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Abstract

As a common generalization of matchings and matroid intersection, W.H. Cunningham and J.F. Geelen introduced the notion of path-matching, which they generalized even further by introducing even factors of weakly symmetric digraphs. Later, a purely combinatorial approach to even factors was given by Gy. Pap and L. Szegő, who showed that the maximum even factor problem remains tractable in the class of hardly symmetric digraphs. The present paper shows a direct polyhedral way to derive weighted integer min-max formulae generalizing those previous results.

1 Introduction

Cunningham and Geelen [2, 3] introduced the notion of *even factor* in digraphs as the edge set of vertex disjoint union of *dicycles* of even length and *dipaths*. Here dicycle (dipath) means directed cycle (directed path), while a cycle (path) can contain both forward and backward edges. The maximum cardinality even factor problem is NP-hard in general (Wang, see [3]) but there are special classes of digraphs where it can be solved.

In the paper digraphs are assumed to have no loops and no parallel edges. An edge of a digraph is called *symmetric* if the reversed edge is also in the edge set of the digraph. A digraph is symmetric if all its edges are symmetric, while a digraph is said to be *weakly symmetric* if its strongly connected components are symmetric. Thus weak symmetry means that the edges that belong to a dicycle are symmetric. Pap and Szegő [10] introduced the more general class of *hardly symmetric* digraphs where the problem remains tractable. A digraph is said to be hardly symmetric if the edges contained in an odd dicycle are symmetric.

Using an algebraic approach, Cunningham and Geelen proved a min-max formula for the maximum cardinality even factor problem in weakly symmetric digraphs [3].

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Later, Pap and Szegő [10] proved a simpler formula for the same problem by a purely combinatorial method and they also observed that the same proof and formula works for the hardly symmetric case.

As the unweighted problem is tractable only in special digraphs, the weight function also has to possess certain symmetries. If $G' = (V', A')$ is a weakly symmetric digraph and $c' : A' \to \mathbb{R}$ is a weight function s.t. $c'(uv) = c'(vu)$ if both uv and vu belong to A', then the *pair* (G', c') is called *weakly symmetric*. Cunningham and Geelen considered a weighted generalization in [3] for weakly symmetric pairs. Using their unweighted formula and a primal-dual method they derived integrality of a polyhedron similar to the perfect matching polyhedron.

The present paper deals with a more general class of digraphs and weight functions which include the above mentioned cases. We show integrality of polyhedra related to even factors and a total dual integral system which generalizes among others the path-matching polyhedron of Cunningham and Geelen [4]. Our proofs use only polyhedral techniques, without relying on earlier weighted and unweighted results.

We conclude this section by stating the main result of the paper. To simplify the notation, the singleton $\{v\}$ will sometimes be denoted by v. For $x : V \to \mathbb{R}$ and $U \subseteq V$ we use the notation $x(U) = \sum_{v \in U} x(v)$ and we identify the functions $x : V \to \mathbb{R}$ with the vectors $x \in \mathbb{R}^V$. A family $\mathcal{F}$ of subsets of V is said to be *laminar* if for any $X, Y \in \mathcal{F}$, at least one of $X \subseteq Y$, $Y \subseteq X$, or $X \cap Y = \emptyset$ holds.

Similarly to the definition of weakly symmetric pairs, a *pair* of a digraph $G' = (V', A')$ and a weight function $c' : A' \to \mathbb{R}$ is said to be *hardly symmetric* if G' is hardly symmetric and $c'(C) = c'(\overline{C})$ for every odd dicycle C of G', where $\overline{C}$ denotes the odd dicycle obtained from C by reversing the direction of its edges.

If $G = (V, A)$ is a directed graph and $U \subseteq V$, then $G[U]$ stands for the subgraph of G induced by U; $\delta^{in}(U)$, $\delta^{out}(U)$ and $i(U)$ denote respectively the set of edges *entering*, *leaving* and *spanned by* U. Although the digraph is not indicated in these notations, it will always be clear from the context. The main result of the paper is as follows.

Theorem 1.1. *If $(G = (V, A), c)$ is a hardly symmetric pair and c is integral, then the optimal solution of the linear program*

$$\max cx \tag{1}$$

$$x \in \mathbb{R}^A,\ x \geq 0 \tag{2}$$

$$x(\delta^{in}(v)) \leq 1 \quad \textit{for every } v \in V \tag{3}$$

$$x(\delta^{out}(v)) \leq 1 \quad \textit{for every } v \in V \tag{4}$$

$$x(i(S)) \leq |S| - 1 \quad \textit{for every } S \subseteq V,\ |S| \textit{ odd} \tag{5}$$

is attained on an integer vector, and the dual of that linear program has an integer optimal solution $(\lambda_v^{in}, \lambda_v^{out}, z_S)$, where $\mathcal{F} = \{S : z_S > 0\}$ is a laminar family and for each $S \in \mathcal{F}$, $G[S]$ is strongly connected and symmetric.

The solution set of (2)-(3)-(4)-(5) is denoted by $\mathrm{EF}(G)$. It is easy to see that for any digraph G, the integer solutions of $\mathrm{EF}(G)$ are exactly the incidence vectors of even factors of G. While this polyhedron is not integer in general, we can guarantee an integral optimum solution for the above class of digraphs and weights. The proof of Theorem 1.1 uses a direct polyhedral technique. First, dual integrality is proved for hardly symmetric pairs, by considering a rational optimal dual solution which is extremal in some sense, and by altering it to reach integrality while optimality is maintained. Primal integrality can then be derived from this result. This technique resembles the way of proving integrality via the powerful notion of total dual integrality introduced by Edmonds and Giles [6]. The difference is that here we have dual integrality only for a restricted class of weight functions.

