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The Path-packing Structure of Graphs

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Abstract

We prove Edmonds-Gallai type structure theorems for Mader's edge- and vertex-disjoint paths including also capacitated variants, and state a conjecture generalizing Mader's minimax theorems on path packings and Cunningham and Geelen's path-matching theorem.

1 Introduction

Let $G = (V, E)$ be a graph, $\mathcal{T} = \{T_1, \dots, T_k\}$ a family of pairwise disjoint subsets of V, $T := T_1 \cup \dots \cup T_k$. The points in T are called *terminal* points. A $\mathcal{T}$-path is a path between two terminal points in different members of $\mathcal{T}$. Let $\mu = \mu(G, \mathcal{T})$ denote the maximum number of (fully vertex-) disjoint $\mathcal{T}$-paths.

A T-path is a path in G with two different endpoints in T and all other points in $V \setminus T$. In the edge-disjoint case we will consider T-paths, and $\mathcal{T}$ will not be defined. Clearly, a given set of paths in G is a set of pairwise edge-disjoint T-paths if and only if the corresponding paths of the line-graph of G are pairwise vertex-disjoint $\mathcal{T}$-paths, where $\mathcal{T} = \{T_x : x \in T\}$, $T_x := \{v_e : e \in \delta(x)\}$ $(x \in T)$, and v_e denotes the vertex of the line-graph corresponding to the edge e, and $\delta(x)$ denotes the set of edges incident to x, and in general, $\delta(X) = \delta_G(X)$ $(X \subseteq V(G))$ is the set of edges with exactly one endpoint in X, $d(X) := d_G(X) = |\delta_G(X)|$. The maximum number of edge-disjoint T-paths of G will be denoted by $\mu^*(G, T)$.

Furthermore, we will use the following notations. In a graph $G = (V, E)$, $E[X]$ denotes the set of edges spanned by X for a subset $X \subseteq V$, and $G(X)$ the subgraph induced by X; $G - X$ denotes the graph obtained from G by deleting the vertices of $X \subseteq V$. For a subset $F \subseteq E$ of edges, $G - F$ denotes the graph obtained from G by deleting the edges in F.

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The main result of this paper is an Edmonds-Gallai type structure theorem for maximum sets of edge-disjoint T-paths and vertex-disjoint $\mathcal{T}$-paths. What does this mean?

A simplest example is accessibility in directed graphs: given a directed graph $G = (V, A)$ and $x_0 \in V$, find a directed path from x_0 to the other vertices of G or provide a certificate that there is none.

Everybody knows that there is no path between x_0 and $x \in V$ if and only if there is a *directed cut separating* x_0 *and* x, that is, a set $X \subseteq V$, $x_0 \in X$, $x \notin X$ so that no arc of G is leaving X. The X will be called a *directed cut.* Let us point out the following phenomenon.

There exists a unique set X_0 *which is a directed cut separating* x_0 *at the same time from every* $x \in V$ *for which there is no* (x_0, x)*-path.*

Indeed, there exists an arborescence with root x_0 and with a path from x_0 to all the vertices accessible from x_0, so that the set X_0 of vertices of this arborescence is a directed cut. This problem is a special case of the more general problem of shortest paths from a fixed vertex x_0 of a weighted directed graph without negative directed circuits, and of the uniquely determined potentials that certify the length of these paths for every vertex.

In order to help the reading we provide a brief introduction to the Edmonds-Gallai theorem. Denote by $\bar{G}$ the graph which arises from G by adding a vertex x_0 and joining x_0 to all the vertices of G.

If M is a maximum matching of G, let $\bar{M}$ arise from M by adding the edges x_0x where x is a vertex not saturated by M. We will call *alternating path* (with respect to $\bar{M}$) a path joining x_0 to a vertex of G starting with an edge in $\bar{M}$, and containing alternatively edges of $\bar{M}$ and not in $\bar{M}$; if the last edge of the path is in $\bar{M}$ we will say the path is *symmetric*, if not, then we will say it is *asymmetric.*

A set $X \subseteq V(G)$ is called a *Tutte-set*, if there exists a maximum matching which covers X and all components of $G-X$ are fully matched within the component except for at most one vertex. (This means that the minimum attains at X in the Tutte-Berge formula about the maximum size of a matching.) If X is a Tutte-set of G, we will denote by $C(X)$ the union of the even (vertex-cardinality) components of $G-X$, and by $D(X)$ the union of the odd components of $G - X$.

Define now $D(G)$ to be the set of vertices which are endpoints of symmetric paths, and $A(G)$ the set of vertices which are not, but are the endpoints of asymmetric paths. Clearly, $D(G) = \{v \in V(G) : \mu(G - v) = \mu(G)\}$, $A(G)$ is the set of neighbors of vertices in $D(G)$, and $C(G)$ is the rest of the vertices.

Exclusion Lemma for matchings: *If* G *is a graph,* M *is a maximum matching and* X *is a Tutte-set, then*

(i) there is no alternating path to the vertices of $C(X)$*, that is,* $C(X) \subseteq C(G)$*,*

(ii) there is no symmetric alternating path to the vertices of X*, that is,* $X \subseteq V(G) \setminus D(G)$*.*

The essence of the Edmonds-Gallai theorem is that there is a unique 'maximal' Tutte-set in the following sense (compare the result with the containments of the lemma):

Theorem 1.1. *For any graph G, $X := A(G)$ is a Tutte-set, and $C(X) = C(G)$, $D(X) = D(G)$.*

When the Edmonds-Gallai theorem is stated in articles or books, often many other (in our view less essential) statements are also added. Those can be useful in the context of the various goals, but are merely easy corollaries of the above statement which provides the main content, which is: there is a most exclusive Tutte-set that excludes everything that can be excluded with the Exclusion Lemma.

Let us mention some uses of the Edmonds-Gallai theorem – for the generalizations not many of these have been worked out: Linear hull of matchings, matching lattice; Cunningham's analysis of the 'slither' game; Cornuéjols, Hartvigsen and Pulleyblank's solution of the k-gon-free 2-matching problem [1], or of the problem of packing complete graphs (equivalently edges and triangles), etc. In Lovász and Plummer's book [11] the theorem is applied as a basic and standard tool throughout. It is useful in cases where the *set* of maximum matchings is important.

