'_a = (x'_a, y'_a)$ with $||p'_a - p_a|| < \epsilon$. Thus replacing p_a by p'_a, where $||p'_a - p_a|| < \epsilon$, does not decrease the rank of the rigidity matrix. Hence (G, p') is also infinitesimally rigid. •

We say that a framework (G, p) is *general* if the points p_v, $v \in V$, are in general position in $\mathbb{R}^2$ (i.e. no three points lie on a line). The facts that isostatic graphs are rigid, and that generic realisations of rigid graphs are both infinitesimally rigid and general, immediately implies the following lemma.

Lemma 2.13. *Let $G = (V, E)$ be isostatic. Then there is an infinitesimally rigid general framework (G, p) on G.*

3 Feasible splittings

Throughout this section we suppose that $G=(V,E)$ is M-independent, and $x,y,z\in V$ are three distinct vertices. An *obstacle (for the ordered triple* (x,y,z)*)* is an ordered triple of critical sets (X,Y,Z) for which $X\cap Y=\{z\}$, $X\cap Z=\{y\}$, and $Y\cap Z=\{x\}$. It follows from Lemmas 2.1 and 2.4 and the fact that G is M-independent that if (X,Y,Z) is an obstacle then $X\cup Y\cup Z$ is also critical, $d(X,Y,Z)=0$ holds, and each of the sets $X\cup Y, X\cup Z, Y\cup Z$ is semi-critical.

A *near obstacle (for the ordered triple* $(x,y;z)$*)* is an ordered triple of sets $(X,Y;Z)$ for which X,Y are critical, Z is semi-critical, $X\cap Y=\{z\}$, $X\cap Z=\{y\}$, $Y\cap Z=\{x\}$, and $d(X,Y,Z)=0$. It follows from Lemma 2.2 and the fact that G is M-independent that if $(X,Y;Z)$ is a near-obstacle then $X\cup Y\cup Z$ is semi-critical. Note that the notation reflects that the only semi-critical member of the near-obstacle is Z and that the pair of vertices from x,y,z that Z contains is $\{x,y\}$.

Henceforth, in this section, we suppose that

$$\text{there is no obstacle in } G \text{ for the ordered triple } (x,y,z). \tag{3}$$

Note that (3) implies that there is no obstacle for all orderings of the vertices x,y,z.

Let v be a vertex of degree three in $V-\{x,y,z\}$ and $a,b\in N(v)$ be a non-adjacent pair of neighbours of v. The pair va,vb is called *suitable* if G_v^{ab} also satisfies (3). The next lemma is easy to verify.

Lemma 3.1. *The pair va,vb is suitable if and only if there is no near obstacle $(P,Q;R)$ in G with $v\in V-(P\cup Q\cup R)$ and $a,b\in R$, for some triple $(p,q;r)$, such that $\{p,q,r\}=\{x,y,z\}$.*

Lemma 3.2. *Let $(X,Y;Z)$ be a near obstacle for $(x,y;z)$ and let W be a critical set with $x,y\in W$. Then $X\cup Y\cup Z\cup W$ is critical.*

Proof: Since $\{x,y\}\subseteq Z\cap W$, we have $|Z\cap W|\geq 2$. It follows from (3) that $Z\cap W$ is not critical, since otherwise $(X,Y,Z\cap W)$ would be an obstacle for (x,y,z). Using (2) and the fact that W is critical and Z is semi-critical, we can deduce

$$2|Z|-4+2|W|-3=i(Z)+i(W)\leq i(Z\cap W)+i(Z\cup W)\leq$$

$$2|Z\cap W|-4+2|Z\cup W|-3.$$

Thus equality holds throughout and $Z\cup W$ is critical.

If $W\cap(X\cup Y)=\{x,y\}$, then (X,Y,W) would be an obstacle for (x,y,z), contradicting (3). Thus we may assume, without loss of generality, that $|W\cap Y|\geq 2$. Hence $|(Z\cup W)\cap Y|\geq 2$, and so $Z\cup W\cup Y$ is critical by Lemma 2.3. Clearly, we have $|(Z\cup W\cup Y)\cap X|\geq 2$, and hence, again by Lemma 2.3, we have that $X\cup Y\cup Z\cup W$ is critical, as claimed. •