2 Preliminaries

The maximal strongly connected subgraphs of a digraph having 2-vertex-connected underlying undirected graph are called *blocks*. The following crucial observation on the structure of hardly symmetric digraphs was made by Z. Király [8].

Lemma 2.1 (Király). *Each block of a hardly symmetric digraph is either symmetric or bipartite.*

It follows that any odd dicycle is entirely spanned by some symmetric block. A dicycle C contained in a symmetric block is symmetric regardless of the parity of $|C|$. A similar statement is true for weight functions. A block is said to be *non-bipartite* if it contains an odd cycle, which by Lemma 2.1 implies that it contains an odd dicycle.

Lemma 2.2. *If (G, c) is a hardly symmetric pair and C is a dicycle contained in some symmetric non-bipartite block, then $c(C) = c(\overline{C})$.*

For a hardly symmetric digraph $G = (V, A)$, let $\mathcal{D}(G)$ denote the family of vertex-sets of its blocks, which partitions into the family of vertex-sets of bipartite blocks $\mathcal{B}(G)$ and the family of vertex-sets of non-bipartite (hence symmetric) blocks $\mathcal{S}(G)$. Let moreover $\mathcal{C}(G)$ be the family of vertex-sets of the strongly connected components of G.

In the following we characterize weight functions in hardly symmetric pairs. Clearly, the modification of the weight of edges which are not spanned by some strongly connected component preserves the hardly symmetric property. The same holds for the edges of $G[U]$ for every $U \in \mathcal{B}(G)$. The following lemma describes weights in non-bipartite blocks.

Lemma 2.3. *Let $(G = (V, A), c)$ be a hardly symmetric pair s.t. $\mathcal{S}(G) = \{V\}$. Then c is of the form*

$$c = \sum_{v \in V} b_v^{in} \chi_{\delta^{in}(v)} + \sum_{v \in V} b_v^{out} \chi_{\delta^{out}(v)} + \sum_{e \in A} b_e \chi_{\{e, \overline{e}\}}$$

for some reals b_v^{in}, b_v^{out} and b_e, where χ stands for the characteristic function.

Let $\mathcal{S}_*(G)$ denote the family of vertex-sets of the components of the hypergraph $(\cup\mathcal{S}(G), \mathcal{S}(G))$. It is easy to see that $\mathrm{EF}(G)$ equals to the solution set of

$$x \in \mathbb{R}^A,\ x \geq 0 \tag{6}$$
$$x(\delta^{in}(v)) \leq 1 \quad \text{for every } v \in V \tag{7}$$
$$x(\delta^{out}(v)) \leq 1 \quad \text{for every } v \in V \tag{8}$$
$$x(i(S)) \leq |S| - 1 \quad \text{for every } S \subseteq U,\ |S| \text{ odd},\ U \in \mathcal{S}_*(G). \tag{9}$$

Thus to prove Theorem 1.1, it is enough to consider this system.

3 Partial dual integrality

This section describes a general framework for obtaining primal integrality results based on the structure of the set of weight functions for which integral dual optimal solutions exist. We consider a bounded primal linear system of the form

$$\max\{cx :\ Ax \leq b,\ x \geq 0\}, \tag{10}$$

where A is a fixed $m \times n$ integer matrix, and b is a fixed integer vector. The dual system is

$$\min\{yb :\ yA \geq c,\ y \geq 0\}. \tag{11}$$

Let x_0 be a vertex of the primal polyhedron, and let $C \subseteq \mathbb{Z}^n$ be the set of integer weight vectors for which x_0 is optimal, and in addition there is an integral optimal dual solution. The strong duality theorem implies that cx_0 is an integer for every $c \in C$. Another easy observation is that C is closed under addition. Indeed, let $c_1, c_2 \in C$, and let y_1, y_2 be integral optimal dual solutions for c_1 and c_2, respectively. Now x_0 and $y_1 + y_2$ satisfy the complementarity conditions for the weight vector $c_1 + c_2$, which means that x_0 is an optimal primal solution and $y_1 + y_2$ is an optimal dual solution for $c_1 + c_2$.

The following Lemma states that in a certain sense the reverse of the above observations is also true.

Lemma 3.1. *Let $C \subseteq \mathbb{Z}^n$ be a set of integer weight vectors that is closed under addition, and suppose that for every $c \in C$ there is an integral optimal solution y_c of the dual system (11). Then for every $c_0 \in C$ there is an optimal solution x_0 of the primal system (10) such that zx_0 is an integer for every $z \in C$ (and thus for every integer combination of vectors in C).*

Proof. Let $c_0 \in C$; we will find an x_0 with the above properties. Let $z_0, z_1, \ldots$ be an enumeration of the elements of C such that $z_0 = c_0$. We will construct a sequence of weight vectors $c_0, c_1, \ldots$ (all of them in C) such that there is a vertex of the polyhedron (10) that is optimal for all of these weight vectors. The general step is the following:

- Suppose that $c_0, c_1, \ldots, c_i$ are already defined, and let the face F_i be the set of points that are optimal primal solutions for c_i. The set of weight vectors for which the optimal primal solutions are a subset of F_i form an open set in $\mathbb{R}^n$. Thus there exists a positive integer k such that the optimal primal solutions for the weight vector $kc_i + z_{i+1}$ form a sub-face of F_i. Let $c_{i+1} := kc_i + z_{i+1}$.

