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A polyhedral approach to even factors

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Abstract

Generalizing path-matchings, W.H. Cunningham and J.F. Geelen introduced the notion of even factor in directed graphs. In weakly symmetric directed graphs they proved a min-max formula for the maximum cardinality even factor by algebraic method and also discussed a primal-dual method for the weighted case. Later, Gy. Pap and L. Szegő proved a simplified formula by purely combinatorial method and derived a Gallai-Edmonds type structure theorem. In this paper, polyhedra related to even factors are considered, integrality and total dual integrality of these linear descriptions are proved directly, without using earlier unweighted results.

1 Introduction

Cunningham and Geelen [2, 3] introduced the notion of *even factor* in *digraphs* as the edge set of vertex disjoint union of *dipaths* and *dicycles* of even length. For short, on a digraph we mean directed graph, dicycle and dipath mean directed cycle and directed path, while in a cycle or in a path, there can be forward and backward edges, too. The maximum cardinality even factor problem is NP-hard in general (Wang, see [3]) but there are special classes of digraphs where it can be solved.

In the paper throughout digraphs with no loops and with no parallel edges are considered, only oppositely directed edges are permitted. An edge of a digraph is called *symmetric*, if the reserved edge is in the edge set of the digraph, too. A digraph is symmetric if all its edges are symmetric, while a digraph is said to be *weakly symmetric* if its strongly connected components are symmetric. Then weak symmetry means that the edges contained in dicycle are symmetric. Pap and Szegő [8] introduced the more general class of *hardly symmetric* digraphs where the problem remains tractable. A digraph is said to be hardly symmetric if the edges contained in odd dicycle are symmetric.

Using an algebraic approach, Cunningham and Geelen proved a min-max formula for the maximum cardinality even factor problem in weakly symmetric digraphs [3]. Later, Pap and Szegő [8] proved a simpler formula for the same problem with purely

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combinatorial methods and they also observed that the same proof and formula works for the hardly symmetric case.

As the unweighted problem is tractable only in special graphs, the weight function also has to possess certain symmetries. If $G' = (V', A')$ is a weakly symmetric graph and $c' : A' \to \mathbb{R}$ is a weight function s.t. $c'(uv) = c'(vu)$ if both uv and vu belong to A', then the *pair* (G', c') is called *weakly symmetric*. Cunningham and Geelen considered weighted generalization in [3] for weakly symmetric pairs. Using their unweighted formula and a primal-dual method they derived integrality of a polyhedron similar to the perfect matching polyhedron.

Here we deal with a more general class of digraphs and weight functions which turn out to be general enough to contain the mentioned cardinality and weighted cases. We prove integrality of polyhedra related to even factors and total dual integrality. We do not use earlier unweighted results, which rather follow as consequences. This approach enables us to derive weighted min-max formulas, nevertheless we omit to present them explicitly, since these are direct consequences of the polyhedral theorems.

Before, let $A \in \mathbb{Z}^{m \times n}$ be an integer matrix, $b \in \mathbb{Z}^m$ an integer vector. The polyhedron defined by $\{x : Ax \leq b\}$ is said to be *integer* if every face of it contains an integer vector. The system $Ax \leq b$ is called *totally dual integral* (TDI) if the dual of the linear program $\max\{cx : Ax \leq b\}$ has an integer valued optimal solution for every integer valued c such that it has an optimal solution. Polyhedra defined by TDI systems are interesting because of the fact that they are integer polyhedra.

To distinguish directed and undirected edges, the directed edge with tail u and head v will be denoted by uv and the undirected edge with endvertices u and v will be denoted by $\{u, v\}$. If $G = (V, E = E_\mathrm{d} \cup E_\mathrm{u})$ is a *mixed graph*, E_d and E_u denote respectively the set of its directed edges and the set of its undirected edges and $y \in \mathbb{R}^E$, then for $u \in V$ and $S \subseteq V$ we will use the notations
$d_y(u) = \sum_{\{u,v\} \in E_\mathrm{u}} y(\{u, v\}) + \sum_{uv \in E_\mathrm{d}} y(uv) + \sum_{vu \in E_\mathrm{d}} y(vu)$,
$i_y(S) = \sum \{y(\{u, v\}) : \{u, v\} \in E_\mathrm{u},\ u, v \in S\} + \sum \{y(uv) : uv \in E_\mathrm{d},\ u, v \in S\}$,
$\varrho_y(S) = \sum\{y(uv) : uv \in E_\mathrm{d}, u \notin S, v \in S\}$ and $\delta_y(S) = \varrho_y(V - S)$. We do not indicate the graph because it will be clear from y and from the context.

For any digraph $G = (V, A)$, one can consider the following system of linear inequalities

$$x \in \mathbb{R}^A,\ x \geq 0 \tag{1}$$
$$\varrho_x(v) \leq 1 \quad \text{for every } v \in V \tag{2}$$
$$\delta_x(v) \leq 1 \quad \text{for every } v \in V \tag{3}$$
$$i_x(S) \leq |S| - 1 \quad \text{for every } S \subseteq U,\ |S| \text{ odd},\ U \in \mathcal{C}(G) \tag{4}$$

whose solution set is denoted by $\mathrm{EF}(G)$ and $\mathcal{C}(G)$ denotes the set of vertex sets of the strongly connected components of G. We do not distinguish a subset of the edges and its characteristic vector, thus we can observe easily, that the integer solutions of $\mathrm{EF}(G)$ are exactly the even factors of G. However, as mentioned above, the polyhedron defined by these inequalities is not integer. After some preliminaries we will define a class of graphs and weight functions where we can state that $\max\{cx : x \in \mathrm{EF}(G)\}$ is attained on an integer vector.