Lemma 3.3. *Let $(X,Y;Z)$ be a near obstacle for $(x,y;z)$, let $v\in V-(X\cup Y\cup Z)$ be a vertex with $N(v)=\{a,b,c\}$, and suppose that $a,b\in Z$. Let Z' be a semi-critical set with $v\notin Z'$ and $x,y,c\in Z'$. Then $z\in Z'$.*

Proof: For a contradiction suppose that $z \notin Z'$. Since $\{y, x\} \subseteq Z \cap Z'$, we have $|Z \cap Z'| \geq 2$. Note that $c \notin Z$ must hold, since otherwise $Z + v$ would be critical and $(X, Y, Z + v)$ would be an obstacle for (x, y, z). The set $Z \cap Z'$ cannot be critical, since otherwise $(X, Y, Z \cap Z')$ would be an obstacle for (x, y, z). The set $Z \cup Z'$ cannot be critical either, since otherwise $i((Z \cup Z') + v) \geq 2|(Z \cup Z') + v| - 2$ would follow, contradicting the fact that G is M-independent. Thus, using the fact that Z and Z' are semi-critical, we can use (2) to deduce that

$$2|Z| - 4 + 2|Z'| - 4 = i(Z) + i(Z') \leq$$

$$i(Z \cap Z') + i(Z \cup Z') \leq 2|Z \cap Z'| - 4 + 2|Z \cup Z'| - 4.$$

Thus $Z \cup Z'$ is semi-critical, and so $(Z \cup Z') + v$ is critical.

We must have $|Z' \cap (X \cup Y)| \geq 3$, otherwise $(X, Y, (Z \cup Z') + v)$ would be an obstacle for (x, y, z). We may assume, without loss of generality, that $|Z' \cap Y| \geq 2$.

The sets $X \cup Y$ and $Z \cup Z'$ are semi-critical, and the set $X \cup Y \cup Z \cup Z'$ cannot be critical, since $d(v, X \cup Y \cup Z \cup Z') = 3$. Thus (2) gives that $T := (X \cup Y) \cap (Z \cup Z')$ is semi-critical. Furthermore, $|T| \geq 3$, and, since $z \notin Z'$ and $d(X, Y) = 0$, $G[T]$ is disconnected. This contradicts Lemma 2.9. •

Lemma 3.4. *Let $(X, Y; Z)$ be a near obstacle for $(x, y; z)$ and let $v \in V - (X \cup Y \cup Z)$ be a vertex with $N(v) = \{a, b, c\}$ and $a, b \in Z$. Then there is no near obstacle $(P, Q; R)$ in G for $(p, q; r)$ with $\{p, q, r\} = \{x, y, z\}$, $v \in V - (P \cup Q \cup R)$ and $b, c \in R$.*

Proof: Suppose that $(P, Q; R)$ is a near obstacle for $(p, q; r)$ with $\{p, q, r\} = \{x, y, z\}$, $v \in V - (P \cup Q \cup R)$ and $b, c \in R$. If $\{p, q\} = \{x, y\}$ and $r = z$ then R is a semi-critical set with $v \notin R$, $x, y, c \in R$, and $z \notin R$. This contradicts Lemma 3.3.

Thus we may assume, without loss of generality, that $p = z, q = x$, and $r = y$. Now P is a critical set with $x, y \in P$, and Y is a critical set with $x, z \in Y$. Thus Lemma 3.2, applied to the near obstacles $(X, Y; Z)$ and $(P, Q; R)$, respectively, gives that $X \cup Y \cup Z \cup P$ and $P \cup Q \cup R \cup Y$ are both critical. Now Lemma 2.3 implies that $T := X \cup Y \cup Z \cup P \cup Q \cup R$ is critical. Since $d(v, T) = 3$, this contradicts the fact that G is M-independent. •

Lemma 3.5. *Let $(X, Y; Z)$ be a near obstacle for $(x, y; z)$ and let $v \in V - (X \cup Y \cup Z)$ be a vertex with $N(v) = \{a, b, c\} \neq \{x, y, z\}$ and $a, b \in Z$. Suppose that the split on va, vb is admissible. Then at least one of the splits va, vc or vb, vc is admissible.*

Proof: Suppose that the splits va, vc and vb, vc are both non-admissible. By Lemma 2.6 there is a pair A, B of critical sets with $v \notin A \cup B$, $A \cap B = \{c\}$, $a \in A$, $b \in B$, $d(A, B) = 0$, such that $A \cup B$ is semi-critical. We may assume that A and B are both maximal subject to $v \notin A \cup B$, $a \in A$, $b \in B$, $c \in A \cap B$.