Since we have $F_0 \supseteq F_1 \supseteq \dots$ for the sequence of faces, $\cap_{i=0}^{\infty} F_i$ is a non-empty face of the primal polyhedron (10). Let x_0 be an arbitrary vertex of this face; then x_0 is optimal for all of $c_0, c_1, \dots$. The strong duality theorem implies that $c_i x_0$ is an integer for every i, and therefore zx_0 is an integer for every integer combination z of those weight vectors. But the elements of C can be obtained as integer combinations of that type. □

If $C = \mathbb{Z}^n$, then the above defined partial dual integrality specializes to the notion of *total dual integrality* introduced by Edmonds and Giles [6], which also extends to the case of unbounded polyhedra. "Total dual integral" is usually abbreviated as *TDI*.

4 A TDI system

The polyhedral results presented in Theorem 1.1 extend the results of the second named author in [9] where primal integrality is proved for a more restricted class of set functions, using total dual integrality of a different linear system. Here we cite these total dual integrality results, and we will show how they follow from Theorem 1.1.

Let $G = (V, A)$ be a hardly symmetric digraph, and let us consider a partition of $\mathcal{D}(G)$ into a set $\mathcal{S}$ of blocks and a set $\mathcal{B}$ of blocks s.t. for any $U \in \mathcal{S}$, $G[U]$ is symmetric and for any $U \in \mathcal{B}$, $G[U]$ is bipartite. Note that these families are not necessarily identical with $\mathcal{S}(G)$ and $\mathcal{B}(G)$ since a block can be both bipartite and symmetric. A weight function $c : A \to \mathbb{R}$ is said to be *evenly symmetric* (w.r.t. some G, $\mathcal{S}$ and $\mathcal{B}$) if $c(uv) = c(vu)$ for any edge uv spanned by some $U \in \mathcal{S}$. Let $\mathcal{S}_*$ be the family of vertex-sets of the components of the hypergraph $(\cup\mathcal{S}, \mathcal{S})$. Evenly symmetric weight functions enable us to handle the symmetric edges spanned by the members of $\mathcal{S}$ as undirected edges, since these oppositely directed edges have the same weight. So a *mixed graph* is considered with vertex set V and edge set $E = E_\text{d} \cup E_\text{u}$,

$$E_\text{u} = \{\{u, v\} : uv, vu \in \cup_{U \in \mathcal{S}} i(U)\},$$
$$E_\text{d} = \{uv \in A : \{u, v\} \notin E_\text{u}\},$$

where uv stands for the directed edge with tail u and head v, and $\{u, v\}$ for the undirected edge with endpoints u and v. For this mixed graph and for some $v \in V$ and $U \subseteq V$, $d(v)$ denotes the set of (directed and undirected) edges adjacent to v, $\delta^{in}(U)$ denotes the set of directed edges entering U, $\delta^{out}(U) = \delta^{in}(V - U)$, and $i(U)$ denotes the set of (directed and undirected) edges spanned by U.

In the following we give a TDI system which describes the polyhedron

$$\text{SEF}(G) = \left\{ y \in \mathbb{R}^E : \exists x \in \text{EF}(G),\ y(e) = \left\{ \begin{array}{rl} x(uv) + x(vu) & \text{if } e = \{u, v\} \in E_\text{u} \\ x(uv) & \text{if } e = uv \in E_\text{d} \end{array} \right\} \right\}.$$

It is easy to see that the integral points of this polyhedron correspond to even factors (however, an integral point does not define a unique even factor).

Theorem 4.1. *For a hardly symmetric digraph $G = (V, A)$ and corresponding families $\mathcal{S}$ and $\mathcal{B}$,* SEF(G) *is the solution set of*

$$y \in \mathbb{R}^E,\ y \geq 0 \tag{12}$$
$$y(d(v)) \leq 2 \quad \text{for every } v \in V \tag{13}$$
$$y(i(S)) \leq |S| - 1 \quad \text{for every } S \subseteq U,\ |S| \text{ odd},\ U \in \mathcal{S}_* \tag{14}$$
$$y(i(S)) + y(\delta^{in}(S)) \leq |S| \quad \text{for every } S \in \mathcal{U} \tag{15}$$
$$y(i(S)) + y(\delta^{out}(S)) \leq |S| \quad \text{for every } S \in \mathcal{U} \tag{16}$$

where $\mathcal{U} = \{U : \exists S \in \mathcal{S}_* \text{ s.t. } U \subseteq S\} \cup \{\{v\} : v \in V - \cup\mathcal{S}\}$.

This system is TDI. Moreover, if $c \in \mathbb{Z}^E$ is an integer vector, then there is an integer optimal solution $(\mu_v^d, \mu_S^i, \mu_S^{i,in}, \mu_S^{i,out})$ of the dual system s.t.

$$\mathcal{G} = \{S : \mu_S^i > 0\} \cup \{S : \mu_S^{i,in} > 0\} \cup \{S : \mu_S^{i,out} > 0\} \text{ is a laminar family,} \tag{17}$$
$$\text{if } \mu_S^i > 0, \mu_T^{i,in} > 0,\ S \cap T \neq \emptyset, \text{ then } S \subseteq T, \tag{18}$$
$$\text{if } \mu_S^i > 0, \mu_T^{i,out} > 0,\ S \cap T \neq \emptyset, \text{ then } S \subseteq T, \text{ and} \tag{19}$$
$$\mu_S^{i,out} > 0, \mu_T^{i,in} > 0 \text{ implies } S \cap T = \emptyset. \tag{20}$$