For a digraph G', $\overline{G'}$ denotes its underlying undirected graph. The maximal strongly connected subgraphs of a digraph having 2-vertex-connected underlying undirected graph are called *blocks*. If $U \in \mathcal{C}(G)$, then let $\mathcal{D}(U)$ denote the collection of vertex-sets of blocks of $G[U]$. where $G[U]$ denotes the subgraph of G induced by U. A digraph G' is said to be bipartite if $\overline{G'}$ is bipartite. Throughout the paper, we deal with digraphs s.t. for any $U \in \mathcal{C}(G)$, $S \in \mathcal{D}(U)$, $G[S]$ is symmetric or bipartite. We consider families $\mathcal{S}, \mathcal{B} \subseteq 2^V$ s.t. $\mathcal{S} \cap \mathcal{B} = \emptyset$, $\mathcal{S} \cup \mathcal{B} = \cup_{U \in \mathcal{C}(G)} \mathcal{D}(U)$, for any $S \in \mathcal{S}$, $G[S]$ is symmetric and for any $S \in \mathcal{B}$, $G[S]$ is bipartite. A digraph is said to be *evenly symmetric* if there exist families satisfying the above criteria. The *pair* (G, c) is said to be *evenly symmetric* if $G = (V, A)$ is an evenly symmetric digraph with respect to $\mathcal{S}$ and $\mathcal{B}$, and $c : A \to \mathbb{R}$ is a weight function s.t. $c(uv) = c(vu)$ if uv and vu belong to A and $\exists S \in \mathcal{S}$ s.t. $u, v \in S$. More precisely we should say that (G, c) is *evenly symmetric* with respect to the families $\mathcal{S}$ and $\mathcal{B}$, but the context will make clear everywhere the families we consider. We let $\mathcal{S}^\star$ be the family of maximal members of

$$\{T : \exists \mathcal{T} \subseteq \mathcal{S}, T = \cup \mathcal{T} \text{ s.t. the hypergraph } (T, \mathcal{T}) \text{ is connected}\}.$$

It is easy to see that $\mathrm{EF}(G)$ equals to the solution set of

$$x \in \mathbb{R}^A,\ x \geq 0 \tag{5}$$
$$\varrho_x(v) \leq 1 \quad \text{for every } v \in V \tag{6}$$
$$\delta_x(v) \leq 1 \quad \text{for every } v \in V \tag{7}$$
$$i_x(S) \leq |S| - 1 \quad \text{for every } S \subseteq U,\ |S| \text{ odd},\ U \in \mathcal{S}^\star. \tag{8}$$

Moreover, with respect to some objective function, this system and (1)-(2)-(3)-(4) have integer dual optimal solution at the same time. A family $\mathcal{F}$ of subsets of V is said to be *laminar* if for any $X, Y \in \mathcal{F}$, at least one of $X \subseteq Y$, $Y \subseteq X$ or $X \cap Y = \emptyset$ holds. Now we are ready to state a result on dual integrality.

Theorem 1.1. *If (G, c) is evenly symmetric and c is an integer vector, then the dual of the linear program $\max\{cx : (5) - (6) - (7) - (8)\}$ has integer optimal solution $(\lambda_v^\varrho, \lambda_v^\delta, z_S)$. Moreover, there is one for which $\{S : z_S > 0\}$ is a laminar family.*

To deal with primal integrality and to exploit the theory of TDI systems, the symmetric edges spanned by some members of $\mathcal{S}$ are considered as undirected edges and we take the *mixed graph* with vertex set V and edge set $E = E_\mathrm{d} \cup E_\mathrm{u}$,

$$E_\mathrm{u} = \{\{u, v\} : uv, vu \in A,\ u, v \in U \text{ for some } U \in \mathcal{S}\},$$
$$E_\mathrm{d} = \{uv \in A : \{u, v\} \notin E_\mathrm{u}\}.$$

We need the description (in the sense of linear inequalities) of the following polyhedron

$$\mathrm{SEF}(G) = \left\{ y \in \mathbb{R}^E : \exists x \in \mathrm{EF}(G),\ y(e) = \left\{ \begin{array}{rl} x(uv) + x(vu) & \text{if } e = \{u, v\} \in E_\mathrm{u} \\ x(uv) & \text{if } e = uv \in E_\mathrm{d} \end{array} \right\} \right\}.$$

Theorem 1.2. *For the evenly symmetric digraph* $G = (V, A)$, $\mathrm{SEF}(G)$ *is the solution set of*

$$y \in \mathbb{R}^E,\ y \geq 0 \tag{9}$$
$$d_y(v) \leq 2 \quad \textit{for every } v \in V \tag{10}$$
$$i_y(S) \leq |S| - 1 \quad \textit{for every } S \subseteq U,\ |S| \textit{ odd, } U \in \mathcal{S}^\star \tag{11}$$
$$i_y(S) + \varrho_y(S) \leq |S| \quad \textit{for every } S \in \mathcal{U} \tag{12}$$
$$i_y(S) + \delta_y(S) \leq |S| \quad \textit{for every } S \in \mathcal{U} \tag{13}$$

where $\mathcal{U} = \{S : \exists U \in \mathcal{S}^\star \textit{ s.t. } S \subseteq U\} \cup \{\{v\} : v \in V - \cup\mathcal{S}^\star\}$.

Observe that if $|\mathcal{C}(G)| = 1$, then this specializes to $2\mathrm{M}(G)$ where $\mathrm{M}(G)$ is the *matching polyhedron* defined by

$$\begin{aligned} y \in \mathbb{R}^E,\ y &\geq 0 \\ d_y(v) &\leq 1 \quad \text{for every } v \in V \\ i_y(S) &\leq \frac{|S|-1}{2} \quad \text{for every } S \subseteq V,\ |S| \text{ odd.} \end{aligned}$$

This system was proved to be a TDI system by Cunningham and Marsh [5]. Another well-known special case is when the strongly connected components are bipartite. In this case

$$\begin{aligned} y \in \mathbb{R}^E,\ y &\geq 0 \\ \varrho_y(v) &\leq 1 \quad \text{for every } v \in V \\ \delta_y(v) &\leq 1 \quad \text{for every } v \in V \end{aligned}$$

is obtained, which is TDI. Also, we can see easily that any integer solution of this system is a vertex disjoint union of dipaths and even dicycles.

