On a generalization of the stable roommates problem

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Abstract

We consider two generalizations of the stable roommates problem: a) we allow parallel edges in the underlying graph and b) we study a problem with multiple partners. We reduce both problems to the classical stable roommates problem and describe an extension of Irving's algorithm that solves the generalized problem efficiently. We give a direct proof of a recent result on the structure of stable b-matchings as a by-product of the justification of the algorithm.

Keywords: stable marriage problem, stable roommates problem, Irving's algorithm

1 Introduction

In this note, we study two generalizations of the stable roommates problem of Gale and Shapley [6]. We have given a finite simple graph $G = (V, E)$ and for each vertex v of V there is a linear order $<_v$ on the set of neighbours $\Gamma(v)$ of v in G. It is convenient to denote the set of linear orders $<_v$ by $\mathcal{O}$, and we say that $(G, \mathcal{O})$ is an instance of the *stable roommates problem* (or SR, for short). A *stable matching of SR instance* $(G, \mathcal{O})$ is a set M of disjoint edges of E such that for each edge e of E either $e \in M$ or e has a vertex (say) v and M has an edge (say) m such that $m <_v e$ holds. Or, equivalently, a stable matching is a matching M of G that admits no *blocking edge*, that is, an edge e outside M such that $e <_v m$, whenever v is a vertex of e and m is an edge of M covering v. It is usual in economics and in game theory to call the vertices of G agents and to think about the linear order $<_v$ as the preference order of agent v on his/her possible partners. Then the notion of stability refers to a partnership situation in which no two agents exist that would prefer to quit their current partnership so as to become partners of each other.

Gale and Shapley in [6] have described the so called *stable marriage problem* (SM, for short), a special case of the stable roommates problem where the underlying graph G is bipartite. They exhibited an efficient algorithm that in this particular case finds a stable matching for any instance of SM. In contrast to this, a SR instance might admit no stable matching, whatsoever. The first efficient algorithm to decide whether for a given SR instance there exists a stable matching or not is due to Irving [8]. Later on, such algorithms were found by Feder [2] and Subramanian [14], and Tan and Hsueh [16]. The last mentioned algorithm could even find a compact characterization described earlier by Tan in [15] for the existence of a stable matching.

We shall describe a more general model (the stable activities problem, or SA), in which several different partnerships may be possible between two agents. The idea behind the name is that the parnership of agents is a common activity (say playing table tennis, chess, partnership in bridge, etc.) that they do on a clubmeeting and each agent has his/her own preference order on the activities depending on the type of activity and the partner. For the bipartite case, multiple edges might represent different marriage or labour contracts. A special case of the SA (a manpower scheduling

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problem for airplanes) was studied by Cechlárová and Ferková in [1]. Namely, they wanted to construct cabin crews from a pool of pilots by selecting captain-copilot pairs in such a way that no instability occurs. In [1], the authors exhibit a generalization of Irving's algorithm to solve the problem. In Section 2 of this note we reduce the SA to the SR.

In Section 3, we study another generalization (the stable multiple activities problem, or SMA) where agents may participate in more than one partnerships. In a stable situation, there is no activity such that each of the two participants would prefer to be involved in that activity either because he/she is involved in less activities than the maximum number or because he/she prefers that particular activity to some other activity he/she is involved in. Again, a special case of our general problem, called the *stable fixtures problem* was studied by Irving and Scott in [10], where a generalization of Irving's algorithm is derived. In Section 3, we reduce the SMA to the SR and indicate how it is possible to use any algorithm for the SR to solve the SMA. In Section 4, we briefly describe that extension of Irving's algorithm that we get by our method. In Section 5, we also exhibit a direct proof of a colouring property in stable b-matching problems observed recently by Fleiner [5, 4] and Teo *et. al.* [17]. We conclude by indicating other generalizations and modifications of the SR that we conjecture to be reducable to the original problem.

The idea of this paper is that we reduce the SMA to the SR by two constructions. While the b-expansion construction that we use for multiple partner matchings is folklore in the area of stable matchings, the other construction is very similar to the one that is used in [3] to prove that any SR instance can be reduced to another SR instance on a 3-colourable graph. (The same related construction is employed in [4] to extend Tan's characterization [15] on the existence of a stable matching.)

2 The stable activities problem

In this section, we shall consider the following generalized stable matching problem. Let finite graph $G = (V, E)$ be given and for each vertex v of V we have a linear order $\prec_v$ on the set of edges $E(v) = E(v, G)$ incident with v. If $\mathcal{O}$ denotes the set of linear orders $\prec_v$ (for $v \in V$) then $(G, \mathcal{O})$ is called an instance of the *stable activities problem*, or SA, for short. We say that subset F of E dominates edge e of E if there is a vertex v of V and an edge f of F such that $f \prec_v e$. Let $\mathcal{D}(F)$ denote the set of edges that are dominated by F. A *stable matching* of SA instance $(G, \mathcal{O})$ is a subset M of E with the property that $\mathcal{D}(M) = E \setminus M$.