If G is symmetric and we choose $\mathcal{S} = \mathcal{D}(G)$, then this specializes to $2\mathrm{M}(G)$ where $\mathrm{M}(G)$ is the *matching polyhedron* defined by

$$y \in \mathbb{R}^E,\ y \geq 0$$
$$y(d(v)) \leq 1 \quad \text{for every } v \in V$$
$$y(i(S)) \leq \frac{|S| - 1}{2} \quad \text{for every } S \subseteq V,\ |S| \text{ odd.}$$

This system was proved to be a TDI system by Cunningham and Marsh [5]. Another well-known special case is when the strongly connected components are all bipartite, i.e. $\mathcal{B} = \mathcal{D}(G)$ is a valid choice. In this case

$$y \in \mathbb{R}^E,\ y \geq 0$$
$$y(\delta^{in}(v)) \leq 1 \quad \text{for every } v \in V$$
$$y(\delta^{out}(v)) \leq 1 \quad \text{for every } v \in V$$

is obtained, which is TDI by *total unimodularity.* Also, we can see easily that any integer solution of this system is a vertex disjoint union of dipaths and even dicycles.

A more general special case is the *path-matching polyhedron* which was introduced by Cunningham and Geelen. For a more detailed description, the reader is referred to the paper of Cunningham and Geelen [4] and to the paper of Frank and Szegő [7]. In short, we have a digraph $G = (V, A)$ and a partition of V into sets T_1, R and T_2, such that T_1 and T_2 are stable sets, $G[R]$ is symmetric, no edge enters T_1, and no edge leaves T_2. Let $\mathcal{S} = \mathcal{D}(G)$. Then the system (12)-(13)-(14)-(15)-(16) defined on this graph specializes to the path-matching polyhedron

$$
\begin{aligned}
y \in \mathbb{R}^E,\ y &\geq 0 \\
y(d(v)) &\leq 2 \quad \text{for every } v \in R \\
y(i(S)) &\leq |S| - 1 \quad \text{for every } S \subseteq R,\ |S| \text{ odd} \\
y(i(S)) + y(\delta^{in}(S)) &\leq |S| \quad \text{for every } S \subseteq R \\
y(i(S)) + y(\delta^{out}(S)) &\leq |S| \quad \text{for every } S \subseteq R \\
y(\delta^{in}(v)) &\leq 1 \quad \text{for every } v \in T_2 \\
y(\delta^{out}(v)) &\leq 1 \quad \text{for every } v \in T_1
\end{aligned}
$$

which was proved to be TDI by Cunningham and Geelen [4].

The rest of the paper is organized as follows. In the next section, we derive the unweighted min-max formula of Pap and Szegő [10], while in the subsequent sections, the detailed proofs of Theorems 1.1 and 4.1 are presented.

5 Unweighted min-max formula

In [3], Cunningham and Geelen derived weighted results for weakly symmetric pairs using their unweighted min-max formula and a primal-dual method. Here we follow an opposite direction and prove the unweighted formula of Pap and Szegő [10] as a consequence of Theorem 1.1. First, to better understand the analogy with older results, let us recall a non-standard form of the well-known Berge-Tutte formula. If $G = (V, E)$ is an undirected graph then $\text{odd}(G)$ denotes the number of connected components of G having an odd number of vertices, and $N_G(X) = \{v \in V - X : \exists u \in X, \{u, v\} \in E\}$.

Theorem 5.1 (Berge and Tutte). *If $G = (V, E)$ is an undirected graph, then the cardinality of a maximum matching of G is*

$$\min_{X \subseteq V} \frac{|V| + |N_G(X)| - \text{odd}(G[X])}{2}.$$

In a digraph $G = (V, A)$, we define $N_G^{out}(X) = \{v \in V - X : \exists u \in X, uv \in A\}$ and let $\text{odd}(G)$ denote the *number of strongly connected components of G with no entering arc (i.e. source components) that have an odd number of vertices.* Using this notation, Pap and Szegő [10] proved the following.

Theorem 5.2 (Pap and Szegő). *If $G = (V, A)$ is a hardly symmetric digraph, then the maximum cardinality of an even factor is*

$$\min_{X \subseteq V} |V| + |N_G^{out}(X)| - \text{odd}(G[X]).$$

Proof. First we prove max $\leq$ min. For a fixed X, any even factor of G has at most $|X| - \text{odd}(G[X])$ edges spanned by X, at most $|N_G^{out}(X)|$ edges leaving X, and at most $|V - X|$ edges having tail in $V - X$. Thus an even factor of G has cardinality at most $|V| + |N_G^{out}(X)| - \text{odd}(G[X])$.

To see max $\geq$ min, we consider integer optimal primal and dual solutions of $\max \sum_{e \in A} x(e)$ subject to (2)-(3)-(4)-(5). We assume that for the dual solution $(\lambda_v^{in}, \lambda_v^{out}, z_S)$, $\mathcal{F} = \{S : z_S > 0\}$ is a laminar family, and $G[S]$ is strongly connected and symmetric for each $S \in \mathcal{F}$. Such a dual solution exists by Theorem 1.1. The primal solution is an even factor, while the dual solution can be modified to prove our formula. Clearly, the dual solution is 0-1 valued, hence $\mathcal{F}$ forms a subpartition of V. Let us choose this dual solution so that $|\{v \in \cup\mathcal{F} : \lambda_v^{in} + \lambda_v^{out} > 0\}|$ is as small as possible.