Let us list some other examples of the same kind of theorem:

A generalization of the Edmonds-Gallai theorem for (f, g)-factors (subgraphs with upper and lower bounds and maybe parity constraints) has been proved by Lovász [9], [10]. The relation of this generalization to the corresponding minimax theorem (of Tutte for f-factors) is the same as the relation of the Berge-Tutte theorem for matchings and of the Edmonds-Gallai theorem. Similarly, the minimax theorems for the Chinese-Postman problem (T-joins) can be sharpened to provide an Edmonds-Gallai type theorem for T-joins [17], which can be further applied to disjoint path and cut packing problems or results on the duality gap or stronger integrality properties in these problems. Recently, Spille and Szegő [19] have developed the corresponding sharpening of the minimax theorems on path-matchings.

The analogous results concerning Mader's path-packings will be worked out in this paper. The main result concerns vertex-disjoint paths (Sect. 4). The result about edge-disjoint paths is a consequence. However, because of its simplicity and for a more gradual understanding of this structure, we will first give a different – much simpler – proof for the edge-disjoint case (Sect. 3).

These structure theorems are usually consequences of the algorithms that solve the problems. There are three problems though for which this use of structure theorems has been reversed. This means that the assertion of the theorems is mechanically converted into an optimality criterion. If the 'canonical dual solution' defined from a list of tentative optimal solutions does not satisfy the criterion, then one of the two following cases can happen: either a new 'optimal solution' can be added to the list, improving the 'canonical dual solution'; or a better solution can be found, showing that the solutions of the list were not optimal. Such a proof does in fact not use the structure theorem but states it as a conjecture, and provides a new, algorithmic proof of it. We refer to such an algorithm under the name *structure algorithm.*

A first structure algorithm has been provided by Lovász [11] for matchings. For (f, g)-factors Lovász worked out the Edmonds-Gallai theorems [11]. The corresponding structure algorithm has been worked out in [16] in the language of accessibility from a fix vertex in a bidirected graph. This algorithm contains structure algorithms for (f, g)-factors and other factorizations or orientation problems with upper, lower bound and parity constraints. The third generalization of Lovász's algorithm concerns minimum T-joins in graphs [15].

For Mader's path packings, or path-matchings not only the structure algorithms are missing, but there is no combinatorial algorithm solving these problems at all! The present paper is intended to be a first step in this direction by providing the statement of the structure theorem. The algorithms and the applications of the developed results are yet to be found.

In this paper a path P is considered to be a set of vertices $P \subseteq V(G)$. We will also speak about the edges of P, and we will denote them by $E(P)$. If $\mathcal{P}$ is a family of paths, $E(\mathcal{P}) = \cup_{P\in\mathcal{P}} E(P)$. Therefore 'disjoint' paths mean pairwise vertex-disjoint paths, unless it is specified that the paths are 'edge-disjoint' only.

A *half-T-path* (or *half path*) is a path with at least one endpoint in T. The other endpoint may or may not be in T, and the path may also consist of only one point in T. However, if the path has more than one vertex, the two endpoints must be different.

Both in the edge- and vertex-disjoint case the main property for classifying vertices for the path structure is the following: given a vertex of the graph is it possible to add a half path joining a terminal with the given vertex, so that deleting all edges, resp. vertices of that path, the maximum number of $\mathcal{T}$-paths does not change?

We will say that an edge or vertex is *covered* by a given set $\mathcal{P}$ of paths if it is contained in a path of $\mathcal{P}$. Otherwise it is *uncovered*.

We finish this introduction by stating Mader's theorems.

First we consider the edge-disjoint case. Let $T := \{t_1, t_2, \ldots t_k\}$. A subpartition $\mathcal{A} := \{A_1, A_2, \ldots A_k\}$ of V is said to be a T-partition if $t_i \in A_i$, $(i = 1, \ldots, k)$. For $X \subseteq V(G)$, a component K of $G - X$ is said to be *odd* (*even*) if $d_G(K)$ is an odd (even) number.

Theorem 1.2 (Mader [12]). *The maximum number of edge-disjoint T-paths equals to the minimum value of*

$$|A_0| + \sum_{C\in\mathcal{C}} \left\lfloor \frac{1}{2} d_G(C) \right\rfloor \quad , \tag{1}$$

taken over all T-partitions $\mathcal{A} := \{A_1, A_2, \ldots A_k\}$, where A_0 is the set of edges whose two endpoints are in A_i and A_j with $i \neq j$; $A := \bigcup_{i=1}^{k} A_i$, and the elements of $\mathcal{C}$ are the vertex-sets of the components of the graph $G - A$.

A T-partition $\mathcal{A} := \{A_1, A_2, \ldots A_k\}$ is called *optimal* if

$$\mathrm{val}^*_{G,T}(\mathcal{A}) := |A_0| + \sum_{C\in\mathcal{C}} \left\lfloor \frac{1}{2} d_G(C) \right\rfloor$$

is minimum among all T-partitions, that is, if it is equal to $\mu^*(G, T)$.

Second we consider the vertex-disjoint case. For a surprisingly short (non-algorithmic) proof, see Schrijver [14].

Theorem 1.3 (Mader [13]). *The maximum number of disjoint $\mathcal{T}$-paths is equal to the minimum value of*

$$|U_0| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|B_i| \right\rfloor \quad , \tag{2}$$

taken over all partitions $U_0, U_1, \ldots, U_n$ of V such that each $\mathcal{T}$-path disjoint from U_0 traverses some edge spanned by some U_i. B_i denotes the set of vertices in U_i that belong to T or have a neighbor in $V - (U_0 \cup U_i)$.

We will use the following reformulation of Theorem 1.3. It has several advantages: First – and this is crucial for us now – it provides the right context for a structure theorem in the sense that the parts of a dual solution correspond exactly to the partition provided by the structure theorem. We also think that this formulation is more natural in view of the edge-version of Mader's theorem (Theorem 1.2), because it has the same form and the latter is sitting in it with a pure specialization. Third, it makes explicit what the sentence 'each $\mathcal{T}$-path disjoint from U_0 traverses some edge spanned by some U_i' means in Theorem 1.3. Indeed, the meaning of this sentence is that the vertices of U_0 and the edges spanned by the U_i $(i = 1, \ldots, k)$ block all the $\mathcal{T}$-paths, that is, deleting these vertices and edges each component of the remaining graph contains vertices from at most one $T_i \in \mathcal{T}$ $(i = 1, \ldots, k)$.