It is useful to see that the SA is a generalization of the SR. Namely, each SR instance $(G, \mathcal{O})$ can be considered to be an instance $(G, \mathcal{O}^*)$ of the SA, if we let the linear order $<_v$ on $\Gamma(v)$ induce the linear order $\prec_v$ on $E(v)$ for each vertex v. Then if M is a stable matching of $(G, \mathcal{O})$, it is easy to check that $\mathcal{D}(M) = E \setminus M$ and so M is a stable matching of $(G, \mathcal{O}^*)$. On the other hand, if $\mathcal{D}(M) = E \setminus M$ then M is a set of disjoint edges, as otherwise some edge of M would be dominated by M. So M is a matching, and each edge outside M is dominated by M, showing that M is a stable matching of SR instance $(G, \mathcal{O})$.

In this section, we shall reduce the SA to the SR in such a way that a bipartite problem is reduced to a bipartite one. For this reason, fix SA instance $(G, \mathcal{O})$. We define a new SR instance $(G', \mathcal{O}')$ where $G' = (V', E')$ is a simple graph and $\mathcal{O}'$ is a set of linear orders $\prec'_{v'}$ for $v' \in V'$.

To define G', substitute each edge $e = uv$ of G by a 6-cycle with two hanging edges as in Figure 1. The linear orders on the stars are represented by the numbers on the edges around a vertex. That is, if edge e was the ith best in $\prec_u$ and jth best in $\prec_v$ then edge uu_0^e is ith best in $\prec'_u$ and edge vv_0^e is jth best in $\prec'_v$.

Theorem 2.1. *Let $(G, \mathcal{O})$ be an instance of the SA, and let SR instance $(G', \mathcal{O}')$ be constructed from $(G, \mathcal{O})$ as described above. There is a stable matching of $(G, \mathcal{O})$ if and only if there is a stable matching of $(G', \mathcal{O}')$. Moreover, any stable matching of $(G', \mathcal{O}')$ induces a stable matching of $(G, \mathcal{O})$ and each stable matching of $(G, \mathcal{O})$ can be induced by some stable matching of $(G', \mathcal{O}')$.*

Proof. Assume that M' is a stable matching of SR instance $(G', \mathcal{O}')$ and let $e = uv$ be an edge of G. As none of the edges of $u_1^e v_2^e, v_2^e v_0^e, v_0^e v_1^e, v_1^e u_2^e, u_2^e u_0^e, u_0^e u_1^e$ block M', we get that M' must cover all six

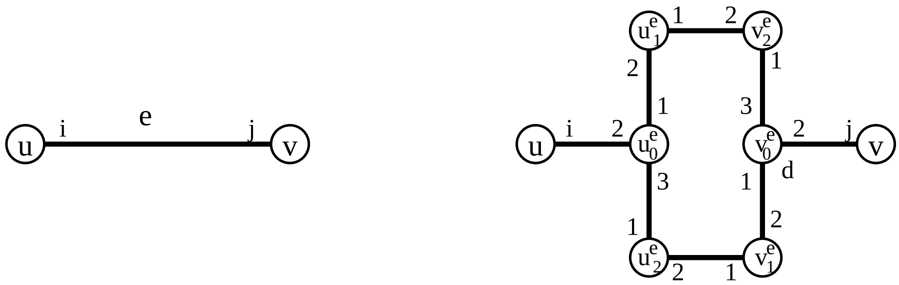

Figure 1: The gadget to be introduced along edge e for the definition of G'

vertices $u_1^e, v_2^e, u_0^e, v_0^e, u_2^e, v_1^e$. This means that either $uu_0^e, u_1^e v_2^e, u_2^e v_1^e, v_0^e v \in M'$, or $u_1^e v_2^e, u_0^e u_2^e, v_0^e v_1^e \in M'$ or $u_1^e u_0^e, v_2^e v_0^e, u_2^e v_1^e \in M'$.

Let $M := \{e = uv \in E : uu_0^e \in M'\}$. As $uu_0^e \in M'$ if and only if $vv_0^e \in M'$, M is a matching, i.e. $\mathcal{D}(M) \subseteq E \setminus M$. To prove the stability of M in SA instance $(G, \mathcal{O})$, we have to show that any edge $e = uv$ of G not in M is dominated by M. Let e be such an edge. As $uu_0^e \notin M'$, by symmetry we may assume that $u_1^e v_2^e, u_0^e u_2^e, v_0^e v_1^e \in M'$. As M' is dominating uu_0^e, there is an edge uu_0^f of M' with $uu_0^f \prec'_u uu_0^e$, that is, $f \in M$ and $f \prec_u e$. Hence stable matching M' of SR instance $(G', \mathcal{O}')$ induces stable matching M of SA instance $(G, \mathcal{O})$.

Now let us assume that M is a stable matching of SA instance $(G, \mathcal{O})$. Let $e = uv$ be an edge of E. Exactly one of the following three cases occurs. Case 1: edge e belongs to M. Case 2: e is dominated by M at u. Case 3: e is not dominated by M at u but e is dominated by M at v. In case 1 let $uu_0^e, u_1^e v_2^e, u_2^e v_1^e, v_0^e v \in M'$, in case 2 let $u_1^e v_2^e, u_0^e u_2^e, v_0^e v_1^e \in M'$ and in case 3 let $u_1^e u_0^e, v_2^e v_0^e, u_2^e v_1^e \in M'$. By these declarations along all edges of E, we define a subset M' of E'. As M is a matching, each vertex of V' is incident with at most one edge of M'. To see stability, we observe that for all three cases, any edge on 6-cycle $u_1^e u_0^e u_2^e v_1^e v_0^e v_2^e$ is either in M' or is dominated by M'. So we only have to check the implication that if edge uu_0^e is not in M' then uu_0^e is dominated by M'. If, in this situation, uu_0^e is not dominated by M' already at u_0^e, then for the corresponding edge e of G we must have that e is dominated by some edge m of M at u. But the definition of M' and linear order $\prec'_u$ implies that uu_0^e is dominated by edge uu_0^m of M' at u. □