Claim 5.3. *If $v \in \cup\mathcal{F}$, then $\lambda_v^{in} + \lambda_v^{out} = 0$.*

Proof. If $z_S = 1$ and $\lambda_v^{in} = 1$ for some $v \in S$, then we could change the dual variables to $z_S = 0$ and $\lambda_u^{in} = 1$ for every $u \in S$, while the other variables remain unchanged, which contradicts our extremal choice. □

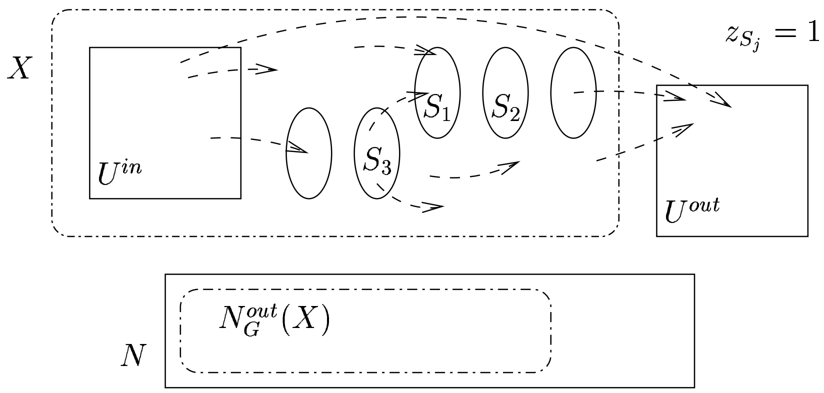

Figure 1: The edges indicated by *dashed* line *cannot* occur.

From the dual solution $(\lambda_v^{in}, \lambda_v^{out}, z_S)$ we can define the set X for which max $\geq$ min. Let $U^{in} = \{v : \lambda_v^{in} = 1, \lambda_v^{out} = 0\}$, $U^{out} = \{v : \lambda_v^{out} = 1, \lambda_v^{in} = 0\}$, $N = \{v : \lambda_v^{in} = \lambda_v^{out} = 1\}$ and $X = V - N - U^{out}$. Then $G[X - U^{in}]$ is composed of symmetric odd components, which are source components in $G[X]$. Clearly, $N_G^{out}(X) \subseteq N$. Therefore the cardinality of a maximum even factor is at least

$$2|N| + \sum_{z_S=1} (|S| - 1) + |U^{out}| + |U^{in}| \geq$$
$$|N| + |V| - \text{odd}(G[X]) \geq |V| + |N_G^{out}(X)| - \text{odd}(G[X]).$$

□

6 Structure of hardly symmetric weights

Let $G' = (V', E')$ be a digraph. An *ear* of G' is a dicycle or a dipath (with different ends) of G', while a *proper ear* of G' is a dipath (with different ends) of G'. The sequence $C_0, P_1, P_2, \dots, P_k$ is an *ear-decomposition* of G' if the following hold:

- C_0 is a dicycle called *initial dicycle*;
- every P_i is an ear;
- $C_0 \cup P_1 \cup \dots \cup P_{i-1}$ has exactly two common vertices with P_i, namely the ends of P_i, if P_i is a dipath, and has exactly one common vertex with P_i, if P_i is a dicycle;
- $C_0 \cup P_1 \cup \dots \cup P_k = G'$.

Similarly, the sequence $C_0, P_1, P_2, \dots, P_k$ is a *proper ear-decomposition* of G' if C_0 is a dicycle of length at least 2, called *initial dicycle*; every P_i is a proper ear; $C_0 \cup P_1 \cup \dots \cup P_{i-1}$ has exactly two common vertices with P_i, namely the ends of P_i; and $C_0 \cup P_1 \cup \dots \cup P_k = G'$.

It is well-known that every strongly connected digraph G' has an ear-decomposition, moreover, every dicycle of G' can be the initial dicycle of this ear-decomposition. Similarly, a strongly connected digraph G' which is 2-vertex-connected in the undirected sense has a proper ear-decomposition, and every dicycle of G' of length at least 2 can be the initial dicycle of a proper ear-decomposition.

Proof of Lemma 2.1. Let $G = (V, A)$ be a hardly symmetric digraph with $\mathcal{D}(G) = \{V\}$. If G is bipartite, then we are done. Thus, G contains a (not necessarily directed) closed walk W having and odd number of edges (with multiplicity). W may have forward and backward edges; consider such a W containing minimum number of backward edges.

Our aim is to find a directed closed walk having an odd number of edges. If W has no backward edge, then we are done. If W has a backward edge uv, then by strong connectivity there is a dipath P from v to u. If P has an even number of edges, then P together with uv is an odd dicycle. Otherwise, uv can be replaced by P in W, and the number of backward edges decreases, contradicting the assumption. Thus there is a directed closed walk having an odd number of edges, which of course contains an odd dicycle C.