A more general special case is the *path-matching polyhedron* which was introduced by Cunningham and Geelen. For a detailed description, the interested reader is referred to the paper of Cunningham and Geelen [4] and Frank and Szegő [6]. Here, V is the disjoint union of T_1, R and T_2, where T_1 and T_2 are stable sets, $G[R]$ is symmetric, no edge enters T_1 and no edge leaves T_2. System (9)-(10)-(11)-(12)-(13) defined on this graph specializes to the path-matching polyhedron

$$\begin{aligned} y \in \mathbb{R}^E,\ y &\geq 0 \\ d_y(v) &\leq 2 \quad \text{for every } v \in R \\ i_y(S) &\leq |S| - 1 \quad \text{for every } S \subseteq R,\ |S| \text{ odd} \\ i_y(S) + \varrho_y(S) &\leq |S| \quad \text{for every } S \subseteq R \\ i_y(S) + \delta_y(S) &\leq |S| \quad \text{for every } S \subseteq R \\ \varrho_y(v) &\leq 1 \quad \text{for every } v \in T_2 \\ \delta_y(v) &\leq 1 \quad \text{for every } v \in T_1 \end{aligned}$$

which was proved to be TDI by Cunningham and Geelen [4]. Our main theorem is the following.

Theorem 1.3. *If $G = (V, A)$ is evenly symmetric, then (9)-(10)-(11)-(12)-(13) is TDI. In addition, if $c \in \mathbb{Z}^E$ is an integer vector, then there is an integer optimal solution $(\mu_v^d, \mu_S^i, \mu_S^{i,\varrho}, \mu_S^{i,\delta})$ of the dual of* $\max\{cy : (9) - (10) - (11) - (12) - (13)\}$ *s.t.*

$$\mathcal{G} = \{S : \mu_S^i > 0\} \cup \{S : \mu_S^{i,\varrho} > 0\} \cup \{S : \mu_S^{i,\delta} > 0\} \text{ is a laminar family,} \tag{14}$$

$$\text{if } \mu_S^i > 0, \mu_T^{i,\varrho} > 0,\ S \cap T \neq \emptyset, \text{ then } S \subseteq T, \tag{15}$$

$$\text{if } \mu_S^i > 0, \mu_T^{i,\delta} > 0,\ S \cap T \neq \emptyset, \text{ then } S \subseteq T, \text{ and} \tag{16}$$

$$\mu_S^{i,\delta} > 0, \mu_T^{i,\varrho} > 0 \text{ implies } S \cap T = \emptyset. \tag{17}$$

Clearly, this generalizes the weakly symmetric case because a weakly symmetric graph G is evenly symmetric with $\mathcal{S} = \cup_{U \in \mathcal{C}(G)} \mathcal{D}(U)$. Due to the following characterization of hardly symmetric graphs, which was observed and proved by Z. Király [7], our model includes the hardly symmetric case, too.

Lemma 1.4 (Király). *Let $G = (V, A)$ be a hardly symmetric digraph which is itself a block. Then G is symmetric or bipartite.*

The rest of the paper is organized as follows. In the next section, we derive the unweighted min-max formula of Pap and Szegő [8], while in the subsequent sections, the detailed proofs of Theorem 1.2 and Theorem 1.3 will be presented.

2 Unweighted min-max formula

In [3], Cunningham and Geelen using their unweighted min-max formula and a primal-dual method, derived polyhedral results. Here we follow an opposite direction and prove the unweighted formula of Pap and Szegő [8] as a consequence of Theorem 1.3. First, for better view of analogy, we recall a non-usual form of the well-known Berge-Tutte formula. If $G = (V, E)$ is an undirected graph then $\mathrm{odd}(G)$ denotes the number of connected components of G having an odd number of vertices, and $N_G(X) = \{v \in V - X : \exists u \in X, \{u, v\} \in E\}$.

Theorem 2.1 (Berge and Tutte). *If $G = (V, E)$ is an undirected graph, then the cardinality of a maximum matching of G is*

$$\min_{X \subseteq V} \frac{|V| + |N_G(X)| - \mathrm{odd}(G[X])}{2}.$$

In a digraph $G = (V, A)$, we define $N_G^+(X) = \{v \in V - X : \exists u \in X, uv \in A\}$ and let $\mathrm{odd}(G)$ denote the *number of strongly connected components of G having an odd number of vertices with no entering arc (i.e. source components).* Using this notation, Pap and Szegő [8] proved the following.

Theorem 2.2 (Pap and Szegő). *If $G = (V, A)$ is a hardly symmetric digraph, then the maximum cardinality of an even factor is*

$$\min_{X \subseteq V} |V| + |N_G^+(X)| - \mathrm{odd}(G[X]).$$

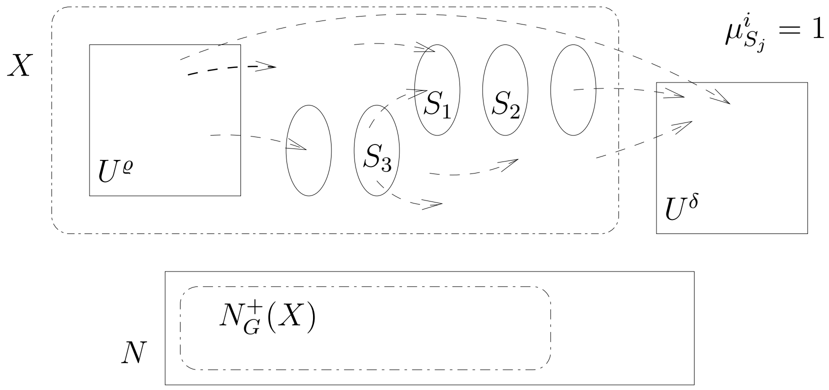

Figure 1: The edges indicated by *dashed* line *cannot* occur.