If $X, X_0 \subseteq V$ are disjoint, and $\{X_1, \ldots, X_k\}$ is a partition of X, then $\mathcal{X} := (X; X_0, X_1, X_2, \ldots X_k)$ is said to be a $\mathcal{T}$-partition if $T_i \subseteq X_0 \cup X_i$, $(i = 1, \ldots, k)$. The *value* of this $\mathcal{T}$-partition $\mathcal{X}$ is defined to be

$$\mathrm{val}_{G,\mathcal{T}}(\mathcal{X}) := |X_0| + \sum_{C \in \mathcal{C}} \left\lfloor \frac{1}{2}|C \cap X| \right\rfloor \quad , \tag{3}$$

where the elements of $\mathcal{C}$ are the vertex-sets of the components of the graph $G - X_0 - \bigcup_{i=1}^{k} E[X_i]$.

Theorem 1.4.

$$\mu(G, \mathcal{T}) = \min \mathrm{val}_{G,\mathcal{T}}(\mathcal{X}) \quad , \tag{4}$$

where the minimum is taken over all $\mathcal{T}$-partitions $\mathcal{X}$.

It is easy to see that the two forms of Mader's theorem are equivalent: indeed, in the original variant Theorem 1.3, the set U_0 and the edges induced by the sets U_i $(i = 1, \ldots, k)$ block all the $\mathcal{T}$-paths, that is, all the components of the remaining graph contain terminal vertices in at most one of the T_i's. It is straightforward now to define the sets of Theorem 1.4, and similarly, a dual solution of the latter is easy to define in terms of the former.

For a $\mathcal{T}$-partition $\mathcal{X} := (X; X_0, X_1, X_2, \ldots X_k)$, a component K of $G - X_0 - \bigcup_{i=1}^{k} E[X_i]$ is said to be *odd* (*even*) if $|K \cap X|$ is an odd (even) number. Set $K \cap X$ is denoted by $B(K)$ and is called the *border* of K. Denote $C(\mathcal{X})$ the union of the even components, $D(\mathcal{X})$ the union of the odd components. A $\mathcal{T}$-partition $\mathcal{X} := (X; X_0, X_1, X_2, \ldots X_k)$ is called a *Mader-partition* if $\mathrm{val}_{G,\mathcal{T}}(\mathcal{X}) = \mu(G, \mathcal{T})$.

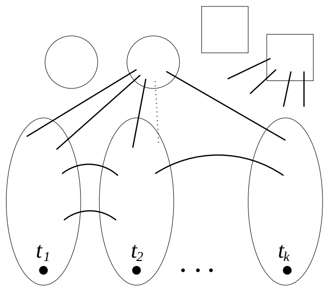

Figure 1: An optimal T-partition

2 What Does an Optimal Partition Exclude?

In every minimax theorem the optimal solutions on the ‘max’ side (primal) and the optimal solutions on the ‘min’ side (dual) have a particular structure; this is a consequence of the trivial $\leq$ inequality between these two, and if the equality holds, then a bunch of equalities are also implied (complementary slackness).

We first wish to exhibit (mainly with the two joined figures) the corresponding combinatorial structure, called ‘complementary slackness’ in linear programming, for Mader’s theorems. Moreover, like in the special cases exhibited in the Introduction for analogy, these conditions do imply some evident exclusions. The main goal of this section is to state these exclusions: like for directed accessibility or matchings the Edmonds-Gallai type structure will be the most exclusive Mader-set.

First consider the edge-disjoint case. Let $\mathcal{P}$ be a family of edge-disjoint T-paths and $\mathcal{A} = \{A_1, \dots, A_k\}$ be a T-partition, and

$$|\mathcal{P}| = |A_0| + \sum_{C \in \mathcal{C}} \left\lfloor \frac{1}{2} d_G(C) \right\rfloor,$$

using the notations of Theorem 1.2.

It can be immediately checked from Mader’s theorem that $P \in \mathcal{P}$ touches only two different A_i-s, and either it goes directly from one to the other through an edge between the two, or it goes to an even or an odd component K of $G - \cup\mathcal{A}$ and has two common edges with the cut $\delta_G(X)$. (See Fig. 1.)

In a maximum family $\mathcal{P}$ of pairwise edge-disjoint T-paths some vertices s might be the endpoints of an additional half T-path P, edge-disjoint from all the paths in the family. A vertex s will be called *i-rooted* ($t_i \in T$), if there is a family $\mathcal{P}$ for which the endpoint of P in T is t_i.

Theorem 2.1 (Exclusion theorem for T-paths). *Let $\mathcal{A} := \{A_1, A_2, \dots A_k\}$ be an optimal T-partition, A_0 the set of edges whose two endpoints are in A_i and A_j respectively, and $i \neq j$; $A := \bigcup_{i=1}^k A_i$, and the elements of $\mathcal{C}$ are the vertex-sets of the components of the graph $G - A$.*

(i) If $v \in C \in \mathcal{C}$, and $d_G(C)$ is even, then v is not i-rooted for any $i = 1, \dots, k$.

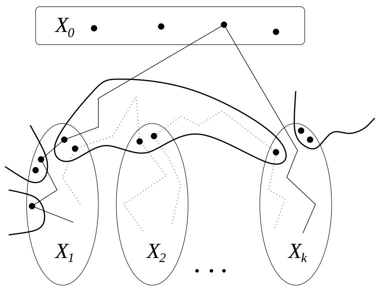

Figure 2: A Mader-partition

(ii) If $v \in A_i$, then v is not j-rooted for $j \in \{1,\dots,k\}$, $j \neq i$.

Proof. Immediately follows from Mader's theorem (Theorem 1.2). □

Next consider the vertex-disjoint case. It is useful to look at Theorem 1.4 for deducing the condition of equality, and then the corresponding exclusions.

Let $\mathcal{P}$ be a maximum family of disjoint $\mathcal{T}$-paths and $\mathcal{X}$ be a $\mathcal{T}$-partition for which equality holds in Theorem 1.4. Then $P \in \mathcal{P}$ either goes through vertices of exactly two different X_i, X_j $(1 \leq i < j \leq k)$ and two vertices of the border of an even or odd component, or contains exactly one vertex of X_0. If P has a vertex v in X_0, then the segment of P between one end-node and v is devided into two parts. The first part (it maybe empty) consists of the segment of P which traverse an odd component K of $\mathcal{X}$ traversing exactly one node of the border of K, then a part which is included in X_i $(i > 0)$ and contains at most one vertex on a border of any odd components and no vertex on a border of any even components.

Vertices s that are not covered by some maximum family of pairwise vertex-disjoint $\mathcal{T}$-paths might be the endpoints of an additional half $\mathcal{T}$-path P vertex-disjoint from all the paths in the family. Denote the endpoint of P in T by t, and let $t \in T_i$ $(i \in \{1,\dots,k\})$. Such a vertex s will be called *i-rooted*. If a vertex is not i-rooted for any $i \in \{1,\dots,k\}$, we will say it is *0-rooted*. If a vertex is not i-rooted, but it is in T_i or has an i-rooted neighbor, we will say it is *i-touched*.