The equivalence of stable matchings of G and G' described in the above proof allows us to decide whether a stable matching exists for an instance $(G, \mathcal{O})$ of the SA and to find a stable matching if one exists, once we have an algorithm that does the same for SR instance $(G', \mathcal{O}')$. So one can generalize his/her favourite stable matching algorithm for the SR to an algorithm for the SA by following what the algorithm does along the gadgets on Figure 1. We illustrate this in the Section 4, where we describe how Irving's algorithm can be generalized to the SA.

3 Multiple partner matchings

In this section, we consider another generalization of the stable roommates problem to the situation where each agent has a nonnegative integral quota on the number of possible partnerships. Formally, an instance of the *stable multiple activities problem* (or SMA, for short) is a triple $(G, \mathcal{O}, b)$, where $G = (V, E)$ is a finite graph, $\mathcal{O}$ is a set of linear orders $\prec_v$ for $v \in V$, $\prec_v$ being an order on set $E(v)$ of edges incident with v and $b : V \to \mathbb{N}$ is a quota function on the vertices. We say that subset F of E *b-dominates* edge e of E if there is a vertex v of e and different elements $f_1, f_2, \ldots, f_{b(v)}$ of F such that $f_i \prec_v e$ for $i = 1, 2, \ldots, b(v)$. We denote by $\mathcal{D}^b(F)$ the set of edges that are b-dominated by F. A subset M of E is a *stable b-matching* of SMA instance $(G, \mathcal{O}, b)$ if $\mathcal{D}^b(M) = E \setminus M$.

Equivalently, we can say that M is a stable b-matching, if $\mathcal{D}^b(M) \subseteq E \setminus M$ and $\mathcal{D}^b(M) \supseteq E \setminus M$ holds. The former condition means that M is a *b-matching*, that is, each vertex v of V is incident with at most $b(v)$ edges of M, while the latter condition says that for any edge e outside M, there is a vertex v and different edges $m_1, m_2, \dots m_{b(v)}$ of M such that $m_i \prec_v e$ for $i = 1, 2, \dots, b(v)$.

It is well known that there always exists a stable b-matching of SMA instance $(G, \mathcal{O}, b)$ whenever G is finite, bipartite and simple, and a straightforward extension of the Gale-Shapley algorithm finds one (see Gusfield and Irving [7] p. 54-55.) Actually, Gale and Shapley in [6] considered the so called college admission problem which is the special case of the SMA where G is simple and bipartite, and moreover the b-values on one colour class of G are uniformly 1. To generalize this feature of the college admission problem, we say that SMA instance $(G, \mathcal{O}, b)$ has the *many-to-one property* if $b(u) = 1$ or $b(v) = 1$ for each edge $e = uv$ of $E(G)$.

The college admission problem can be reduced to the stable marriage problem by the *b-expansion* of the underlying instance defined below (see Gusfield and Irving [7] p. 38 or Roth and Sotomayor [11] p. 131-132). Let $(G, \mathcal{O}, b)$ be an instance of the SMA. Define the instance $(G^b, \mathcal{O}^b)$ of the SA by

$$\begin{aligned} V(G^b) &:= \{v^i : v \in V(G) \text{ and } i = 1, 2, \dots, b(v)\} \\ E(G^b) &:= \{e^{i,j} = u^i v^j : e = uv \in E(G) \text{ and } u^i, v^j \in V(G^b)\} \\ \mathcal{O}^b &:= \{\prec_{v^i} : v(i) \in V(G^b)\}, \text{ where} \\ u^i v^j \prec_{u^i} u^i w^k &\iff \begin{cases} uv \prec_u uw \text{ or} \\ v = w \text{ and } j < k. \end{cases} \end{aligned}$$

That is, instead of each vertex v of G, we introduce $b(v)$ equivalent copies, and the linear orders are essentially inherited from the SMA instance. (We only have to take special care to define the linear order if the two adjacent edges originate from the same edge of G.) In the reduction of the college admission problem to the stable marriage problem, Gale and Shapley implicitly use the following fact.

Lemma 3.1. *Let $(G, \mathcal{O}, b)$ be an instance of the SMA with the many-to-one property. There is a stable b-matching of $(G, \mathcal{O}, b)$ if and only if there is a stable matching of $(G^b, \mathcal{O}^b)$. Moreover, any stable matching of $(G^b, \mathcal{O}^b)$ induces a stable b-matching of $(G, \mathcal{O}, b)$ and each stable b-matching of $(G, \mathcal{O}, b)$ can be induced by a stable matching of $(G^b, \mathcal{O}^b)$.*

Proof. Let M^b be a stable matching of $(G^b, \mathcal{O}^b)$. We claim that

$$M := \{e = uv \in E : e^{i,j} \in M^b \text{ for some } 1 \le i \le b(u) \text{ and } 1 \le j \le b(v)\} \tag{1}$$

is a stable b-matching of $(G, \mathcal{O}, b)$. As each v^i is incident with at most one edge of M^b, projection M of M^b is a b-matching, and we only have to check that each edge $e = uv$ of G outside M is b-dominated by M. By the many-to-one property, we may assume that $b(u) = 1$. If, for some $j = 1, 2, \dots, b(v)$, $e^{1,j}$ is dominated at u by edge $m^{1,k}$ of M^b then e is b-dominated at u by edge m of M. Otherwise, each $e^{1,j}$ is dominated at v by some edge ${m_j}^{j,k_j}$ of M^b. As M^b is a matching and $(G, \mathcal{O}, b)$ has the many-to-one property, edges $m_1, m_2, \dots, m_{b(v)}$ of M are different and $m_i \prec_v e$ holds. So e is b-dominated at v by M.