Proof. First we prove max $\leq$ min. For a fixed X, any even factor of G has at most $|X| - \text{odd}(G[X])$ edges spanned by X, at most $|N^+_G(X)|$ edges leaving X, and at most $|V - X|$ edges having tail in $V - X$. Then, an even factor of G has cardinality at most $|V| + |N^+_G(X)| - \text{odd}(G[X])$.

To see max $\geq$ min, we consider integer optimal primal and dual solutions of $\max \sum_{e \in E} y(e)$ subject to (9)-(10)-(11)-(12)-(13), suppose moreover that the dual solution $(\mu^d_v, \mu^i_S, \mu^{i,\varrho}_S, \mu^{i,\delta}_S)$ satisfies (14)-(15)-(16)-(17). Such a dual solution exists by Theorem 1.3. A primal solution is an even factor, while the dual solution may have to be modified.

Claim 2.3. *The family $\mathcal{G} = \{S : \mu^i_S > 0\} \cup \{S : \mu^{i,\varrho}_S > 0\} \cup \{S : \mu^{i,\delta}_S > 0\}$ form a subpartition of V. Moreover, we can assume that $\mathcal{G} \cup \{v : \mu^d_v = 1\}$ is a subpartition.*

Proof. The first statement is a consequence of (14)-(15)-(16)-(17) and that the solution is 0-1 valued. Consider now a solution s.t. this subpartition covers a minimum number of vertices v with $\mu^d_v = 1$. If $\mu^d_v = 1$, $\mu^i_S = 1$ and $v \in S$, then we can decrease μ^i_S by 1 and increase μ^i_{S-u-v} and μ^d_u by 1 for some $u \in S - v$, contradicting the minimality. If $\mu^d_v = 1$, $\mu^{i,\varrho}_S = 1$ and $v \in S$ then we can decrease $\mu^{i,\varrho}_S$ by 1 and increase $\mu^{i,\varrho}_{S-v}$ by 1 which would be a dual solution with strictly smaller value. $\square$

From the dual solution μ we can define X for which max $\geq$ min. After setting $U^\delta = \cup\{S : \mu^{i,\delta}_S = 1\}$, $U^\varrho = \cup\{S : \mu^{i,\varrho}_S = 1\}$ and $N = \{v : \mu^d_v = 1\}$ we can define $X = V - N - U^\delta$. Then $G[X - U^\varrho]$ is composed by symmetric odd components, which are source components in $G[X]$. Clearly, $N^+_G(X) \subseteq N$. If $uv \in A$ leaves U^ϱ, then $v \in N$, and similarly, if $uv \in A$ enters U^δ, then $u \in N$. Then the cardinality of a maximum even factor is at least

$$2|N| + \sum_{\mu^i_S = 1} (|S| - 1) + |U^\delta| + |U^\varrho| = |N| + |V| - \text{odd}(G[X]) \geq |V| + |N^+_G(X)| - \text{odd}(G[X]).$$

$\square$

3 Structure of hardly symmetric digraphs

Let $G' = (V', E')$ be a digraph. An *ear* of G' is a dicycle or a dipath (with different ends) of G', while a *proper ear* of G' is a dipath (with different ends) of G'. Then the sequence $C_0, P_1, P_2, \ldots, P_k$ is an *ear-decomposition* of G' if C_0 is a dicycle called *initial dicycle*; every P_i is an ear; $C_0 \cup P_1 \cup \cdots \cup P_{i-1}$ has exactly two common vertices with P_i, namely the ends of P_i, if P_i is a dipath, and has exactly one common vertex with P_i, if P_i is a dicycle; and $C_0 \cup P_1 \cup \cdots \cup P_k = G'$. Similarly, the sequence $C_0, P_1, P_2, \ldots, P_k$ is a *proper ear-decomposition* of G' if C_0 is a dicycle of length at least 2, called *initial dicycle*; every P_i is a proper ear; $C_0 \cup P_1 \cup \cdots \cup P_{i-1}$ has exactly two common vertices with P_i, namely the ends of P_i; and $C_0 \cup P_1 \cup \cdots \cup P_k = G'$.

It is well-known that a strongly connected digraph G' possesses an ear-decomposition, moreover, every dicycle of G' can be the initial dicycle of this ear-decomposition. It is much more non-trivial that a strongly connected digraph G' s.t. $\overline{G'}$ is 2-vertex-connected, has proper ear-decomposition, and every dicycle of G' of length at least 2 can be the initial dicycle of a proper ear-decomposition.

Proof of Lemma 1.4. If $\overline{G}$ is bipartite, then we are done. Thus, G contains a (not necessarily directed) closed walk W, having and odd number of edges (with multiplicity). W may have forward and backward edges. Consider such a W containing minimum number of backward edges.

Now, we are looking for a directed closed walk having an odd number of edges. If W has no backward edge, then we are done. If W has a backward edge uv, then by strong connectivity there is a dipath P from v to u. If P has an even number of edges, then P together with uv is an odd dicycle. Otherwise, uv can be exchanged to P in W, and the number of backward edges decreases. Finally, we have a directed closed walk having an odd number of edges. After decomposing it into dicycles, we get an odd dicycle C.