A vertex $t \in T_i$ is i-rooted if and only if it is not covered by a maximum family of vertex disjoint $\mathcal{T}$-paths. If it is covered by every maximum family, then by definition, it is i-touched.

Theorem 2.2 (Exclusion theorem for $\mathcal{T}$-paths). *Let $\mathcal{X} := (X; X_0, X_1, X_2, \dots X_k)$ be a Mader-partition, and $\mathcal{C}$ the set of components with cardinality at least 2 of the graph $G - X_0 - \bigcup_{i=1}^{k} E[X_i]$.*

(i) If $v \in C \in \mathcal{C}$, where C is an even component, or $v \in X_0$, then v is not i-rooted for any $i = 1,\dots,k$.

(ii) If $v \in X_i$ and is not in a border of an odd component, then for $j \in \{1,\dots,k\}$, $j \neq i$, v is neither j-rooted nor j-touched.

(iii) If $v \in X_i$ and is in a border of an even component, then for $j \in \{1,\dots,k\}$, $j \neq i$, v is not j-touched.

(iv) If $v \in X_i$ and is in a border of an odd component, then for $j \in \{1, \ldots, k\}$, $j \neq i$, v is not j-rooted.

Proof. Immediately follows from Mader's theorem (Theorem 1.4). □

3 Edge-disjoint Paths

In this section we prove the structure theorem for maximum sets of edge-disjoint paths.

The terms and notations we introduce here are local in this section. (We will use the same or similar terminology in the rest of the paper, but in a different sense: in the definition of the terms 'edge-disjoint' will be replaced by 'vertex-disjoint'.)

The reader who aims at a quick direct understanding of the most general claims can skip this section: the results we are proving will be particular cases of the theorems on vertex-disjoint paths in the following section. However, it may be helpful to see the occurrence of the main ideas in a particular case.

Let $T = \{t_1, t_2, \ldots, t_k\}$. The set of vertices that are i-rooted and not j-rooted if $j \neq i$ will be denoted by V_i. Let us denote the vertices that are both i-rooted and j-rooted for some $i \neq j$ by V_∞. Vertices that are not i-rooted for any $i \in T$ are called 0-*rooted*. V_0 denotes the union of these vertices. Let $\mathcal{V}$ denote the T-partition $\{V_1, \ldots, V_k\}$.

By using Mader's Theorem 1.2, we will prove the following structure theorem.

Theorem 3.1 (Structure theorem of edge-disjoint paths). *The set $\mathcal{V}$ is an optimal T-partition,*

V_∞ is the union of the odd components of $\mathcal{V}$ and V_0 is the union of the even components of $\mathcal{V}$.

Proof. Let $\mathcal{X} := \{X_1, X_2, \ldots X_k\}$ be an optimal T-partition (i.e., whose value is μ^*) for which the cardinality of the union of the odd components is minimal and among these minimize the cardinality of the union of $\bigcup_{i=1}^k X_i$. By Mader's Theorem 1.2 such a T-partition exists.

We will prove that $\mathcal{V} = \mathcal{X}$. Let $D(\mathcal{X})$ ($C(\mathcal{X})$) denote the union of the odd (resp. even) components of $\mathcal{X}$.

$V - T$ is partitioned by the three sets $C(\mathcal{X}), \cup_{i=1}^k X_i - T, D(\mathcal{X})$, and on the other hand it is also partitioned by the three sets $V_0, \cup_{i=1}^k V_i, V_\infty$. We will prove the following three containment relations

(1)* $C(\mathcal{X}) \subseteq V_0$,

(2)* $X_i \subseteq V_i$ for $1 \leq i \leq k$,

(3)* $D(\mathcal{X}) \subseteq V_\infty$.

It follows that each of the three classes of the first partition is contained in a corresponding class of the second partition, hence equalities follows througout, that is, in (1)*, (2)* and (3)* as well.

$(1)^*$ follows by Mader's theorem immediately. (See Theorem 2.1 (i).)

Now we prove $(2)^*$.

If there are edges between X_i and X_j, then subdivide them with single nodes, so we may assume that there are no edges between X_i and X_j for $1 \leq i < j \leq k$ (by Mader's theorem these vertices of degree 2 are always covered by a maximum set of edge-disjoint T-paths).

Now we define graph $G^* = (V^*, E^*)$. V^* is obtained by shrinking the components of $V - \bigcup_{i=1}^k X_i$ to single nodes and E^* is obtained by deleting the loops after the shrinkings.

Let us define the auxiliary graph $G' = (V', E')$ and capacities on the edges. Let $V' := V^* \cup \{r, s\}$, $E' := E^* \cup \{rt_i : 1 \leq i \leq k\} \cup \{ct : c \text{ is a node of } G^* - \bigcup_{i=1}^k X_i\}$. Let the capacity of the new edges incident to r be infinity, of an edge cs be $2 \cdot \left\lfloor \frac{d_{G^*}(c)}{2} \right\rfloor$, and of the other edges be one.

Let Y be the minimal (r,s)-cut in G' for which the cardinality of Y is minimum. Obviously, it has capacity $2\text{val}^*_{G,T}(\mathcal{X}) = 2\mu^*$.

Claim 3.2. *$X_i \subseteq Y$ for all $1 \leq i \leq k$.*

Proof. Since Y is a minimal minimum cut, it is clear that Y does not contain any node c which was obtained by shrinking an even component, furthermore if node $c \in Y$ and c was obtained by shrinking an odd component, then every edge $e \in E \cap E'$ incident to c has other endnode in $Y \cap \bigcup_{i=1}^k X_i$.

Suppose indirectly that $v \in X_i$ is not in Y. Then let $\mathcal{X}_Y := \{X_i \cap Y : 1 \leq i \leq k\}$. $\mathcal{X}_Y$ is a T-partition and $|\bigcup_{X \in \mathcal{X}_Y} X|$ is strictly smaller than $|\bigcup_{X \in \mathcal{X}} X|$. Let l denote the number of nodes of G' in Y obtained by shrinking an odd component of $\mathcal{X}$. Let $q_{G^*}(\mathcal{X}_Y)$ denote the number of odd components of T-partition $\mathcal{X}_Y$ in G^*.