Next, let M be a stable b-matching of $(G, \mathcal{O}, b)$. As M is a b-matching, there exists a matching M^b of G^b such that (1) holds. If $e^{i,j}$ blocks M^b in $(G^b, \mathcal{O}^b)$ then e cannot be b-dominated by M, hence M^b is a stable matching. □

Note that although by Lemma 3.1, we can reduce the SMA with the many-to-one property to the SA, it is not true that the same b-expansion reduces the general SMA to the SA. The reason is that disjoint copies a fixed edge of SMA instance $(G, \mathcal{O}, b)$ may appear in a stable matching of $(G^b, \mathcal{O}^b)$. The main observation in this section is that an extension of the construction in Section 2 reduces the SMA to the SMA with the many-to-one property.

Definition 3.2. For SMA instance $(G, \mathcal{O}, b)$, define SMA instance $(G', \mathcal{O}', b')$ by defining G' and $\mathcal{O}'$ as in Section 2, and let $b'(v) := b(v)$ if v is a vertex of G and let $b'(v) := 1$ otherwise.

As different vertices of G are not adjacent in G', we have the following observation.

Observation 3.3. SMA instance $(G', \mathcal{O}', b')$ in Definition 3.2 has the many-to-one property.

The next lemma gives the reduction of the SMA to the SMA with the many-to-one property.

Lemma 3.4. *Let $(G, \mathcal{O}, b)$ be an instance of the SMA and $(G', \mathcal{O}', b')$ be the instance of the SMA in Definition 3.2. There is a stable b-matching of $(G, \mathcal{O}, b)$ if and only if there is a stable b'-matching of $(G', \mathcal{O}', b')$. Moreover, any b'-matching of $(G', \mathcal{O}', b')$ induces a stable b-matching of $(G, \mathcal{O}, b)$ and each stable b-matching of $(G, \mathcal{O}, b)$ can be induced by some stable b'-matching of $(G', \mathcal{O}', b')$.*

Proof. The proof of Theorem 2.1 can be used here mutatis mutandis with the following replacements: SR $\mapsto$ SMA with the many-to-one property; SA $\mapsto$ SMA; $(G, \mathcal{O}) \mapsto (G, \mathcal{O}, b)$; $(G', \mathcal{O}') \mapsto (G', \mathcal{O}', b')$; and matching $\mapsto$ b-matching, or b'-matching, whichever applies. □

If we put together Lemma 3.1, Observation 3.3 and Lemma 3.4 we get a reduction of the SMA to the SR.

Theorem 3.5. *Let $(G, \mathcal{O}, b)$ be an instance of the SMA. There is a stable b-matching of instance $(G, \mathcal{O}, b)$ if and only if there is a stable matching of instance $((G')^{b'}, (\mathcal{O}')^{b'})$ of the SR. Moreover, any stable matching of $((G')^{b'}, (\mathcal{O}')^{b'})$ induces a stable b-matching of $(G, \mathcal{O}, b)$ and each stable b-matching of $(G, \mathcal{O}, b)$ can be induced by a stable matching of $((G')^{b'}, (\mathcal{O}')^{b'})$.* □

Proof. We only have to check that $(G')^{b'}$ is simple, but it follows from the simplicity of G'. □

4 An extension of Irving's algorithm

We have seen in Section 3 that the SMA (just like its special case, the SA) can be reduced to the SR. Hence, if we have an algorithm to find a stable matching for finite simple graphs, then we can find a stable (b-)matching in a finite (multi)graph. (Note that our constructions preserve bipartiteness, hence for bipartite SMA instances we can even use the Gale-Shapley algorithm.) Due to the constructions in Section 3, the complexity of the algorithm for the SA will grow with a constant factor (in case of the SMA, the maximum value of b (which should be less than m) also comes in). If we want to avoid this constant factor or if we want an algorithm that works directly with the general problem then we can translate the algorithm on the construction back to the language of the original problem. By this, we can describe a new algorithm that is based on the old one, but solves a more general problem, namely the SA or the SMA instead of the SR.

To illustrate the above idea, we extend the first algorithm for finding stable matchings for the SR by Irving [8] to an algorithm for the SMA. A natural way to justify the correctness of the extended algorithm, would be to check what Irving's original algorithm does on $((G')^{b'}, (\mathcal{O}')^{b'})$. This certainly is a possible way to find the appropriate extension of Irving's algorithm to the SMA. But to justify the correctness of the extension is too tedious because of the complexity of the constructions, so, instead of that, we prove this from scratch.