As above, $c \in Z$ would imply that $(X, Y, Z + v)$ is an obstacle for (x, y, z), so we must have $c \notin Z$. Observe that $(A \cup B) \cap Z$ is not critical, since $a, b \in (A \cup B) \cap Z$ and

va, vb is an admissible pair. The set $A \cup B \cup Z$ is not critical, since $d(v, A \cup B \cup Z) = 3$ and G is M-independent. Thus it follows from (2) that

$$2|A \cup B| - 4 + 2|Z| - 4 = i(A \cup B) + i(Z) \leq i((A \cup B) \cap Z) + i(A \cup B \cup Z) \leq$$
$$2|(A \cup B) \cap Z| - 4 + 2|A \cup B \cup Z| - 4.$$

This implies that $Q := A \cup B \cup Z$ and $T := (A \cup B) \cap Z$ are both semi-critical. Now $G[T]$ is disconnected (since $d(A, B) = 0$ and $c \notin Z$). By Lemma 2.9 this gives $|T| = 2$. Since $X \cup Y \cup Z$ is semi-critical, a similar counting argument, using (2), gives that $(A \cup B) \cap (X \cup Y \cup Z)$ is semi-critical. Since Q is semi-critical, $Q + v$ is critical. This implies, by (3), that $(A \cup B) \cap ((X \cup Y) - Z) \neq \emptyset$, and hence $|(A \cup B) \cap (X \cup Y \cup Z)| \geq 3$. By symmetry, we may assume that $|B \cap X| \geq 2$. Lemma 2.9 implies that $(A \cup B) \cap (X \cup Y \cup Z)$ is disconnected. Using the facts that $|T| = 2$, $G[T]$ is disconnected, and $d(X, Y, Z) = 0$, we deduce that $T = \{x, y\}$ holds.

Since $N(v) \neq \{x, y, z\}$ we have $c \neq z$. Since $|B \cap X| \geq 2$, and B and X are critical, and $\{v, a\} \cap (X \cup B) = \emptyset$, Lemma 2.3 and the maximality of B implies $X \subseteq B$. If $|B \cap Y| \geq 2$ then $B \cup Y$ is critical, which is impossible (since G is M-independent and $d(v, B \cup Y) \geq 3$). Thus $B \cap Y = \{z\}$. Since $c \neq z$, we have $c \in B - Y$. Similarly, if $|A \cap Y| \geq 2$, then $Y \subseteq A$ follows from the maximality of A. This would imply $\{z, c\} \subset A \cap B$, contradicting the fact that $A \cap B = \{c\}$. So $A \cap Y = \{x\}$. Since A, B, Y are all critical, $|A \cap Y| = 1$, $|B \cap Y| = 1$, and $|A \cap Y| = 1$, Lemma 2.2 implies that $A \cup B \cup Y$ is also critical. But $d(v, A \cup B \cup Y) = 3$, contradicting the fact that G is M-independent. This proves the lemma. •

A split at v is *feasible* if it is admissible and suitable.

Lemma 3.6. *Let $v \in V - \{x, y, z\}$ be a vertex with $d(v) = 3$ and $N(v) \neq \{x, y, z\}$. Then there is a feasible split at v.*

Proof: Let $N(v) = \{a, b, c\}$. It follows from Lemma 2.7 that there is an admissible split, say va, vb, at v. If this split is suitable then the lemma follows. Otherwise, by Lemma 3.1, there is a near obstacle $(X, Y; Z)$ for $(x, y; z)$ with $a, b \in Z$. By Lemma 3.5 we may assume that the split va, vc is also admissible. If this split is suitable, we are done. Otherwise, by Lemma 3.1, there is a near obstacle $(P, Q; R)$ for $(p, q; r)$ with $\{p, q, r\} = \{x, y, z\}$, $v \in V - (P \cup Q \cup R)$ and $b, c \in R$. This contradicts Lemma 3.4. •

For two vertices $x, y \in V$ in an isostatic graph let $C_{x,y}$ denote the (unique) minimal critical set which contains the pair x, y. Since the graph is isostatic, V is critical. Thus $C_{x,y}$ exists for all pairs x, y. Uniqueness follows from Lemma 2.3. Since each set in an obstacle (X, Y, Z) is critical, we can deduce the following characterisation.