Now we have

$$2\mu^* = d_{G'}(Y) = \sum_{i=1}^{k} d_{G'}(X_i \cap Y) - l \geq$$

$$\sum_{i=1}^{k} d_{G^*}(X_i \cap Y) - q_{G^*}(\mathcal{X}_Y) = 2\text{val}^*_{G^*,T}(\mathcal{X}_Y) \geq 2\mu^*.$$

Hence equivality holds throughout, in particular, $\text{val}^*_{G^*,T}(\mathcal{X}_Y) = \mu^*$ and furthermore $l = q_{G^*}(\bigcup_{i=1}^k X_i)$, that is, after blowing up the shrinked components, T-partition $\mathcal{X}_Y$ contradicts the choice of $\mathcal{X}$. □

Proof of $(2)^*$. Suppose indirectly that $u \in X_i$ is not in V_i. Let e denote a new edge us. Then the maximum value of an (r,s)-flow in G' does not change, hence there is a minimum cut Z which does not contain u. Since $Y \cap Z$ is also a minimum cut, it contradicts to Claim 3.2. □

Proof of $(3)^*$. We may assume that $X_i = \{t_i\}$ for all i and $G - T$ is a single odd component C.

Let v be a vertex in C. We may suppose that there is a node t_i such that after adding edge $e = vt_i$ to G, $\mu^*_{G+e,T} = \mu^*_{G,T} = \frac{d_G(C)-1}{2}$, that is, by Mader's Theorem 1.2, there is a T-partition $\mathcal{F}$ such that $\text{val}_{G,T}(\mathcal{F}) = \frac{d_G(C)-1}{2} = \frac{d_{G+e}(C)-2}{2}$. Since the union of the odd components of $\mathcal{F}$ is strictly smaller than $|C|$, it contradicts the choice of $\mathcal{X}$. □ □

4 Vertex-disjoint Paths

In this section we state and prove the structure theorem of vertex-disjoint paths. We first complete the (uniquely determined) classification of the vertices – started in Sect. 2 that leads to such a theorem.

In order to avoid a superfluous distinction of cases, it might be useful to note that the above distinction of touched terminal and non-terminal points is not really necessary: for every $t \in T$ introduce two new vertices, t_1, t_2 and join both to t and to nothing else. Replace $t \in T_i$ by $\{t_1, t_2\}$ for all $t \in T_i$, to get T_i', and do this for all $i = 1, \ldots, k$. Now every terminal point is i-rooted for some i (that is, it is in V^+, which will be defined soon), and the status of the points in G did not change. In particular, if $t \in T_i$, then t certainly has the i-rooted neighbors t_1, t_2, so it is at least i-touched, and it is i-rooted if and only if it was before, that is, if and only if t is not covered by every maximum path packing. (The gadget does not do anything else than realize physically what the definition provides anyway.)

Clearly, if $v \in V$ is i-rooted, then $\mu(G, T_1, \ldots, T_j \cup \{v\}, \ldots, T_k) = \mu(G, \mathcal{T}) + 1$ for all $j \neq i$. If $\mathcal{X}$ is an optimal $\mathcal{T}$-partition, and $v \in X_i$, then the statement is reversible, since then v cannot be rooted in any other class but T_i.

Define the following sets of vertices.

- $C^* := \{v \in V : v$ is 0-rooted and is not i-touched for any $i\}$
- $D^* := \{v \in V : v$ is both i-rooted and j-rooted for some $i \neq j\}$
- $V_0 := \{v \in V : v$ is 0-rooted, and is both i-touched and j-touched for some $i \neq j\}$
- $V_i^* := \{v \in V : v$ is i-rooted and neither j-rooted, nor j-touched for any $j \neq i\}$
- $V_i^C := \{v \in V : v$ is 0-rooted, i-touched, and not j-touched for any $j \neq i\}$
- $V_i^D := \{v \in V : v$ is i-rooted, not j-rooted for all $j \neq i$, and j-touched for some $j \neq i\}$
- $V_i := V_i^* \cup V_i^C \cup V_i^D$
  $V^+ := \bigcup_{i=1}^k V_i$, $C := C^* \cup \bigcup_{i=1}^k V_i^C$, $D := D^* \cup \bigcup_{i=1}^k V_i^D$.

It is easy to see that the sets above define a partition of V. We will see that $\mathcal{V} := \mathcal{V}(G, \mathcal{T}) := \{V^+; V_0, V_1, \ldots, V_k\}$ is a $\mathcal{T}$-partition. Theorem 2.2 is the 'trivial' exclusion part of this.

Theorem 4.1 (Structure theorem of vertex-disjoint paths). $\mathcal{V}(\mathcal{T}) = (V^+; V_0, V_1, V_2, \ldots, V_k)$ *is a* $\mathcal{T}$*-partition,* $\mu(G, T) = val_{G,T}(\mathcal{V})$*. Furthermore,*

C is the union of the even components of $G - V_0 - \bigcup_{i=1}^k E[V_i]$*, and*

D is the union of the odd components of $G - V_0 - \bigcup_{i=1}^k E[V_i]$.

Proof. Let $\mathcal{X} := \{X; X_0, X_1, X_2, \dots X_k\}$ be a $\mathcal{T}$-partition of V of value μ for which $X_0 \cup C(\mathcal{X})$ is inclusionwise maximal, and among these, maximize X (again, with respect to inclusion).

By Mader's Theorem 1.3 such a $\mathcal{T}$-partition $\mathcal{X}$ exists. We will prove that $\mathcal{X} = \mathcal{V}(\mathcal{T})$. We keep the notation $\mathcal{V}(\mathcal{T}) = (V^+; V_0, V_1, \dots, V_k)$, $C = C(\mathcal{T})$, $D = D(\mathcal{T})$; in order to avoid confusion look again at the definition of these sets and realize that for the moment $C = C(\mathcal{T})$, $D = D(\mathcal{T})$ do not have anything to do with even or odd components; on the other hand $C(\mathcal{X})$ $(D(\mathcal{X}))$ is the union of the even (odd) components of the graph $G - X_0 - \bigcup_{i=1}^{k} E[X_i]$.

We proceed by induction on $|V(G) \setminus T|$. If this is 0, then the theorem simplifies to the original Edmonds-Gallai structure theorem.

The proof will realize a project based on the following facts generalizing the edge-disjoint case: $V(G)$ is partitioned by the three sets $X_0 \cup C(\mathcal{X})$, $X \setminus C(\mathcal{X})$, and $D(\mathcal{X}) \setminus X$; on the other hand, $V(G)$ is also partitioned by the three sets $V_0 \cup C$, $V^+ \setminus C$, and D^*. We will prove the following three containment relations

(1) $X_0 \cup C(\mathcal{X}) \subseteq V_0 \cup C$,

(2) $X_i \setminus C(\mathcal{X}) \subseteq V_i \setminus V_i^C$ for all $i = 1, \dots, k$,

(3) $D(\mathcal{X}) \setminus X \subseteq D^*$.