Claim 3.1. *There is a sequence of symmetric digraphs $G_0 \subset G_1 \subset \cdots \subset G_l = G$ s.t. G_0 is a symmetric odd cycle, and for any two vertices s and t of G_i, G_i contains odd and also even length dipaths from s to t.*

Proof. We consider a proper ear-decomposition of G with initial cycle C. Then C is odd, hence symmetric. Moreover, there exist odd and also even dipaths between any two vertices of C. Suppose, by induction, that a subgraph $G_i \subseteq G$ is built up by the proper ear-decomposition, G_i is symmetric and between any two vertices of G_i, there exist odd and also even dipaths contained in G_i. Then the edges of the following proper ear Q are contained in odd dicycle, hence are symmetric. Moreover, between any two vertices of Q, there exists odd dipath and also even dipath. □

The claim completes the proof. □

4 Projection

For the proof of Theorem 1.2 we invoke a *projection technique* of Balas and Pulleyblank from [1]. We consider the polyhedron defined by

$$P = \{(x, y) : A_1x + B_1y \le b_1, A_2x + B_2y = b_2,\ x, y \ge 0\}$$

where our aim is to describe

$$P' = \{y : \exists x \text{ s.t. } (x, y) \in P\}$$

in the sense of linear inequalities. After defining the cone $W = \{(w_1, w_2) : w_1 \ge 0, w_1A_1 + w_2A_2 \ge 0\}$, it can be seen via Farkas' lemma that

$$P' = \{y : y \ge 0, (w_1B_1 + w_2B_2)y \le w_1b_1 + w_2b_2 \text{ for every } (w_1, w_2) \in W\}.$$

To get a polyhedral description of P', finitely many inequalities have to be chosen which define the same polyhedron. To this end, a finite set $\widehat{W} \subseteq W$ will be determined s.t.

$$P' = \{y : y \ge 0, (w_1B_1 + w_2B_2)y \le w_1b_1 + w_2b_2 \text{ for every } (w_1, w_2) \in \widehat{W}\}.$$

Proof of Theorem 1.2. To apply this projection technique, consider the following system in the above role of P.

$$x \in \mathbb{R}^A, y \in \mathbb{R}^E,\ x, y \ge 0 \tag{18}$$

$$\varrho_x(v) \le 1 \quad \text{for every } v \in V \tag{19}$$

$$\delta_x(v) \le 1 \quad \text{for every } v \in V \tag{20}$$

$$i_x(S) \le |S| - 1 \quad \text{for every } S \subseteq U,\ |S| \text{ odd},\ U \in \mathcal{S}^\star \tag{21}$$

$$-x(uv) + y(uv) = 0 \quad \text{for every } uv \in E_\mathrm{d} \tag{22}$$

$$-x(uv) - x(vu) + y(\{u, v\}) = 0 \quad \text{for every } \{u, v\} \in E_\mathrm{u}. \tag{23}$$

Let $(w^\varrho_v : v \in V, w^\delta_v : v \in V, w^i_S : S \subseteq U, U \in \mathcal{S}^\star, w_{uv} : uv \in E_\mathrm{d}, w_{\{u,v\}} : \{u, v\} \in E_\mathrm{u})$ be a member of the cone determined by inequalities $w^\varrho_v \ge 0$, $w^\delta_v \ge 0$, $w^i_S \ge 0$ and by

$$w^\varrho_v + w^\delta_u \ge w_{uv} \quad \text{for every } uv \in E_\mathrm{d}$$

$$w^\varrho_v + w^\delta_u + \sum_{u,v \in S} w^i_S \ge w_{\{u,v\}} \quad \text{for every } \{u, v\} \in E_\mathrm{u}$$

$$w^\varrho_u + w^\delta_v + \sum_{u,v \in S} w^i_S \ge w_{\{u,v\}} \quad \text{for every } \{u, v\} \in E_\mathrm{u}.$$

Thus we have inequalities $y \ge 0$ and

$$\sum_{e \in E} w_e y(e) \le \sum_{v \in V} w^\varrho_v + \sum_{v \in V} w^\delta_v + \sum_{S \subseteq U,\ |S| \text{ odd},\ U \in \mathcal{S}^\star} w^i_S(|S| - 1). \tag{24}$$

Our goal is to decompose (24) into the nonnegative combination of inequalities of (9)-(13).

We can observe that if $w^i_S > 0$ then (24) can be decomposed into the nonnegative combination of an inequality of type (11) and an inequality of form (24) but with fewer positive component w^i_S.

Similarly, if $w^\varrho_v > 0$ and $w^\delta_v > 0$ for some v, this can be removed by (10). If $w^\varrho_v > 0$ or $w^\delta_v > 0$ for some $v \in V - \cup\{S : S \in \mathcal{S}\}$, this also can be removed by (12) or by (13).

Moreover, the constraint can be decomposed into the sum of inequalities, s.t. for each of the new inequalities there exists $U \in \mathcal{S}^\star$ so that $w^\varrho_v = 0$ and $w^\delta_v = 0$ if $v \notin U$. Furthermore, by $y \geq 0$, it is enough to consider inequalities in the special case when $w_{uv} \geq 0$ and $w_{\{u,v\}} \geq 0$.

Hence, any remaining constraint is of form

$$\sum_{e\in E} w_e y(e) \leq \sum_{v\in U} w^\varrho_v + \sum_{v\in U} w^\delta_v \tag{25}$$

for some $U \in \mathcal{S}^\star$ and for weights s.t.

$$w^\varrho_v + w^\delta_u \geq w_{uv} \quad \text{for every } uv \in E_\mathrm{d} \tag{26}$$

$$w^\varrho_v + w^\delta_u \geq w_{\{u,v\}} \quad \text{for every } \{u,v\} \in E_\mathrm{u} \tag{27}$$

$$w^\varrho_u + w^\delta_v \geq w_{\{u,v\}} \quad \text{for every } \{u,v\} \in E_\mathrm{u} \tag{28}$$

where $w^\varrho_v \geq 0$, $w^\delta_v \geq 0$, $v \in U$ and, for any $v \in U$, at most one of the two quantities is strictly positive. An edge $uv \in A$ is said to be *tight* if $uv \in E_\mathrm{d}$ and equality holds in (26) or if $\{u,v\} \in E_\mathrm{u}$ and equality holds in (27). The set of tight edges is denoted by $A^=$. By increasing the value of $w_{\{u,w\}}$ and w_{uv}, it can be assumed that every $e \in E_\mathrm{d}$ is tight and for $\{u,v\} \in E_\mathrm{u}$ at least one of uv and vu is tight.