The input of the extension of Irving's algorithm (the *EI algorithm*, for short) is an instance $(G, \mathcal{O}, b){=}(G_0, \mathcal{O}_0, b)$ of the SMA, and its output is either a conclusion that no stable b-matching of $(G, \mathcal{O}, b)$ exists or a stable b-matching of $(G, \mathcal{O}, b)$. The EI algorithm has two phases. In both phases, it transforms an instance $(G_i, \mathcal{O}_i, b)$ to another instance $(G_{i+1}, \mathcal{O}_{i+1}, b)$ in such a way that

$$G_{i+1} \text{ is a proper subgraph of } G_i, \tag{2}$$

$$\text{if } (G_i, \mathcal{O}_i, b) \text{ has a stable } b\text{-matching then } (G_{i+1}, \mathcal{O}_{i+1}, b) \text{ has one,} \tag{3}$$

$$\text{any stable } b\text{-matching of } (G_{i+1}, \mathcal{O}_{i+1}, b) \text{ is a stable } b\text{-matching of } (G_i, \mathcal{O}_i, b). \tag{4}$$

Define $B(u, G_i)$ as the set of the $b(u)$ best edges of G_i in $\prec_u$ and let $D(u, G_i)$ denote those edges f of G_i that can only be b-dominated at u:

$$\begin{aligned} B(u, G_i) &:= \{f = ux \in E(u, G_i) : |\{g \in E(u, G_i) : g \prec_u f\}| < b(u)\} \\ D(u, G_i) &:= \{f = ux \in E(u, G_i) : f \in B(x, G_i)\} \ . \end{aligned}$$

We say that SMA instance $(G_i, \mathcal{O}_i, b)$ has the *first-last property* if for each vertex u of G_i and for each edge $e \in E(G_i)$ incident with u

$$|\{f \in D(u, G_i) : f \prec_u e\}| < b(u) \tag{5}$$

holds. The name of the first-last property comes from the SR where it means that each agent is the last choice of his/her first choice. Consequently, each agent is the first choice of his/her last choice. This is generalized in the following lemma.

Lemma 4.1. *If SMA instance $(G_i, \mathcal{O}_i, b)$ has the first-last property then $|B(u,G_i)| = |D(u,G_i)|$ for each vertex u of G_i.*

Proof. Observe that $|D(u,G_i)| \le |B(u,G_i)|$ for each vertex u of G_i. This is because if $|B(u,G_i)| < b(u)$ then $D(u,G_i) \subseteq E(u,G_i) = B(u,G_i)$. Otherwise $|B(u,G_i)| = b(u)$, and $|D(u,G_i)| \le b(u)$ by the first-last property. By double counting, $\sum\{|B(u,G_i)| : u \in G_i\} = \sum\{|D(u,G_i)| : u \in G_i\}$, so Lemma 4.1 follows. □

If instance $(G_i, \mathcal{O}_i, b)$ does not have the first-last property then the algorithm makes a first phase step. That is, it finds an edge $e = uv$ that violates (5) and deletes e from G_i to get G_{i+1}. To construct $\mathcal{O}_{i+1}$, we restrict each order of $\mathcal{O}_i$ to the remaining edges. The motivation is that each agent selects his/her best possible partners according to his/her quota and proposes to them. If agent v receives at least $b(v)$ proposals than he/she will never be a partner of another agent who is worse than the $b(v)$th proposer of v.

Lemma 4.2. *If $(G_{i+1}, \mathcal{O}_{i+1}, b)$ is constructed from $(G_i, \mathcal{O}_i, b)$ by deleting e in a first phase step then properties (2–4) hold.*

Proof. Property (2) holds trivially for $(G_{i+1}, \mathcal{O}_{i+1}, b)$. Assume that M is a stable b-matching of $(G_i, \mathcal{O}_i, b)$. By the definition of the first phase step, there are different edges $f_1, f_2, \ldots, f_{b_u} \in D(u,G_i)$ such that $f_j \prec_u e$ for $j = 1, 2, \ldots, b(u)$. The definition of $D(u,G_i)$ implies that either all f_j's belong to M or M must b-dominate some f_j at u. In both cases, e is b-dominated by M at u, hence $e \notin M$, so M is a stable b-matching of $(G_{i+1}, \mathcal{O}_{i+1}, b)$. This proves (3).

Observe that after the deletion of e, we still have $f_j \in D(u, G_{i+1})$ for $j = 1, 2, \ldots, b(u)$. So the above argument applies to any stable b-matching M' of $(G_{i+1}, \mathcal{O}_{i+1}, b)$, and shows that M' b-dominates e. Hence M' is a stable b-matching of $(G_i, \mathcal{O}_i, b)$, justifying (4). □

The above proof justifies the following observation.

Observation 4.3. If edge e is deleted in the first phase of the EI algorithm then e does not belong to any stable b-matching of $(G_0, \mathcal{O}_0, b)$. □

If the first phase step cannot be executed, i.e. $(G_i, \mathcal{O}_i, b)$ has the first-last property then either the edges of graph G_i form a b-matching M which (by property (4)) is a stable b-matching of $(G_0, \mathcal{O}_0, b)$, or the algorithm makes a second phase step, that is, it finds and eliminates a so called rotation to get $(G_{i+1}, \mathcal{O}_{i+1}, b)$. A *rotation* of $(G_i, \mathcal{O}_i, b)$ is a pair of edge sets $R = (\{e_1, e_2, \ldots, e_k\}, \{f_1, f_2, \ldots, f_k\})$ such that $e_j = u_j v_j$, $f_j = u_j v_{j+1}$ (here and further on, addition in the indices is modulo k), e_j is maximal (i.e. worst) in $\prec_{v_j}$ and f_j is the $(b(u_j)+1)$st least (i.e. $(b(u_j)+1)$st best) element of $\prec_{u_j}$.