It follows that each of the three classes of the first partition is contained in a corresponding class of the second partition. The equalities follow throughout, that is, in (1), (2) and (3) as well. To prove the theorem we only have to prove $X_0 = V_0$ in addition to these equalities, and again, because of the equality in (1), this is equivalent to $X_0 \subseteq V_0$ and $C(\mathcal{X}) \subseteq C$. Let us prove these first, they are relatively simpler:

We prove $X_0 \subseteq V_0$, admitting (1), (2), (3). Let $\mathcal{P}$ be an optimal path packing and $x \in X_0$. By complementary slackness (Fig. 2) x is contained in exactly one $P \in \mathcal{P}$, and P is vertex-disjoint from $C(\mathcal{X})$ and of $X_0 \setminus \{x\}$. Hence either one of the neighbors of x on P is in $D(\mathcal{X}) \setminus X$ – and by (3) in D^*–, or the two neighbors of x on P, denote them by a and b, are in $X_i \setminus C(\mathcal{X})$ and $X_j \setminus C(\mathcal{X})$ respectively, where $i \neq j$ (Fig. 2). Now by (2) a is then i-rooted, and b is j-rooted. Thus x is 0-rooted, i-touched and j-touched, that is, $x \in V_0$, as claimed.

We prove now $C(\mathcal{X}) \subseteq C$. Since $C(\mathcal{X}) \setminus X \subseteq C^*$ is obvious, all we have to prove is $C(\mathcal{X}) \cap X_i \subseteq V_i^C$ $(i = 1, \dots, k)$. Fix $i \in \{1, \dots, k\}$, and let $v \in C(\mathcal{X}) \cap X_i$. By Theorem 2.2 (i) v is 0-rooted, and by (iii) it is neither j-rooted nor j-touched for $j \neq i$, proving the assertion.

Since $v \in X_0 \cup C(\mathcal{X})$ implies that v is 0-rooted, (1) is trivial.

We are remained with (2) and (3).

Let us prove now the most difficult part, (2). Let $v \in X_i \setminus C(\mathcal{X})$; then v is not j-rooted for any $j \neq i$ (since Theorem 2.2 (ii) and (iv)), whence the containment we want to prove is equivalent to the following:

Claim: Let $\mathcal{T}_v := (T_1, \dots, T_k, \{v\})$. Then $\mu(G, \mathcal{T}_v) = \mu(G, \mathcal{T}) + 1$.

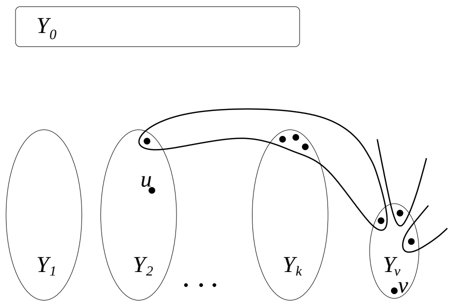

Figure 3: Mader-partition if $Y_v \neq \emptyset$

Suppose for a contradiction $\mu(G, \mathcal{T}_v) = \mu(G, \mathcal{T})$. We know by induction that $\mathcal{V}(\mathcal{T}_v) =: (Y; Y_0, Y_1, \dots, Y_k, Y_v)$ is an optimal $\mathcal{T}_v$-partition. We show that

$$X_0 \cup C(\mathcal{X}) \subseteq Y_0 \cup C(\mathcal{T}_v) \cup U, \tag{4}$$

where U is the union of Y_v and of those components of $D(\mathcal{T}_v) = G - Y_0 - (E[Y_v] \cup \cup_{i=1}^{k} E[Y_i])$ that have a common vertex with Y_v.

The Claim will then be proved, since the optimal $\mathcal{T}$-partition $\mathcal{Y}$ defined from $\mathcal{V}(\mathcal{T}_v)$ by deleting Y_v from the classes (and replacing Y by $Y \setminus Y_v$) has

$$X_0 \cup C(\mathcal{X}) \cup U \subseteq Y_0 \cup C(\mathcal{Y}).$$

Since $v \notin X_0 \cup C(\mathcal{X})$ and $v \in Y_0 \cup C(\mathcal{Y})$, $\mathcal{Y}$ contradicts the maximal choice of $X_0 \cup C(\mathcal{X})$ if it is an optimal partition which we now prove.

If each optimal path packing in $(G, \mathcal{T}_v)$ covers v, then $v \in Y_0 \cup C(\mathcal{T}_v)$ (and $U = \emptyset$). Since $\mathrm{val}_{G,\mathcal{T}}(\mathcal{Y}) = \mu$, $\mathcal{Y}$ contradicts the maximal choice of $X_0 \cup C(\mathcal{X})$.

If there is an optimal path packing in $(G, \mathcal{T}_v)$ which does not cover v, then $v \in Y_v$. Hence every component K of $G - Y_0 - (E[Y_v] \bigcup \cup_{i=1}^{k} E[Y_i])$ having a nonempty intersection with Y_v is in $D(\mathcal{Y})$, and $|K \cap Y_v| = 1$. Hence $U \in C(\mathcal{Y})$, $\mathrm{val}_{G,\mathcal{T}}(\mathcal{Y}) = \mu$, and $\mathcal{Y}$ contradicts the maximal choice of $X_0 \cup C(\mathcal{X})$.

In order to prove the claim it is sufficient now to prove (4). We prove it by showing that the complement of the left hand side with respect to $V(G)$ contains the complement of the right hand side, that is, the following suffices:

Supposing that $u \notin U$ is j-rooted in $(G, \mathcal{T}_v)$ for some $j \in \{1, \dots, k\}$, we show that it is also j-rooted in $(G, \mathcal{T})$.

Define $\mathcal{T}_{u,v} := (T_1, \dots, T_k, \{u, v\})$. We have $\mu(G, \mathcal{T}_{u,v}) = \mu(G, \mathcal{T}) + 1$, and if we have an optimal path packing that leaves either u or v uncovered, then we are done.

So u and v are covered by every maximal path packing in $(G, \mathcal{T}_{u,v})$. $\mathcal{V}(\mathcal{T}_v)$ shows that u (v) cannot be l-touched in $(G, \mathcal{T}_{u,v})$ for any $l \neq j$ ($l \neq i$). Hence by induction u and v are in $C(\mathcal{T}_{u,v})$. If u and v are not in the same even component of $C(\mathcal{T}_{u,v})$, then

$$\mu(G, \mathcal{T}) \leq \mu(G, \mathcal{T}_{u,v}) - 2,$$

which is impossible, therefore u, v are on the boundary of a component K of $C(\mathcal{T}_{u,v})$.