Claim 6.1. *There is a sequence of symmetric digraphs $G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_l = G$ s.t. G_0 is a symmetric odd cycle, and for any two vertices s and t of G_i, G_i contains both odd and even length dipaths from s to t.*

Proof. We consider a proper ear-decomposition of G with initial cycle C. Then C is odd, hence symmetric, and clearly there are both odd and even dipaths between any two vertices of C. Suppose, by induction, that a subgraph $G_i \subseteq G$ is built up by the proper ear-decomposition, G_i is symmetric, and there exist both odd and even dipaths in G_i between any two vertices of G_i. Then the edges of the next proper ear

Q are contained in an odd dicycle, hence they are symmetric. It is easy to see that there are both odd and even dipaths between any two vertices of Q. □

The claim completes the proof. □

Proof of Lemma 2.2. We can assume that G is itself a symmetric non-bipartite block. Our goal is to prove that $c(C) = c(\overline{C})$ for every even cycle C of length at least 4 ($c(C) = c(\overline{C})$ automatically holds for a cycle of length 2). G has a proper ear decomposition $C, P_1, P_2, \ldots, P_i$ with initial cycle C. Since G is non-bipartite, there is a first ear, say P_j, whose addition violates the bipartite property. By 2-vertex-connectivity, $C \cup P_1 \cup P_2 \cup \cdots \cup P_{j-1}$ contains vertex disjoint (all the inner and end vertices are different) paths (in the undirected sense) from the endpoints of P_j to C; let these be Q_1 and Q_2. Now $C \cup Q_1 \cup Q_2 \cup P_j$ is non-bipartite, and it has two odd cycles containing P_j. Using the symmetry of G and the property that on odd dicycles the weight-sum is the same in the two directions, this yields that $c(C) = c(\overline{C})$. □

Proof of Lemma 2.3. Clearly, $\chi_{\delta^{in}(v)}$, $\chi_{\delta^{out}(v)}$ and $\chi_{\{e,\overline{e}\}}$ are hardly symmetric weight functions. Fix a root $s \in V$. If P and Q are dipaths from s to t, then $c(P) - c(\overline{P}) = c(Q) - c(\overline{Q})$ by Lemma 2.2, where $\overline{P}$ is the dipath obtained by reversing the direction of the edges of P. Thus we can define $y : V \to \mathbb{R}$ with $y(t) = c(P) - c(\overline{P})$ where P is any $s - t$ dipath. Moreover, it is easy to see that $b = c - \sum_{v \in V} y(v)\chi_{\delta^{in}(v)}$ is a symmetric weight function i.e. $b(e) = b(\overline{e})$. Using y and b we can obtain the desired decomposition of c. □

7 Proof of Theorem 1.1

Proof of Theorem 1.1. First we prove dual integrality. Let $\lambda_v^{in}, \lambda_v^{out} \geq 0$, $z_S \geq 0$ be a rational optimal dual solution (for $v \in V$ and $S \subseteq U$, $U \in \mathcal{S}_*(G)$, $|S|$ odd). We choose a positive integer k (which is fixed throughout the proof) s.t. $\tilde{\lambda}_v^{in} = k\lambda_v^{in}$, $\tilde{\lambda}_v^{out} = k\lambda_v^{out}$, $\tilde{z}_S = kz_S$ are all integers. For fixed k, let us choose moreover this solution so that the vector $\left(K = \sum_{v \in V} \tilde{\lambda}_v^{in} + \sum_{v \in V} \tilde{\lambda}_v^{out}, L = \sum_S \tilde{z}_S |S|^2\right)$ is lexicographically as large as possible. For optimal solutions and a fixed k, K and L are bounded from above. We show that under these conditions, the dual solution is almost integer and it can be transformed easily into an integer optimal one. The steps of the proof are motivated by the work of Balas and Pulleyblank [1].

Claim 7.1. *$\mathcal{F} = \{S : z_S > 0\}$ is a laminar family.*

Proof. If $\mathcal{F}$ was not laminar, then the following simple uncrossing technique could be applied. Suppose that $z_S, z_T > 0$, and none of $S \cap T$, $S - T$, $T - S$ is empty. If $|S \cap T|$ is odd, then we can decrease z_S and z_T, and increase $z_{S \cap T}$ and $z_{S \cup T}$ by $\frac{1}{k}$, in which case K does not change and L increases, which is a contradiction. If $|S \cap T|$ is even, then we can decrease z_S and z_T by $\frac{1}{k}$, increase z_{S-T} and z_{T-S} by $\frac{1}{k}$ and increase λ_v^{in} and λ_v^{out} by $\frac{1}{k}$ if $v \in S \cap T$, so that K increases, which is a contradiction. □

For a graph G and a laminar family $\mathcal{F}$, let $G \times \mathcal{F}$ denote the graph obtained from G by *shrinking the maximal members* of $\mathcal{F}$. For any $S \subseteq V$ s.t. $\{S\} \cup \mathcal{F}$ is again a laminar family, we let $\mathcal{F}[S] = \{U \in \mathcal{F} : U \subsetneq S\}$. Thus $G[S] \times \mathcal{F}[S]$ is the graph obtained from $G[S]$ by shrinking the maximal elements of $\mathcal{F}$ properly contained in S. Let $A^=$ denote the set of *tight* edges, i.e. $A^= = \{uv \in A : \lambda_u^{out} + \lambda_v^{in} + \sum_{u,v \in S} z_S = c(uv)\}$ and let $G^= = (V, A^=)$.