After setting $G^= = (V, A^=)$, we can observe that $G^=[U]$ forms a weakly symmetric digraph, since it is easy to see that if a dicycle of length at least three is contained in $A^=$, then this dicycle in the opposite orientation also belongs to $A^=$, while for a dicycle of length two this is obvious.

Claim 4.1. *If for some $u \in K \in \mathcal{C}(G^=[U])$, at least one of w^ϱ_u and w^δ_u is nonzero, then, for any vertex $v \in K$, $w^\varrho_v = w^\varrho_u$ and $w^\delta_v = w^\delta_u$.*

Proof. Assume w.l.o.g. that $uv, vu \in A^=$, $w^\delta_u > 0$ and $w^\varrho_u = 0$. Then $w^\delta_v = w_{\{u,v\}} > 0$, $w^\varrho_v = 0$ and $w^\delta_u = w_{\{u,v\}}$. □

We set $p = \left|\left\{Y \in \mathcal{C}(G^=[U]) : w^\delta_y + w^\varrho_y > 0 \text{ for } y \in Y\right\}\right|$. If $p = 0$, then we are done. If $p = 1$, then our inequality (25) is of form (12) or (13), more precisely constant times (12) or (13). Thus we suppose $p \geq 2$. If $uv, vu \in A$, $u, v \in U$ and $uv \in A^=$, $vu \notin A^=$ then $w^\delta_v \geq w^\delta_u$, $w^\varrho_v \leq w^\varrho_u$ and at least one of these inequalities holds with strict inequality. We may assume w.l.o.g. the existence of a vertex $u \in U$ with $w^\delta_u > 0$. Let Z be a sink component (a sink in the acyclic digraph obtained by shrinking the strongly connected components) of $G^=[U]$ s.t. $w^\delta_z > 0$ and $w^\varrho_z = 0$ for (every) $z \in Z$. $p \geq 2$ implies that there are vertices $z \in Z$ and $u \in U - Z$ s.t.

$uz \in A^=$ (and $zu \notin A^=$). For such z and u, $w_z^\delta > w_u^\delta$, so we can define $b := w_z^\delta$ and $a := \max\{w_u^\delta : z \in Z, u \in U - Z, uz \in A^=\}$. Then our inequality (25) can be decomposed into the sum of an inequality which is $b - a$ times (13) and another inequality of form (25) but with smaller p. Iterating such steps (and the similar one with (12)), the decomposability of (25) follows. □

5 Proof of Theorem 1.1 and Theorem 1.3

Proof of Theorem 1.1. Let $\lambda_v^\varrho, \lambda_v^\delta \geq 0$, $z_S \geq 0$ be an optimal rational dual solution where $v \in V$, $S \subseteq U$, $|S|$ is odd, $U \in \mathcal{S}^\star$. We take a positive integer k s.t. $\tilde{\lambda}_v^\varrho = k\lambda_v^\varrho$, $\tilde{\lambda}_v^\delta = k\lambda_v^\delta$, $\tilde{z}_S = kz_S$ are integers and we let $\varepsilon = \frac{1}{k}$. For fixed k, we choose moreover this solution so that the vector $\left(A = \sum_{v\in V} \tilde{\lambda}_v^\varrho + \sum_{v\in V} \tilde{\lambda}_v^\delta, B = \sum_S \tilde{z}_S |S|^2\right)$ is lexicographically as large as possible. Between optimal solutions, for fixed k, A and B are upper bounded. We show that under these conditions, the dual solution is almost integer and finally it can be transformed easily to an integer optimal one. The steps of the proof are motivated by the work of Balas and Pulleyblank [1].

Claim 5.1. $\mathcal{F} = \{S : z_S > 0\}$ *is a laminar family.*

Proof. If $\mathcal{F}$ is not laminar, then a simple uncrossing technique can be applied. Suppose that $z_S, z_T > 0$, none of $S \cap T$, $S - T$, $T - S$ is empty. If $|S \cap T|$ is odd, then we can decrease z_S and z_T, and increase $z_{S\cap T}$ and $z_{S\cup T}$ by ε, A does not change, B increases which is a contradiction. If $|S \cap T|$ is even, then we can decrease z_S and z_T by ε, increase z_{S-T} and z_{T-S} by ε and increase λ_v^ϱ and λ_v^δ by ε if $v \in S \cap T$, A increases, which is a contradiction. □

For a graph G and a laminar family $\mathcal{F}$, let $G \times \mathcal{F}$ denote the graph obtained from G by *shrinking the maximal members* of $\mathcal{F}$. For any $S \in \mathcal{F}$, we let $\mathcal{F}[S]$ denote the subfamily of $\mathcal{F}$ consisting of all members of $\mathcal{F}$ properly contained in S. Thus $G[S] \times \mathcal{F}[S]$ is the graph obtained from $G[S]$ by shrinking the maximal elements of $\mathcal{F}$ properly contained in S. Let $A^=$ denote the set of *tight* edges i.e. $A^= = \{uv \in A : \lambda_u^\delta + \lambda_v^\varrho + \sum_{u,v\in S} z_S = c(uv)\}$ and $G^= = (V, A^=)$.

It is straightforward to see that $G^=[S]$ is weakly symmetric for any $S \in \mathcal{S}^\star$.