Lemma 3.7. *Let $x, y, z \in V$ be three distinct vertices in an isostatic graph $G = (V, E)$. Then there exists an obstacle for (x, y, z) if and only if $|C_{xy} \cap C_{xz}| = |C_{xy} \cap C_{yz}| = |C_{xz} \cap C_{yz}| = 1$.*

Note that $C_{x,y}$ is the set of vertices induced by the edges in the unique circuit of the rigidity matroid $\mathcal{R}_2(G)$.

4 Infinitesimally rigid realizations with collinear vertices

Throughout this section let $G = (V, E)$ be an isostatic graph and let $x, y, z \in V$ be distinct vertices. We say that $v \in V$ is *special* if $d(v) = 3$, $v \notin \{x, y, z\}$ and $N(v) \neq \{x, y, z\}$. In the inductive proof of our main result we shall either delete vertices of degree two or use splitting off (and Lemma 3.6) at some special vertex to reduce the graph, unless the graph and the triple x, y, z forms one of the following four exceptional configurations G_i, $3 \leq i \leq 6$.

Let $K_{s,t}$ denote the complete bipartite graph on $s + t$ vertices, $s, t \geq 1$. Recall that K_m denotes the complete graph on m vertices. In the first special configuration G_3 we have $G = K_3$ with $V(G) = \{x, y, z\}$. In the second one, denoted by G_4, the graph is obtained from K_4 by deleting an edge, and $G[\{x, y, z\}] = K_{1,2}$. The graph of G_5 is obtained from $K_{2,3}$ by adding an edge (connecting two vertices of degree two), and $G[\{x, y, z\}] = K_2$. Finally, the graph of G_6 is $K_{3,3}$, and $E(G[\{x, y, z\}]) = \emptyset$.

Lemma 4.1. *Let $G = (V, E)$ be isostatic with $|V| \geq 3$ and let $x, y, z \in V$ be distinct vertices of G. Then at least one of the following holds:*
(a) there is a special vertex in G,
(b) there is a vertex v with $d(v) = 2$ and $v \notin \{x, y, z\}$,
(c) $d(x) = d(y) = d(z) = 2$ and $E(G[\{x, y, z\}]) = \emptyset$,
(d) G and x, y, z form a G_i configuration for some $3 \leq i \leq 6$.

Proof: A vertex v with $d(v) = 3$ and $N(v) = \{x, y, z\}$ will be called *bad*. Observe that G has at most three bad vertices, since otherwise G has a $K_{3,4}$ subgraph, whose vertex set violates (1). Let D_i denote the number of vertices of degree i in G. Since G is isostatic, we have $\sum_{v \in V} d(v) = 4|V| - 6$ and $d(v) \geq 2$ for all $v \in V$. Thus

$$2D_2 + D_3 \geq 6. \tag{4}$$

We shall verify the lemma for each possible value of D_3 by partitioning the alternatives into four subcases.

$D_3 \geq 6$. Now either (a) holds or we have exactly three bad vertices, $d(x) = d(y) = d(z) = 3$, and $G = K_{3,3}$. In the latter case (d) must hold.

$D_3 \in \{4, 5\}$. In this case $D_2 \geq 1$ by (4), so either (b) holds, or, without loss of generality, we have $d(z) = 2$. In the latter case we have at most two bad vertices. If $D_3 = 5$ then this implies (a). So suppose $D_3 = 4$. Then we must have two bad vertices v_1, v_2, and $d(x) = d(y) = 3$. If, in addition, $xy \in E$ then (d) holds. If $xy \notin E$ then $V - S \neq \emptyset$ and $d(S) = 2$, where $S = \{x, y, z, v_1, v_2\}$. Now either $|V - S| = 1$, which implies $|E| = 2|V| - 4$, or $|V - S| \geq 2$, which implies that G is not essentially 3-edge-connected. Both cases contradict the fact that G is isostatic.

$D_3 \in \{2, 3\}$. In this case $D_2 \geq 2$ by (4), so either (b) holds, or, without loss of generality, $d(y) = d(z) = 2$. In the latter case we have at most two bad vertices. So either (a) holds, or each vertex of degree three is either bad or is equal to x. Suppose (a) does not hold. If $D_3 = 3$ then G has two bad vertices v_1, v_2, $d(x) = 3$, $G[S] = K_{2,3}$,

$V - S \neq \emptyset$, and $d(S) = 1$, where $S = \{x, y, z, v_1, v_2\}$. This contradicts the fact that G is 2-connected. Hence we may assume that $D_3 = 2$. Then G has at least one bad vertex v_1. If $d(x) = 2$ then there must be another bad vertex v_2 and hence $G = K_{2,3}$ must hold. This contradicts the fact that G is isostatic.