Lemma 4.4. *If $(G_i, \mathcal{O}_i, b)$ has the first-last property and $E(G_i)$ is not a b-matching then there exists a rotation R of $(G_i, \mathcal{O}_i, b)$ such that $d_{G_i}(v) > b(v)$ for each vertex v covered by R.*

Proof. Define arc set A on $V(G_i)$ by introducing an arc $\vec{a} = \vec{uv}$ if $e = uw$ is the $\prec_u$-maximal edge and $f = wv$ is the $(b(w)+1)$st least edge of $\prec_w$. Observe that if $d_{G_i}(u) > b(u)$ then for the $\prec_u$-maximal edge $e = uw$ we have $e \notin B(u,G_i)$, hence $e \notin D(w,G_i)$, that is, $d_{G_i}(w) > b(w)$ by Lemma 4.1. This means that whenever $d_{G_i}(u) > b(u)$ then there is an arc $\vec{a}$ of A going from u to some vertex v with $d_{G_i}(v) > b(v)$. As $E(G_i)$ is not a b-matching, A is nonempty and contains a cycle. This A-cycle defines a rotation in a natural way. □

If the EI algorithm finds a rotation $R = (\{e_1, e_2, \ldots, e_k\}, \{f_1, f_2, \ldots, f_k\})$ as in Lemma 4.4 such that $\{e_1, e_2, \ldots, e_k\} = \{f_1, f_2, \ldots, f_k\}$ then it concludes that no stable matching of instance $(G, \mathcal{O})$ exists. Otherwise the algorithm *eliminates* rotation R, i.e. it constructs G_{i+1} and $\mathcal{O}_{i+1}$ by deleting edges $\{e_1, e_2, \ldots, e_k\}$ from G_i and by restricting orders of $\mathcal{O}_i$ to the remaining edges, respectively. The following lemma justifies the correctness of the second phase step.

Lemma 4.5. *Let SMA instance $(G_i, \mathcal{O}_i, b)$ have the first-last property and let $R = (\{e_1, e_2, \dots, e_k\}, \{f_1, f_2, \dots, f_k\})$ be a rotation of $(G_i, \mathcal{O}_i, b)$ as in Lemma 4.4.*

A Sets $\{e_1, e_2, \dots, e_k\}$ and $\{f_1, f_2, \dots, f_k\}$ are disjoint or identical. In the latter case, $(G_i, \mathcal{O}_i, b)$ has no stable b-matching.

B If $(G_{i+1}, \mathcal{O}_{i+1}, b)$ is the SMA instance after the elimination of rotation R then properties (2–4) hold.

Proof of part A. Assume that $e_j = f_l$, for some j, l. As $d_{G_i}(v_j) > b(v_j)$ and e_j is the $\prec_{v_j}$-maximal edge, $e_j \in B(u_j, G_i)$ by the first-last property of $(G_i, \mathcal{O}_i, b)$. But $f_l \notin B(u_l, G_i)$, so $u_j \neq u_l$ hence $u_j = v_{l+1}$ and $u_l = v_j$ must hold. So $d_{G_i}(u_l) = b(u_l) + 1$ as e_j is maximal and f_l is the $(b(u_l) + 1)$st least element of $\prec_{u_l}$. In particular, $b(u_l) = |B(u_l, G_i)| = |D(u_l, G_i)|$ by Lemma 4.1. In other words, $|B(u_l, G_i) \cap D(u_l, G_i)| = b(u_l) - 1$, $D(u_l, G_i) \setminus B(u_l, G_i) = \{e_j\}$ and $B(u_l, G_i) \setminus D(u_l, G_i) = \{e\}$ for a unique edge e in $E(u_l, G_i)$.

By the definition of R, both edges e_l and f_{j-1} are incident with u_l and by the degree condition of Lemma 4.4, $e_l, f_{j-1} \notin D(u_l, G_i)$. This yields that $e_l = e = f_{j-1}$. That is, $e_j = f_l$ implies $e_l = f_{j-1}$, and by induction $\{e_1, e_2, \dots, e_k\} = \{f_1, f_2, \dots, f_k\}$ follows. From the structure of R, we also get that edges $e_1, e_2, \dots, e_k$ (in cyclic order $e_j, e_l, e_{j-1}, e_{l-1} \dots, e_{l+1}$) form an odd cycle C of G_i.