We know that K has an even number, say $2m$ boundary points including u and v; in a system of $\mu(G,\mathcal{T}_{u,v}) = \mu(G,\mathcal{T})+1$ feasible $\mathcal{T}_{u,v}$-paths these boundary points are matched by m paths, and in a system of $\mu(G,\mathcal{T})$ $\mathcal{T}$-paths the $2(m-1)$ boundary points are matched by $m-1$ paths (u and v are not terminal points any more, and therefore they are not boundary points).

By switching between the two sets of paths we get a bijection between maximum collections of $\mathcal{T}$-paths and maximum collections of $\mathcal{T}_{u,v}$-paths. Both cover the same set of terminal vertices, except that the latter covers u, v, whereas the former does not.

It follows that $\mathcal{V}(\mathcal{T}_{u,v})$ is not only an optimal $\mathcal{T}_{u,v}$-partition, but also an optimal $\mathcal{T}$-partition, and $\mathcal{V}(\mathcal{T}_{u,v}) = \mathcal{V}(\mathcal{T})$, contradicting the choice of $\mathcal{X}$.

(3) is now simpler to prove:

Indeed, let $v \in D(\mathcal{X})\setminus X$, let K be the (odd) component of $D(\mathcal{X})$ containing v, and define the path packing problem $(K,\mathcal{S}^i)$ as follows: $\mathcal{S}^i = (S_1,\dots,S_i\cup\{v\},\dots,S_k)$, where $S_i = X_i \cap V(K)$ $(i=1,\dots,k)$. Denote $2m+2$ the number of terminal points in $\mathcal{S}^i$; since $\mathcal{X}$ is an optimal $\mathcal{T}$-partition, $\mu(K,\mathcal{S}) = m$ or $m+1$.

If for some $i=1,\dots,k$, $\mu(K,\mathcal{S}^i) = m$, then there exists in K an $\mathcal{S}^i$-partition $\mathcal{Z}$ of value m. Since the $\mathcal{S}^i$-partition $\mathcal{Y} = (\{v\}\cup\cup_{i=1}^k S_i; Y_0=\emptyset,\mathcal{S}^i)$ has value $m+1$, in $\mathcal{Z}$ either $Z_0 \neq \emptyset$ or there exists $i\in\{1,\dots,k\}$ such that $|Z_i| > |S_i|$.

Combining $\mathcal{Z}$ with $\mathcal{X}$ in the obvious way, we get in the former case a $\mathcal{T}$-partition where $X_0\cup C(\mathcal{X})$ increases, whereas in the latter case a $\mathcal{T}$-partition where X increases, contradicting in both cases the choice of $\mathcal{X}$.

Therefore, for all $i=1,\dots,k$, $\mu(K,\mathcal{S}^i) = m+1$. For a fixed i, $\mu(K,\mathcal{S}^i) = m+1$ means exactly that there exists $j \neq i$ so that v is j-rooted in K, and this j is not unique, since $\mu(K,\mathcal{S}^j) = m+1$ also holds. Since according to the Claim every $s \in S_i$ is i-rooted, we can combine the maximum path packing and the half path proving that s is i-rooted with the appropriate path packings in $(K,\mathcal{S}^i)$ showing that v is j-rooted in $(G,\mathcal{T})$. Using that v is also l-rooted for some $l\neq j$ in K, we get a path packing proving that v is l-rooted in $(G,\mathcal{T})$.

Now the claim and the theorem is proved. □

5 Generalizations

Suppose we are given a function $c: V(G) \longrightarrow \mathbb{N}$, and we want to maximize the number of $\mathcal{T}$-paths so that vertex $v\in V(G)$ is contained in at most $c(v)$ paths. Let us denote by $\nu(G,T,c)$ the maximum.

Theorem 5.1.

$$\nu(G,T,c) = \min c(V_0) + \sum_{C\in\mathcal{C}} \left\lfloor \frac{c(\cup_{i=1}^k V_i \cap C)}{2} \right\rfloor,$$

where $V_0 \subseteq V(G)$, V_i is a subpartition of $V(G)\setminus V_0$, $V_0\cup V_i \supseteq T_i$, and $\mathcal{C}$ is the collection of the components of $G - V_0 - \cup_{i=1}^k E[V_i]$.

Proof. Replicate each vertex – vertex v $c(v)$ times. (All copies are adjacent to the same vertex-set, and non-adjacent among them.) Note that all the copies of a vertex are in the same class of the structure theorem, and the theorem follows. The statement is also easy directly from Mader's Theorem 1.4. □ □

In other words, putting weights on vertices does not generalize the problem, but the statement can be useful in algorithmic considerations.

In the following we formulate a common generalization of Mader's $\mathcal{T}$-path problem and Cunningham and Geelen's path-matching theorem [3]. We call this problem the $\mathcal{T}$-path-matching problem.

Let $G = (V, E)$ be a graph and $T_1, \ldots, T_k$ disjoint subsets of V, denote $\mathcal{T} := \{T_1, \ldots, T_k\}, T := \cup_{i=1}^k T_i$. A $\mathcal{T}$-path-matching is a union of edges (called matching edges) and $\mathcal{T}$-paths and paths with one end in T and the other not in T, all vertex-disjoint. The value of a path-matching is the number of its edges plus the number of its matching edges. For a $\mathcal{T}$-path-matching or any subset $F \subseteq E(G)$ *we will simply denote by* $|F|$ *the value of the edges that are in the set. This is defined as the number of edges in it plus the number of matching edges counted a second times, that is, the total number of edges contained in paths + twice the number of edges contained in matchings.* The perfect version of this problem was introduced by Z. Szigeti [20].

A $\mathcal{T}$-partition is a family of pairwise disjoint sets $X_0, X_1, \ldots, X_k \subseteq V(G)$ so that $X_0 \cup X_i \supseteq T_i$. (It is not necessarily a partition of $V(G)$.) The components of the graph $(G - X_0) - \cup_{i=1}^k E[X_i]$ will be called the *components* of this $\mathcal{T}$-partition. Let us denote by $\mathcal{C}$ the family of these components.