Suppose $d(x) = 3$. Then either $V = S$, where $S = \{x, y, z, v_1\}$ (in which case we have a G_4 configuration, and hence (d) holds), or $V - S \neq \emptyset$. If $|V - S| = 1$ then we can deduce that $|V| = 5$ and $|E| = 6$. If $|V - S| \geq 2$ then we get $|E(G[V - S])| = 2|V| - 3 - i(S) - d(S) \geq 2|V - S| - 2$. Both cases contradict the fact that G is isostatic.

Thus we may assume that $d(x) \geq 4$. Then G has two bad vertices v_1, v_2, $G[S] = K_{2,3}$, where $S = \{x, y, z, v_1, v_2\}$, $V - S \neq \emptyset$, and $d(S - \{x\}, V - S) = 0$, contradicting the fact that G is 2-connected.

$D_3 \leq 1$. Now $D_2 \geq 3$ by (4). Hence either (b) holds or $d(x) = d(y) = d(z) = 2$. In the latter case, since G is essentially 3-edge-connected, we can deduce that either (c) holds or $G = G_3 = K_3$, in which case (d) holds. This completes the proof of the lemma. •

A framework (G, p) is *nearly general (with respect to $x, y, z \in V$)* if for every collinear triple p_a, p_b, p_c we have $\{a, b, c\} = \{x, y, z\}$. We also say that (G, p) is nearly general when the set $\{x, y, z\}$ is clear from the context. Let (p_1, p_2, p_3) be an ordered triple of three points of $\mathbb{R}^2$ and let $\lambda \in \mathbb{R} - \{0, 1\}$. We say that the triple is λ-*collinear*, if $(p_3 - p_2) = \lambda(p_3 - p_1)$.

Theorem 4.2. *Let $G = (V, E)$ be an isostatic graph, let $x, y, z \in V$ be distinct vertices, and let $\lambda \in \mathbb{R} - \{0, 1\}$. Then G has an infinitesimally rigid realisation (G, p), which is nearly general with respect to x, y, z and for which (p_x, p_y, p_z) are λ-collinear if and only if G contains no obstacle for the triple (x, y, z).*

Proof: First we prove the 'only if' direction. We prove the stronger statement that if G has an obstacle for the triple (x, y, z) then G has no infinitesimally rigid realisation such that p_x, p_y, p_z are collinear. To see this suppose that (G, p) is an infinitesimally rigid realisation of G such that p_x, p_y, p_z are collinear, and, for a contradiction, suppose also that (X, Y, Z) is an obstacle for (x, y, z) in G. Since $X \cup Y \cup Z$ is critical, $H = G[X \cup Y \cup Z]$ is isostatic. Since G is isostatic, it follows that the restriction (H, p) is an infinitesimally rigid realisation of H. On the other hand, $R(H, p)$ has rank at most $2|V(H)| - 4$. This can be seen by defining 'instantaneous velocities' $v : V(H) \to \mathbb{R}^2$ as follows. Let $v(a) = 0$ for all vertices of Y (in particular, let $v(z) = v(x) = 0$), let $v(y) \neq 0$ be orthogonal to the line of p_x, p_y, p_z, and let us choose the other vectors so that vertices of X and Z rotate about points p_z and p_x, respectively. Since (X, Y, Z) is an obstacle, it is easy to see that this is possible. Thus $R(H, p)v = 0$. Since v is not identically zero and leaves at least two points fixed, it follows that v does not correspond to a 'rigid congruence' of $\mathbb{R}^2$. Hence $R(H, p)$ has rank at most $2|V(H)| - 4$. Thus (H, p) is not infinitesimally rigid, a contradiction.

Next we prove the 'if' direction by induction on $|V|$. To this end let $G = (V, E)$ be an isostatic graph, let $x, y, z \in V$ be distinct vertices, and suppose that G has no obstacle for (x, y, z). Since G is isostatic and has no obstacle for (x, y, z), we can deduce that $|V| \geq 4$, and that if $|V| = 4$ then, without loss of generality, we have $V = \{x, y, z, w\}$ and $E = \{xw, wy, yx, zx, zw\}$. The required infinitesimally rigid framework (G, p) can be obtained by taking three non-collinear points p_x, p_y, p_w for x, y, w, and then choosing p_z in such a way that the framework is λ-collinear with respect to (x, y, z). Clearly, p_z will not be on the line of p_x, p_w, and hence the framework is indeed infinitesimally rigid by Lemma 2.10 (and the fact that any non-collinear realization of K_3 is infinitesimally rigid). It is clearly nearly general as well.