Claim 5.2. *For every* $S \in \mathcal{F}$*,* $G^=[S] \times \mathcal{F}[S]$ *is strongly connected. As a consequence,* $G^=[S]$ *is strongly connected for any* $S \in \mathcal{F}$*.*

Proof. If $G^=[S] \times \mathcal{F}[S]$ is not strongly connected, then it has a strongly connected component with an odd number of vertices, moreover, S has a partition into U^δ, U and U^ϱ, s.t. $|U|$ is odd and any strongly connected component of $G^=[S] \times \mathcal{F}[S]$ is contained in some of the partition classes and if $uv \in A^=$ connects two different classes, then one of the following three possibilities holds: $u \in U$ and $v \in U^\varrho$; or $u \in U^\delta$ and $v \in U$;

or $u \in U^\delta$ and $v \in U^\varrho$. Now we can modify the dual solution in the following way

$$z'_W = \begin{cases} z_W - \varepsilon & \text{if } W = S \\ z_W + \varepsilon & \text{if } W = U \\ z_W & \text{otherwise} \end{cases}$$
$$(\lambda_v^\varrho)' = \begin{cases} \lambda_v^\varrho + \varepsilon & \text{if } v \in U^\varrho \\ \lambda_v^\varrho & \text{otherwise} \end{cases}$$
$$(\lambda_v^\delta)' = \begin{cases} \lambda_v^\delta + \varepsilon & \text{if } v \in U^\delta \\ \lambda_v^\delta & \text{otherwise.} \end{cases}$$

This step yields a new dual solution with larger A which is a contradiction. □

Claim 5.3. *For any $S \in \mathcal{F}$, $G^=[S] \times \mathcal{F}[S]$ is nonbipartite (i.e. contains an odd cycle C, but by strong connectivity and by symmetry of $G^=[S]$, C is a symmetric odd cycle in $A^=$).*

Proof. For contradiction, we choose an S for which $G^=[S] \times \mathcal{F}[S]$ is bipartite. Thus S has a bipartition into S_1 and S_2 so that any member of $\mathcal{F}[S]$ is a subset of some S_i and $G^=[S_i] \times \mathcal{F}_i$ does not contain tight edge, where $\mathcal{F}_i$ denotes the family of maximal members of $\mathcal{F}[S]$ contained in S_i. We can assume moreover that $|S_1 - \cup_{S\in\mathcal{F}_1} S| + |\mathcal{F}_1| \le |S_2 - \cup_{S\in\mathcal{F}_2} S| + |\mathcal{F}_2|$. We apply the following modification of the dual solution

$$z'_W = \begin{cases} z_W + \varepsilon & \text{if } W \in \mathcal{F}_2 \\ z_W - \varepsilon & \text{if } W \in \mathcal{F}_1 \text{ or } W = S \\ z_W & \text{otherwise} \end{cases}$$
$$(\lambda_v^\varrho)' = \begin{cases} \lambda_v^\varrho + \varepsilon & \text{if } v \in S_1 \\ \lambda_v^\varrho & \text{otherwise} \end{cases}$$
$$(\lambda_v^\delta)' = \begin{cases} \lambda_v^\delta + \varepsilon & \text{if } v \in S_1 \\ \lambda_v^\delta & \text{otherwise.} \end{cases}$$

It can be verified that a dual solution is obtained, with larger A, which leads to a contradiction. □

The notation $a \equiv b$ is used if two integers a and b are congruent modulo k (i.e. $a-b$ is divisible by k). For the equivalence class of a under the modulo k equivalence relation we put $\overline{a}$.

Claim 5.4. *For any $S \in \mathcal{F}$, $\tilde{\lambda}_v^\varrho \equiv \tilde{\lambda}_u^\varrho$ and $\tilde{\lambda}_v^\delta \equiv \tilde{\lambda}_u^\delta$, whenever $u, v \in S$.*

Proof. For contradiction, we can choose a minimal $S \in \mathcal{F}$ for which the statement does not hold. Now $G^=[S]$ is symmetric, $G^=[S] \times \mathcal{F}[S]$ is strongly connected, symmetric and nonbipartite. So let uv and vu be edges of $G^=[S] \times \mathcal{F}[S]$ where $u, v \in S$. We let $\tilde{\lambda}_u^\delta \equiv a$, $\tilde{\lambda}_u^\varrho \equiv b$, $\tilde{\lambda}_v^\delta \equiv c$ and $\tilde{\lambda}_v^\varrho \equiv d$. Then for every pair of edges $u'v'$ and $v'u'$ of $G^=[S] \times \mathcal{F}[S]$ we have the same whenever $u', v' \in S$. Using the statement for the maximal members of $\mathcal{F}[S]$, we get that if S' is a maximal member of $\mathcal{F}[S]$, then $\tilde{\lambda}_u^\delta \equiv a$ and $\tilde{\lambda}_u^\varrho \equiv b$ $\forall u \in S'$; or $\tilde{\lambda}_u^\delta \equiv c$ and $\tilde{\lambda}_u^\varrho \equiv d$ $\forall u \in S'$. $G^=[S] \times \mathcal{F}[S]$ is nonbipartite, the symmetric odd cycle C proving this have vertices of the same type, say $\tilde{\lambda}^\delta \equiv c$ and

$\tilde{\lambda}^{\varrho} \equiv d$. The directed ear decomposition with initial cycle C, forces all the vertices of S to have the same. $\square$

We let $\mathcal{F}_{\max}$ denote the maximal members of $\mathcal{F}$. As a consequence we get the following.

Claim 5.5. *If $S \in \mathcal{F} - \mathcal{F}_{\max}$, then z_S is integer.*

From now on, the property that (A, B) is lexicographically as large as possible is not used further, the actual solution easily will be transformed to an integer one. To this end, the number of nonzero modulo k remainders of the variables is decreased sequentially (if any), and finally, the desired integer optimal dual solution is obtained.