Claim 7.2. *For every $S \in \mathcal{F}$, $G^=[S] \times \mathcal{F}[S]$ is strongly connected. As a consequence, $G^=[S]$ is strongly connected for any $S \in \mathcal{F}$.*

Proof. If $G^=[S] \times \mathcal{F}[S]$ is not strongly connected, then it has a strongly connected component U with an odd number of vertices, moreover, S has a partition into (possibly empty) sets U^{out}, U and U^{in}, s.t. any strongly connected component of $G^=[S] \times \mathcal{F}[S]$ is contained in some of the partition classes, and if $uv \in A^=$ connects two different classes, then one of the following three possibilities holds: $u \in U$ and $v \in U^{in}$; or $u \in U^{out}$ and $v \in U$; or $u \in U^{out}$ and $v \in U^{in}$. Now we can modify the dual solution in the following way:

$$z'_W = \begin{cases} z_W - \frac{1}{k} & \text{if } W = S \\ z_W + \frac{1}{k} & \text{if } W = U \\ z_W & \text{otherwise} \end{cases}$$

$$(\lambda_v^{in})' = \begin{cases} \lambda_v^{in} + \frac{1}{k} & \text{if } v \in U^{in} \\ \lambda_v^{in} & \text{otherwise} \end{cases}$$

$$(\lambda_v^{out})' = \begin{cases} \lambda_v^{out} + \frac{1}{k} & \text{if } v \in U^{out} \\ \lambda_v^{out} & \text{otherwise.} \end{cases}$$

This step yields a new dual solution with bigger K which is a contradiction. □

Claim 7.3. *For any $S \in \mathcal{F}$, $G^=[S] \times \mathcal{F}[S]$ is non-bipartite.*

Proof. For contradiction, we choose an S for which $G^=[S] \times \mathcal{F}[S]$ is bipartite. Thus S has a bipartition $\{S_1, S_2\}$ so that any member of $\mathcal{F}[S]$ is a subset of some S_i, and $G^=[S_i] \times \mathcal{F}_i$ does not contain tight edges, where $\mathcal{F}_i$ is the family of maximal members of $\{U \in \mathcal{F} : U \subseteq S_i\}$. We can assume moreover that $|S_1 - \cup_{S \in \mathcal{F}_1} S| + |\mathcal{F}_1| < |S_2 - \cup_{S \in \mathcal{F}_2} S| + |\mathcal{F}_2|$. We apply the following modification of the dual solution:

$$z'_W = \begin{cases} z_W + \frac{1}{k} & \text{if } W \in \mathcal{F}_2 \\ z_W - \frac{1}{k} & \text{if } W \in \mathcal{F}_1 \text{ or } W = S \\ z_W & \text{otherwise} \end{cases}$$

$$(\lambda_v^{in})' = \begin{cases} \lambda_v^{in} + \frac{1}{k} & \text{if } v \in S_1 \\ \lambda_v^{in} & \text{otherwise} \end{cases}$$

$$(\lambda_v^{out})' = \begin{cases} \lambda_v^{out} + \frac{1}{k} & \text{if } v \in S_1 \\ \lambda_v^{out} & \text{otherwise.} \end{cases}$$

A dual solution is obtained, with larger K, which leads to a contradiction. □

The notation $a \equiv b$ is used if two integers a and b are congruent modulo k (i.e. $a - b$ is divisible by k). For the equivalence class of a under the modulo k equivalence relation we write $\overline{a}$.

Claim 7.4. *For any $S \in \mathcal{F}$, $\tilde{\lambda}_v^{in} \equiv \tilde{\lambda}_u^{in}$ and $\tilde{\lambda}_v^{out} \equiv \tilde{\lambda}_u^{out}$ whenever $u, v \in S$.*

Proof. For contradiction, we can choose a minimal $S \in \mathcal{F}$ for which the statement does not hold. Now $G^{=}[S]$ is symmetric, $G^{=}[S] \times \mathcal{F}[S]$ is strongly connected, symmetric and non-bipartite. So let uv and vu be edges of $G^{=}[S] \times \mathcal{F}[S]$ where $u, v \in S$. We let $\tilde{\lambda}_u^{out} \equiv a$, $\tilde{\lambda}_u^{in} \equiv b$, $\tilde{\lambda}_v^{out} \equiv c$ and $\tilde{\lambda}_v^{in} \equiv d$ (in short, u is of type (a, b) and v is of type (c, d)). Using the statement for the maximal members of $\mathcal{F}[S]$, and the fact that $G^{=}[S] \times \mathcal{F}[S]$ is strongly connected, we get that every vertex of S is either of type (a, b) or of type (c, d), and the vertices in a maximal member of $\mathcal{F}[S]$ are of the same type. In addition, every edge of $G^{=}[S] \times \mathcal{F}[S]$ has end-vertices of both types. But $G^{=}[S] \times \mathcal{F}[S]$ is non-bipartite, hence $a \equiv c$ and $b \equiv d$. □

We let $\mathcal{F}_{\max}$ denote the maximal members of $\mathcal{F}$. As a consequence we get the following.

Claim 7.5. *If $S \in \mathcal{F} - \mathcal{F}_{\max}$, then z_S is integer.*

From now on, the property that (K, L) is lexicographically as large as possible will not be used; the actual solution can be easily transformed to an integer one. To this end, the number of nonzero modulo k remainders of the variables is decreased repeatedly until the desired integer optimal dual solution is obtained.