Now we turn to the induction step. Consider an isostatic graph $G = (V, E)$ with $|V| \geq 5$. This graph must satisfy one of the four alternatives (a),(b),(c),(d) of Lemma 4.1. Since $|V| \geq 5$, we only have to consider G_5 and G_6 when (d) holds. We shall verify the theorem for each of these alternatives.

(a) In this case G has a special vertex v. By Lemma 3.6 there is a feasible splitting at v which yields an isostatic graph G_v with no obstacle for (x, y, z). By induction, G_v has a nearly general infinitesimally rigid realisation (G, p) such that p_x, p_y, p_z are λ-collinear. Since v is special, $N_G(v) \neq \{x, y, z\}$. Thus we can use Lemma 2.11 to add vertex v and determine p_v by a 1-extension such that the resulting realisation of G is infinitesimally rigid. By applying Lemma 2.12 to p_v we can make the realisation nearly general and preserve rigidity, as required. (Note that to ensure near generality, we only have to make sure that p_v avoids a set of finitely many lines.)

(b) Now there is a vertex $v \in V - \{x, y, z\}$ with $d(v) = 2$. Since G has no obstacle for (x, y, z), $G - v$ has no obstacle for (x, y, z) either. Since G is isostatic, Lemma 2.7 implies that $G - v$ is also isostatic. Hence, by induction, $G - v$ has a nearly general infinitesimally rigid realisation $(G - v, p)$ where the triple (p_x, p_y, p_z) is λ-collinear. Now we can use Lemma 2.10 to add vertex v and determine p_v by a 0-extension in such a way that the resulting framework is nearly general and infinitesimally rigid, as required.

(c) In this case x, y, z are pairwise non-adjacent vertices of degree two. Thus $H = G - \{x, y, z\}$ is isostatic by Lemma 2.7. By Lemma 2.13, H has a general infinitesimally rigid realisation (H, p). Now we can apply Lemma 2.10 three times to add vertices x, y, z and determine the points p_x, p_y, p_z so that the resulting realization of G is nearly generic, infinitesimally rigid, and the triple (p_x, p_y, p_z) is λ-collinear. (This can be done by choosing a small enough ball B such that each point in B is in general position together with the points of H, fixing p_x as an arbitrary point in B, then choosing a line L through p_x so that it contains no other points of H, and then fixing p_y, p_z on $L \cap B$ so that (p_x, p_y, p_z) are λ-collinear.)

(d) As remarked above, we have two subcases to consider when Lemma 4.1(d) holds. First suppose G and x, y, z form a G_5 configuration on five vertices $\{x, y, z, a, b\}$, with $xy \in E$. We choose the points p_x, p_y, p_z so that they are λ-collinear, and choose p_a so that it is not on the line of p_x, p_y. Then we choose p_b, not on the p_x, p_y line, so that the intersection of the line of p_a, p_b and the line of p_x, p_y is different from p_x, p_y, p_z. We claim that with these positions the framework (G, p) is infinitesimally rigid. This follows from the fact that it can be built from the framework of the triangle x, y, a (with the chosen positions p_x, p_y, p_a) by a 0-extension, which adds b on the vertices

x, y, and another 0-extension, which adds z on the vertices a, b. It follows from Lemma 2.10 and the choice of p that the resulting framework (G, p) is infinitesimally rigid, nearly general, and (p_x, p_y, p_z) are λ-collinear.

The last subcase is when we have a G_6 configuration. Then $G = K_{3,3}$ and x, y, z are pairwise non-adjacent. Let $V(G) = \{x, y, z, a, b, c\}$. We fix the points of the framework as follows. First we choose p_x, p_y, p_z so that (p_x, p_y, p_z) are λ-collinear, and then we choose distinct points p_a, p_b, p_c so that p_b is not on the line of p_x, p_y; p_a is on the line of p_b, p_x; and p_c is on the line of p_b, p_z. We claim that with these positions (G, p) is infinitesimally rigid. This follows from the fact that the framework can be built by 0- and 1-extensions, starting from the triangle a, b, y (with the fixed positions p_a, p_b, p_y): first we add vertex c on vertices b, y by a 0-extension, then vertex x on edge ab and vertex c by a 1-extension, and finally vertex z on edge bc and vertex a by a 1-extension. The choice of the points and Lemmas 2.10 and 2.11 guarantee that (G, p) is infinitesimally rigid and (p_x, p_y, p_z) are λ-collinear. Then we can use Lemma 2.12 to move the points p_b and p_c and hence find the required infinitesimally rigid framework which is nearly general with respect to x, y, z. This completes the proof of the theorem. •

Theorem 4.2 has the following corollaries. A direct proof for the first one appears to be non-trivial.