It can be seen that if λ_v^{δ} and λ_v^{ϱ} are integers for every vertex v then we are at an integer solution. Thus, consider an integer $a \not\equiv 0$ s.t. $-a \equiv \tilde{\lambda}_v^{\delta}$ or $a \equiv \tilde{\lambda}_v^{\varrho}$ for some $v \in V$ and define $U^{\varrho} = \{v \in V : \tilde{\lambda}_v^{\varrho} \equiv a\}$, $U^{\delta} = \{v \in V : \tilde{\lambda}_v^{\delta} \equiv -a\}$, $Z^{\varrho} = \{S : S \in \mathcal{F}_{\max}, \forall v \in S \; \tilde{\lambda}_v^{\varrho} \equiv a$ and $\tilde{\lambda}_v^{\delta} \not\equiv -a\}$ and $Z^{\delta} = \{S : S \in \mathcal{F}_{\max}, \forall v \in S \; \tilde{\lambda}_v^{\delta} \equiv -a$ and $\tilde{\lambda}_v^{\varrho} \not\equiv a\}$. By symmetry we can suppose that $|U^{\varrho}| - \sum_{S \in Z^{\varrho}}(|S| - 1) \geq |U^{\delta}| - \sum_{S \in Z^{\delta}}(|S| - 1)$. Then we can apply the following change in the dual solution

$$(\lambda_v^{\varrho})' = \begin{cases} \lambda_v^{\varrho} - \eta & \text{if } v \in U^{\varrho} \\ \lambda_v^{\varrho} & \text{otherwise} \end{cases}$$

$$(\lambda_v^{\delta})' = \begin{cases} \lambda_v^{\delta} + \eta & \text{if } v \in U^{\delta} \\ \lambda_v^{\delta} & \text{otherwise} \end{cases}$$

$$z'_W = \begin{cases} z_W + \eta & \text{if } S \in Z^{\varrho} \\ z_W & \text{otherwise} \end{cases}$$

$$z'_W = \begin{cases} z_W - \eta & \text{if } S \in Z^{\delta} \\ z_W & \text{otherwise.} \end{cases}$$

η can be chosen to be an positive integer multiple of $\frac{1}{k}$ so that $\left|\left\{\overline{\tilde{\lambda}_v^{\varrho}} : v \in V\right\}\right| + \left|\left\{\overline{\tilde{\lambda}_v^{\delta}} : v \in V\right\}\right| + \left|\left\{\overline{\tilde{z}_S} : S \in \mathcal{F}\right\}\right|$ decreases, thus after a finite application of this step, we obtain an integer optimal dual solution. $\square$

Proof of Theorem 1.3. Consider the evenly symmetric pair (G, c), an integer optimal dual solution $w_v^{\varrho}, w_v^{\delta}, w_S^i$ and an optimal primal solution x of $\max\{cx : (1)-(2)-(3)-(4)\}$. Obviously, this gives a primal solution y of $\max\{c'y : (9)-(10)-(11)-(12)-(13)\}$ with objective function value $c'y = cx$. Next, we construct an integer optimal solution $(\mu_v^d, \mu_S^i, \mu_S^{i,\varrho}, \mu_S^{i,\delta})$ of the dual of $\max\{c'y : (9)-(10)-(11)-(12)-(13)\}$ from w with objective function value cx. This can be done easily, only the steps of the proof of Theorem 1.2 have to be copied.

It remains to prove the existence of an integer optimal dual solution with the special laminar property. To this end, we choose a dual solution among the integer optimal dual solutions so that $\sum_S(\mu_S^i + \mu_S^{i,\varrho} + \mu_S^{i,\delta})|S|(|V - S| + 1)$ is as small as possible.

Claim 5.6. *(14)-(15)-(16)-(17) hold.*

Proof. Suppose contrary that $\mu_S^i, \mu_T^i > 0$, none of $S \cap T$, $S - T$, $T - S$ is empty and set $\varepsilon = \min\{\mu_S^i, \mu_T^i\}$. If $|S \cap T|$ is odd, then we can decrease μ_S^i and μ_T^i, and increase

$\mu^i_{S\cap T}$ and $\mu^i_{S\cup T}$ by ε. If $|S\cap T|$ is even, then we can decrease μ^i_S and μ^i_T by ε, increase μ^i_{S-T} and μ^i_{T-S} by ε and increase μ^d_v by ε if $v \in S\cap T$.

Next, suppose that $\mu^i_S, \mu^{i,\varrho}_T > 0$, $S\cap T$, $S-T$ are nonempty, $\varepsilon = \min\{\mu^i_S, \mu^{i,\varrho}_T\}$. If $|S\cap T|$ is odd, then we can decrease μ^i_S and $\mu^{i,\varrho}_T$, and increase $\mu^i_{S\cap T}$ and $\mu^{i,\varrho}_{S\cup T}$ by ε. If $|S\cap T|$ is even, then we can decrease μ^i_S and $\mu^{i,\varrho}_T$ by ε, increase μ^i_{S-T} and $\mu^{i,\varrho}_{T-S}$ by ε and increase μ^d_v by ε if $v \in S\cap T$.

If $\mu^{i,\varrho}_S, \mu^{i,\varrho}_T > 0$ and $S\cap T$, $S-T$, $T-S$ are nonempty, then we can decrease $\mu^{i,\varrho}_S$ and $\mu^{i,\varrho}_T$, and increase $\mu^{i,\varrho}_{S\cap T}$ and $\mu^{i,\varrho}_{S\cup T}$ by ε.

Last, if $\mu^{i,\varrho}_S, \mu^{i,\delta}_T > 0$ and $S\cap T$ is nonempty, then we can decrease $\mu^{i,\varrho}_S$ and $\mu^{i,\delta}_T$, and increase $\mu^{i,\varrho}_{S-T}$ and $\mu^{i,\delta}_{T-S}$ by ε and increase μ^d_v by ε if $v \in S\cap T$.

The remaining cases are similar. In every case $\sum_{S\in\mathcal{G}} \mu^i_S |S|(|V-S|+1)$ decreases. $\square$

$\square$

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