It can be seen that if λ_v^{out} and λ_v^{in} are integers for every vertex v then we are at an integer solution. Let us consider an integer $a \not\equiv 0$ s.t. $-a \equiv \tilde{\lambda}_v^{out}$ or $a \equiv \tilde{\lambda}_v^{in}$ for some $v \in V$, and define $U^{in} = \{v \in V : \tilde{\lambda}_v^{in} \equiv a\}$, $U^{out} = \{v \in V : \tilde{\lambda}_v^{out} \equiv -a\}$, $Z^{in} = \{S : S \in \mathcal{F}_{\max}, \forall v \in S\ \tilde{\lambda}_v^{in} \equiv a \text{ and } \tilde{\lambda}_v^{out} \not\equiv -a\}$ and $Z^{out} = \{S : S \in \mathcal{F}_{\max}, \forall v \in S\ \tilde{\lambda}_v^{out} \equiv -a \text{ and } \tilde{\lambda}_v^{in} \not\equiv a\}$. By symmetry we can assume that $|U^{in}| - \sum_{S \in Z^{in}}(|S|-1) \geq |U^{out}| - \sum_{S \in Z^{out}}(|S|-1)$. Then we can apply the following change in the dual solution:

$$(\lambda_v^{in})' = \begin{cases} \lambda_v^{in} - \eta & \text{if } v \in U^{in} \\ \lambda_v^{in} & \text{otherwise} \end{cases}$$

$$(\lambda_v^{out})' = \begin{cases} \lambda_v^{out} + \eta & \text{if } v \in U^{out} \\ \lambda_v^{out} & \text{otherwise} \end{cases}$$

$$z'_W = \begin{cases} z_W + \eta & \text{if } S \in Z^{in} \\ z_W & \text{otherwise} \end{cases}$$

$$z'_W = \begin{cases} z_W - \eta & \text{if } S \in Z^{out} \\ z_W & \text{otherwise.} \end{cases}$$

The value η can be chosen to be a positive integer multiple of $\frac{1}{k}$ so that the value $\left|\left\{\overline{\tilde{\lambda}_v^{in}} : v \in V, \tilde{\lambda}_v^{in} \not\equiv 0\right\} \cup \left\{\overline{-\tilde{\lambda}_v^{out}} : v \in V, \tilde{\lambda}_v^{out} \not\equiv 0\right\}\right| + |\{S \in \mathcal{F} : \tilde{z}_S \not\equiv 0\}|$ decreases by at least one. This means that after finitely many applications of this step we obtain an integer optimal dual solution.

Next we prove primal integrality. We apply Lemma 3.1 with C being the set of integer hardly symmetric weight functions. Clearly, this set is additive and contains the vectors $\chi_{\{e\}}$ for $e \notin \cup_{U\in\mathcal{S}(G)} i(U)$ and $\chi_{\{e,\overline{e}\}}$ for $e \in \cup_{U\in\mathcal{S}(G)} i(U)$. Thus there is an optimal solution x s.t. $x(e)$ is integer (0 or 1) if $e \notin \cup_{U\in\mathcal{S}(G)} i(U)$ and $x(e)+x(\overline{e})$ is integer (0, 1 or 2) if $e \in \cup_{U\in\mathcal{S}(G)} i(U)$. An even factor of weight at least cx can easily be constructed from x. □

8 Proof of Theorem 4.1

Proof of Theorem 4.1. Let $G = (V, A)$ be a hardly symmetric digraph (with given $\mathcal{B}$ and $\mathcal{S}$), and c an integral evenly symmetric weight function. The system $\max\{cx : x \in \mathrm{EF}(G)\}$ has an integer dual optimal solution $(\lambda_v^{in}, \lambda_v^{out}, z_S)$ with the properties described in Theorem 1.1. From this, we will obtain a dual optimal solution $(\mu_v^d, \mu_S^i, \mu_S^{i,in}, \mu_S^{i,out})$ of $\{(12)-(13)-(14)-(15)-(16)\}$, satisfying the conditions of Theorem 4.1. During the construction, we consider the union of the two systems, and we start with $\mu_v^d = \mu_S^i = \mu_S^{i,in} = \mu_S^{i,out} = 0$ for every v, S, and end with $\lambda_v^{in} = \lambda_v^{out} = z_S = 0$ for every v, S. The steps of the construction are the following:

1. For $v \notin \cup\mathcal{S}$, let $\mu_v^{i,in} := \lambda_v^{in}$, $\mu_v^{i,out} := \lambda_v^{out}$, and set $\lambda_v^{in} = \lambda_v^{out} := 0$. For every $S \subseteq V$, we let $\mu_S^i := z_S$, and set $z_S := 0$.

2. Let us consider a set $T \in \mathcal{S}_*(G)$, and let $T_1, \ldots, T_k$ denote the strongly connected components of $G^=[T]$. Then $G^=[T_i]$ is symmetric for every i. As a consequence, $\lambda_v^{in} - \lambda_v^{out}$ is constant inside a set T_i. We also know that if $S \in \mathcal{F}$ and $S \cap T \neq \emptyset$, then S is subset of some T_i.

3. For every $v \in T$, let $\mu_v^d := \min\{\lambda_v^{in}, \lambda_v^{out}\}$, and decrease λ_v^{in} and λ_v^{out} by μ_v^d. Thus for every T_i either $\lambda_v^{in} = 0$ and λ_v^{out} is constant for every $v \in T_i$, or vice versa.

4. We can assume that $\lambda_v^{in} > 0$ for some $v \in T$. Let T_j be the strongly connected component on which λ_v^{in} is maximal. We increase $\mu_{T_j}^{i,in}$ by one, and decrease λ_v^{in} by one for every $v \in T_j$. There is no edge $uv \in G^=[T]$ which enters T_j, since then vu would also be in $G^=[T]$ by the maximality of λ_v^{in}, which contradicts the fact that T_j is a strongly connected component. Thus the obtained solution is feasible, and it is easy to see that it is also dual optimal. In addition, no edge disappears from $G^=$.

5. We repeat the above steps 2.–4. with the new $G^=$, until we obtain $\lambda_v^{in} = \lambda_v^{out} = 0$ for every v. The laminarity properties described in Theorem 4.1 are ensured by the fact that edges are never removed from $G^=$, so the strongly connected components can only grow.

□

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