Corollary 4.3. *Let $G = (V, E)$ be an isostatic graph, let $x, y, z \in V$ be distinct vertices, and let $\lambda \in \mathbb{R} - \{0, 1\}$. Then G has a infinitesimally rigid realisation in which p_x, p_y, p_z are collinear if and only if it has an infinitesimally rigid realisation in which (p_x, p_y, p_z) are λ-collinear.*

The next corollary extends (the difficult part of) Theorem 1.3. Note that every isostatic graph on at least four vertices contains non-adjacent pairs of vertices.

Corollary 4.4. *Let $G = (V, E)$ be isostatic with $|V| \geq 4$ and let $x, y \in V$ be non-adjacent vertices. Then there is a vertex $z \in V - \{x, y\}$ such that G has an infinitesimally rigid realisation in which p_x, p_y, p_z are collinear.*

Proof: Consider $C_{x,y}$ (the unique minimal critical set containing the pair x, y). Since $G[C_{x,y}]$ is not complete, there is a vertex $z \in C_{x,y} - \{x, y\}$. Now it follows from Lemma 3.7 that there is no obstacle for the triple (x, y, z) in G, and hence G has the required realisation by Theorem 4.2. •

5 Rigid graphs

In this section we extend Theorem 4.2 to arbitrary (rigid) graphs G. Since G has an infinitesimally rigid realisation if and only if G has an isostatic spanning subgraph H with an infinitesimally rigid realisation, Theorem 4.2 implies that we need to find necessary and sufficient conditions for the existence of an isostatic spanning subgraph H which contains no obstacle for a given set of three vertices.

To state (and prove) this condition, it will be convenient to use matroidal methods and terminology. (We refer the reader to [6] for the basic concepts of matroid theory.) Let G be a graph. A subgraph $H = (W, C)$ is said to be an *M-circuit* in G if C is a circuit (i.e. a minimal dependent set) in $\mathcal{R}_2(G)$. In particular, G is an *M-circuit* if E is a circuit in $\mathcal{R}_2(G)$. It is easy to deduce from (1) that G is an M-circuit if and only if $|V| \geq 2$, $|E| = 2|V| - 2$ and $i(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $2 \leq |X| \leq |V| - 1$. Since G is rigid if E has rank $2|V| - 3$ in $\mathcal{R}_2(G)$, M-circuits are rigid.

Given a matroid $\mathcal{M} = (E, r)$, we can define a relation on E by saying that $e, f \in E$ are related if $e = f$ or if there is a circuit C in $\mathcal{M}$ with $e, f \in C$. It is well-known that this is an equivalence relation. The equivalence classes are called the *components* of $\mathcal{M}$. If $\mathcal{M}$ has at least two elements and only one component then $\mathcal{M}$ is said to be *connected*. We say that a graph $G = (V, E)$ is *M-connected* if $\mathcal{R}_2(G)$ is connected. Then M-connected graphs are also rigid (see [3, Lemma 3.1]). For more details on these concepts and for examples see [3].

Lemma 5.1. *Let C_1, C_2 be M-connected graphs with $|V(C_1) \cap V(C_2)| \geq 2$. Then $C_1 \cup C_2$ is M-connected.*

Proof: Since the M-connected components of $C_1 \cup C_2$ are pairwise edge-disjoint, we only need to show that there exists an M-circuit C with $E(C) \cap E(C_1) \neq \emptyset \neq E(C) \cap E(C_2)$. If there is an edge f in $G[V(C_1) \cap V(C_2)]$ then we can simply choose M-circuits D_i in C_i with $e \in E(D_i)$, $i = 1, 2$, and apply the circuit axiom[1] to D_1, D_2 and e. Otherwise pick two vertices $a, b \in V(C_1) \cap V(C_2)$, let $e = ab$, and consider $C_1 + e$. Since C_1 is M-connected, and hence rigid, $C_1 + e$ is also M-connected. Similarly, $C_2 + e$ is M-connected. Now we may choose two M-circuits D_i in $C_i + e$, $i = 1, 2$, as above, and use the circuit axiom to deduce the existence of the required M-circuit in $C_1 \cup C_2$. •