Let $X := \cup_{i=1}^k X_i$, and

$$\mathcal{C}^* := \{C \in \mathcal{C} : C \cap X \neq \emptyset\},\ \mathcal{C}' := \{C \in \mathcal{C} : C \cap X = \emptyset\}.$$

$$\bar{X} := \cup\{C : C \in \mathcal{C}^*\},\ U := \cup\{C : C \in \mathcal{C}'\}.$$

We denote by ω the number of odd components that do not meet X. Clearly, $\bar{X} \supseteq X$, (but $\bar{X}$ does not necessarily contain T, since X_0 can also contain vetices of T).

Fact 6. There is no edge between a vertex of U and a vertex of $\bar{X}$.

For a component $C \in \mathcal{C}^*$, $\omega_i(C)$ denotes the number of components of $C - X$ of odd cardinality having neighbors only in $X_i \cup X_0$ for $i = 1, 2, \ldots k$. Denote $\Omega_i(C)$ the union of these components, and $\Omega(C) := \cup_{i=1}^k \Omega_i(C)$. Let $\omega(C) := \sum_{i=1}^k \omega_i(C)$.

The value of a $\mathcal{T}$-partition is defined to be

$$\mathrm{val}_{G,\mathcal{T}}(\mathcal{X}) := |V \setminus T| + (|X_0| - \omega) + \sum_{C \in \mathcal{C}} \left\lfloor \frac{|C \cap X| - \omega(C)}{2} \right\rfloor.$$

Conjecture 6.1. *The maximum value of a* $\mathcal{T}$*-path-matching is equal to*

$$\min val_{G,\mathcal{T}}(\mathcal{X}),$$

where the minimum is taken over all $\mathcal{T}$*-partitions* $\mathcal{X}$.

We provide the proof of the so called 'trivial' part (which is indeed, easy, but requires more concentration than for simpler min-max theorems). The assertion is that *the maximum does not exceed the minimum.*

Let F be a $\mathcal{T}$-path-matching, $X_0, X_1, \dots, X_k$ a $\mathcal{T}$-partition. $\{X_0, \bar{X}, U\}$ is a partition of V, and $\{\cup_{v\in X_0}\delta(v), E[\bar{X}], E[U]\}$ is a partition of E. Indeed, by the Fact there is no edge between U and $\bar{X}$. We have:

$$|F| = |F \cap \cup_{v\in X_0}\delta(v)| + |F \cap E[\bar{X}]| + |F \cap E[U]|.$$

Estimate each term:

$$|F \cap \cup_{v\in X_0}\delta(v)| \le 2|X_0 \setminus T| + |X_0 \cap T|,$$

because every $x \in X_0 \setminus T$ is incident either to a matching edge or to at most two path-edges. $x \in X_0 \cap T$ is incident to at most one path edge (and no matching edge).

Let M be the set of matching edges of F, and P the set of its path-edges. Then

$$|F \cap E[\bar{X}]| \le |\bar{X} \setminus T| + \sum_{C\in\mathcal{C}} \left\lfloor \frac{|C \cap X| - \omega(C)}{2} \right\rfloor,$$

which we prove in the following paragraph.

Clearly, $|M \cap E[\bar{X}]| \le |V(M) \cap (\bar{X} \setminus T)|$. Furthermore, let P' be the edge-set of a path in P. Since P' is a forest,

$$|E(P') \cap E[\bar{X}]| \le |V(P') \cap \bar{X}| - 1,$$

hence either $|E(P') \cap E[\bar{X}]| \le |V(P') \setminus T|$, or P' is a $\mathcal{T}$-path and entirely in $\bar{X}$, when we have to add 1 to the right hand side as 'correcting term', that is, we have then $|E(P') \cap E[\bar{X}]| \le |V(P') \setminus T| + 1$. However, the sum of the correcting terms can be estimated from above by $\sum_{C\in\mathcal{C}} \left\lfloor \frac{|C\cap X| - \varphi_1(C) - \varphi_2(C)}{2} \right\rfloor$, where $\varphi_1(C)$ ($\varphi_2(C)$) denotes the number of matching (path) edges joining $\Omega(C)$ to $C \cap X$, since for every $\mathcal{T}$-path entirely contained in $\bar{X}$ there exists at least one $C \in \mathcal{C}^*$ with at least two vertices of P'. Indeed, the two endpoints of P' are contained in sets T_i and T_j respectively ($i \ne j$). Summing up, we get the following (where $\varphi_3(C)$ denotes the number of components K of $\Omega(C)$ for which there is no edge of F between K and X):

$$\begin{aligned}
|F \cap E[\bar{X}]| &\le |\bar{X} \setminus T| + \sum_{C\in\mathcal{C}} \left\lfloor \frac{|C \cap X| - \varphi_1(C) - \varphi_2(C)}{2} \right\rfloor - \varphi_3(C) \\
&= |\bar{X} \setminus T| + \sum_{C\in\mathcal{C}} \left\lfloor \frac{|C \cap X| - \varphi_1(C) - \varphi_2(C) - 2\varphi_3(C)}{2} \right\rfloor \\
&\le |\bar{X} \setminus T| + \sum_{C\in\mathcal{C}} \left\lfloor \frac{|C \cap X| - \omega(C)}{2} \right\rfloor.
\end{aligned}$$

Finally,

$$|F \cap E[U]| \leq |U| - \omega,$$

because $|F \cap E[U]| = \sum_{C \in \mathcal{C}'} |F \cap E(C)|$, and $|F \cap E(C)| \leq |V(C)|$,

moreover, if $C \in \mathcal{C}'$, and $|V(C)|$ is odd, then $|F \cap E(C)| \leq |V(C)| - 1$,

because there is either at least one path entering and leaving the component, or at least one uncovered vertex (or both).

So we finally got:

$$|F| \leq 2|X_0 \setminus T| + |X_0 \cap T| + |U| - \omega + |\bar{X} \setminus T| + \sum_{C \in \mathcal{C}} \left\lfloor \frac{|C \cap X| - \omega(C)}{2} \right\rfloor$$

$$= |X_0 \setminus T| + |\bar{X} \setminus T| + |U| + |X_0 \setminus T| + |X_0 \cap T| - \omega + \sum_{C \in \mathcal{C}} \left\lfloor \frac{|C \cap X| - \omega(C)}{2} \right\rfloor$$

$$= |V \setminus T| + (|X_0| - \omega) + \sum_{C \in \mathcal{C}} \left\lfloor \frac{|C \cap X| - \omega(C)}{2} \right\rfloor.$$

Isn't it challenging to search for a structure algorithm for $\mathcal{T}$-path-matchings (in polynomial time)? This conjecture can be reduced to matchings in the particular case when every class of $\mathcal{T}$ has at most one element, which motivates an adaptation of Schrijver's approach [14].

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