Now let M be a stable b-matching of $(G_i, \mathcal{O}_i, b)$. Pick some vertex (say $u_l = v_j$) of C. If $f \in E(u_l, G_i)$ and $e_j \neq f \neq e_l$ then $f \in B(u_l, G_i) \cap D(u_l, G_i)$, so $f \in M$. At most one of e_j and e_l can belong to M because $d_{G_i}(u_l) > b(u_l)$ and M is a b-matching. As $e_l \in D(u_l, G_i)$, if $e_l \notin M$ then M must b-dominate e_l at u_l, so $e_j \in M$ follows. We got that from two neighbouring edges of C exactly one belongs to C. As C is an odd cycle, this is impossible. That is, no stable b-matching of $(G_i, \mathcal{O}_i, b)$ exists. □

Proof of part B. Property (2) holds trivially. Let N be a stable b-matching of $(G_i, \mathcal{O}_i, b)$. Clearly, if N contains none of the e_j's then N is a stable b-matching of $(G_{i+1}, \mathcal{O}_{i+1}, b)$. Assume that $e_j \in N$. Then by Lemma 4.4, $d_{G_i}(v_j) > b(v_j)$, hence $|D(v_j, G_i)| = b(v_j)$. Moreover, $D(v_j, G_i) \subseteq N$ because e_j being maximal in $\prec_{v_j}$, no edge of G_i is b-dominated by N at v_j. By definition, $f_{j-1} \notin D(v_j, G_i)$, so $f_{j-1} \notin N$, implying that f_{j-1} is b-dominated by N at u_{j-1}. But f_{j-1} is preceeded by exactly $b(u_{j-1})$ edges in $\prec_{u_{j-1}}$, so all of those edges (in particular e_{j-1}) must belong to N. We got that $e_j \in N$ implies $e_{j-1} \in N \not\ni f_{j-1}$, hence $\{e_1, e_2, \dots e_k\} \subseteq N$ and $\{f_1, f_2, \dots f_k\}$ is disjoint from N.

Define $M := N \cup \{f_1, f_2, \dots f_k\} \setminus \{e_1, e_2, \dots e_k\}$. As the e_j's and the f_j's cover the same set of vertices, M is a b-matching. The above argument also shows that M contains the best $b(u_j)$ edges of $\prec_{u_j}$ in G_{i+1}. So if some edge e of G_{i+1} is b-dominated by N at u_j then e is still b-dominated by M at u_j. As N b-dominates no edge of G_i at v_j (because $e_j \in N$ is $\prec_{v_j}$-maximal), M is a stable matching of $(G_{i+1}, \mathcal{O}_{i+1}, b)$. This proves property (3).

Let M be a stable b-matching of $(G_{i+1}, \mathcal{O}_{i+1}, b)$ and $e_j = u_j v_j$ be an edge that has been deleted in the elimination of R. Assume that e_j is not b-dominated by M at v_j. By Lemma 4.4, $d_{G_i}(v_j) > b(v_j)$, so $|D(v_j, G_i)| = b(v_j)$ and $e_j \in D(v_j, G_i)$ by property (5) and Lemma 4.1. If $e \notin M$ for some edge $e \in D(v_j, G_i)$ other than e_j then e has to be b-dominated by M at v_j, and this means that e_j is also b-dominated by M at v_j, a contradiction. So $D(v_j, G_i) \setminus \{e_j\} \subseteq M$. This yields that $f_{j-1} \notin M$ because otherwise e_j would be b-dominated by M at v_j, as $f_{j-1} \notin D(v_j, G_i)$ and $f_{j-1} <_{v_j} e_j$. Hence f_{j-1} has to be b-dominated by M at its other vertex u_{j-1}. But this is impossible as after the deletion of e_{j-1} in the elimination of R, there are only $b(u_j) - 1$ edges left in G_{i+1} that preceed f_{j-1} in $\prec_{u_j}$. The contradiction shows that no e_j can block M, so M is a stable b-matching of $(G_i, \mathcal{O}_i, b)$. This justifies property (4). □

After each rotation elimination, the algorithm returns to the first phase. The following pseudocode summarizes the algorithm.

From Lemmas 4.2, 4.4 and 4.5 we get the following theorem.

Theorem 4.6. *Let $(G_0, \mathcal{O}_0, b)$ be an instance of the SMA and let m denote the number of edges of G_0. The EI algorithm finds a stable b-matching of $(G_0, \mathcal{O}_0, b)$ or concludes that no stable matching of $(G_0, \mathcal{O}_0, b)$ exists in $O(m^2)$ time.*

The EI algorithm
Input: SMA instance $(G_0, \mathcal{O}_0, b)$
Output: stable b-matching of $(G_0, \mathcal{O}_0, b)$, if one exists

```
begin
  i:=0
  while E(G_i) is not a b-matching do
    begin
      if(G_i, O_i, b) does not have the first-last property
        then find e ∈ E(G_i) that violates (5)
          delete e to get (G_{i+1}, O_{i+1}, b)
        else find rotation ({e_1, e_2, ..., e_k}, {f_1, f_2, ..., f_k})
          if{e_1, e_2, ..., e_k} = {f_1, f_2, ..., f_k}
            then STOP: No stable b-matching of (G_0, O_0, b) exists
            else delete {e_1, e_2, ..., e_k} to get (G_{i+1}, O_{i+1}, b)
          end if
      end if
      increase i by 1
    end
  STOP: Output stable b-matching E(G_i) of (G_0, O_0, b)
end
```

Table 1: Pseudocode of the extension of Irving's algorithm

Proof. In step i, the algorithm deletes at least one edge of G_i. So there are at most m steps. Finding all sets $B(u, G_i)$ takes $O(m)$ time. Each edge deletion costs constant time if edges are stored in doubly linked lists along orders $\prec_v$. We can update sets $B(u, G_i)$ after the deletion of k edges in $O(k)$ time. We can recognize the edge to be deleted in the first phase in $O(m)$ time. Finding a rotation according to Lemma 4.4 takes $O(\min(m, n))$ time, so altogether we get $O(m^2)$ for the complexity of the EI algorithm. □

5 A splitting property of the SMA

In this section, we show an interesting splitting property of the SMA. This property was observed recently independently by Fleiner [5] and Teo *et. al.* [17] for bipartite instances. Fleiner generalized the result to nonbipartite instances by reducing it to the bipartite case in [4]. Here, we give a direct proof of this as a by-product of the justification of the EI algorithm.