In what follows we shall consider a rigid graph $G = (V, E)$ with three designated vertices $x, y, z \in V$. For a subgraph H with $x, y, z \in V(H)$ we use H^* to denote $H + xy, yz, zx$. Note that, if there exists an edge in H between x, y, z, then H^* will contain a pair of parallel edges which will induce an M-circuit in H^*.

Lemma 5.2. *Let $G = (V, E)$ be rigid. Then each isostatic spanning subgraph H of G has an (x, y, z)-obstacle if and only if the edges xy, yz, zx belong to three different M-connected components in G^*.*

Proof: First we prove the theorem in the special case when G is isostatic. Suppose that G has an (x, y, z)-obstacle. Then we have three edge-disjoint M-circuits C_1, C_2, C_3 in G^* with $xy \in E(C_1), yz \in E(C_2)$ and $zx \in E(C_3)$ by Lemma 3.7. For a contradiction suppose that there is an M-circuit C in G^* with $|E(C) \cap \{xy, yz, zx\}| \geq 2$. If $\{xy, yz, zx\} \subset E(C)$ then the circuit axiom, applied to C and C_3, gives an M-circuit C' with $E(C') \subseteq E(C) \cup E(C_3)$ and $zx \notin E(C')$. Since C_1 (resp. C_2) is the unique M-circuit in $G + xy$ ($G + yz$, resp.), C_1, C_2, C_3 are edge-disjoint, and $E(C_i)$

[1] If C, C' are circuits of matroid $\mathcal{M}$ and $x \in C \cap C'$ then there exists a circuit C'' in $\mathcal{M}$ such that $C'' \subset C \cup C'$ and $x \notin C''$.

cannot be a subset of $E(C)$ for $i = 1, 2$, we must have $xy, yz \in E(C')$. Thus we may assume that $E(C)$ contains xy and yz but not zy. As above, the circuit axiom applied to C and C_2 implies that there is an M-circuit C'' with $E(C'') \subset E(C) \cup E(C_2)$ and $yz \notin E(C'')$. Hence C'' is an M-circuit in $G + xy$, $C'' = C_1$, and $E(C_1) \subseteq E(C)$ follows, a contradiction.

Now suppose that G has no (x, y, z)-obstacle. By Lemma 3.7 we may assume that there exist two M-circuits C_1, C_2 in G^* with $xy \in E(C_1), yz \in E(C_2)$ and $|V(C_1) \cap V(C_2)| \geq 2$. Then xy and yz belong to the same M-connected component of G^* by Lemma 5.1. This completes the proof when G is isostatic.

In the rest of the proof we consider an arbitrary rigid graph G. Suppose that there is an M-circuit C in G^* with $xy, yz \in E(C)$. Let H be an isostatic spanning subgraph of G obtained by extending $E(C) - \{xy, yz, zx\}$ to a basis of $\mathcal{R}_2(G)$. Since xy and yz belong to the same M-connected component of H^*, it follows from the first part of the proof that H has no (x, y, z)-obstacle.

Conversely, suppose that the edges xy, yz, zx belong to three different M-connected components in G^*. Then they belong to different M-connected components in H^* for each isostatic spanning subgraph H of G. So it follows from the first part of the proof that there is an (x, y, z)-obstacle in each isostatic spanning subgraph of G. •

Our main result is the following theorem. It follows from Theorem 4.2 and Lemma 5.2.

Theorem 5.3. *Let $G = (V, E)$ be a rigid graph, let $x, y, z \in V$ be distinct vertices, and let $\lambda \in \mathbb{R} - \{0, 1\}$. Then G has a infinitesimally rigid realisation (G, p), which is nearly general with respect to x, y, z and for which (p_x, p_y, p_z) are λ-collinear if and only if there is an M-connected component C of G^* with $|E(C) \cap \{xy, yz, zx\}| \geq 2$.*

There exist efficient algorithms for finding the M-connected components of a graph G. See [1] for a recent $O(|V|^2)$ algorithm and related results. Thus the necessary and sufficient condition in Theorem 5.3 can be tested in $O(|V|^2)$ time.

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