Let G be a graph and u, v be vertices of G. The *contraction* of u and v in G results in the graph in which u and v are merged to one vertex and all edges incident to u or v are joined to the supervertex instead. The contraction may result in parallel edges and loops, so the number of edges does not change after the operation. Graph G' is the *contraction-minor* of G if G' can be constructed from G in a sequence of contractions.

We say that SA instance $(G^{(b)}, \mathcal{O}^{(b)})$ is a *b-split* of SMA instance $(G, \mathcal{O}, b)$ if G is a contraction minor of $G^{(b)}$ such that each vertex v of G is contracted out of $b(v)$ vertices of $G^{(b)}$, moreover, the linear ordes of $\mathcal{O}^{(b)}$ are restricted orders of $\mathcal{O}$ on the supervertices. (As an abuse of notation, we shall call graph $G^{(b)}$ also the b-split of G.) The contraction construction gives a bijection between edges of G and that of $G^{(b)}$, hence for any subset F of edges of G we can talk about the corresponding subset $F^{(b)}$ of edges of $G^{(b)}$. An equivalent form of the following theorem is formulated for bipartite SMA instances in [5] and for bipartite SMA instances with the many-to-one property in [17].

Theorem 5.1. *Any SMA instance $(G, \mathcal{O}, b)$ has a b-split $(G^{(b)}, \mathcal{O}^{(b)})$ such that for any stable b-matching M of $(G, \mathcal{O}, b)$, $M^{(b)}$ is a stable matching of $(G^{(b)}, \mathcal{O}^{(b)})$.*

Proof. To prove the theorem, we shall construct a b-split $G^{(b)}$ of G in such a way that for any stable b-matching M of $(G, \mathcal{O}, b)$, $M^{(b)}$ is a matching of $G^{(b)}$. If we do so, then the induced $(G^{(b)}, \mathcal{O}^{(b)})$ will automatically satisfy the stability requirements of Theorem 5.1 because if edge e is b-dominated by M at v then the corresponding edge $e^{(b)}$ of $G^{(b)}$ will be dominated by $M^{(b)}$ at that vertex of $e^{(b)}$ that is contracted into supervertex v.

We may assume that $(G, \mathcal{O}, b)$ has a stable b-matching, otherwise Theorem 5.1 is trivial. So let the EI algorithm with input $(G_0, \mathcal{O}_0, b) = (G, \mathcal{O})$ terminate at $(G_k, \mathcal{O}_k, b)$ where $E(G_k)$ is a (stable) b-matching. Theorem 5.1 for $(G_k, \mathcal{O}_k, b)$ is trivial: we only have to find a b-split $G_k^{(b)}$ of G_k that has no adjacent edges. Then $G_k^{(b)}$ induces $(G_k^{(b)}, \mathcal{O}_k^{(b)})$. We shall construct $G^{(b)} = G_0^{(b)}$ by induction, i.e. we assume that we know b-split $(G_{i+1}^{(b)}, \mathcal{O}_{i+1}^{(b)})$ of $(G_{i+1}, \mathcal{O}_{i+1}, b)$ that satisfies Theorem 5.1 and we construct b-split $(G_i^{(b)}, \mathcal{O}_i^{(b)})$ of $(G_i, \mathcal{O}_i, b)$ with the same property.

If the EI algorithm has executed a first phase step and deleted $e = uv$ to get $(G_{i+1}, \mathcal{O}_{i+1}, b)$ from $(G_i, \mathcal{O}_i, b)$ then we construct $G_i^{(b)}$ by introducing an arbitrary edge $e^{(b)}$ in $G_{i+1}^{(b)}$ that corresponds to e via the contraction. By Observation 4.3, e is not in any stable b-matching of $(G, \mathcal{O}, b)$, so $(G_i^{(b)}, \mathcal{O}_i^{(b)})$ satisfies the requirements of Theorem 5.1.

If $(G_{i+1}, \mathcal{O}_{i+1}, b)$ is coming from $(G_i, \mathcal{O}_i, b)$ after the elimination of rotation $(\{e_1, e_2, \ldots, e_l\}$, $\{f_1, f_2, \ldots, f_l\})$ then there are edges $f_1^{(b)}, f_2^{(b)} \ldots, f_l^{(b)}$ of $G_{i+1}^{(b)}$ that correspond to $f_1, f_2, \ldots f_l$. To construct G_i, introduce edges $e_1^{(b)}, e_2^{(b)} \ldots, e_l^{(b)}$ in such a way that $e_1^{(b)}, f_1^{(b)}, e_2^{(b)}, f_2^{(b)}, \ldots, e_l^{(b)}, f_l^{(b)}$ form a circuit.

Let M be a stable b-matching of $(G_i, \mathcal{O}_i, b)$ and $M^{(b)}$ be the corresponding edge set in G_i. If M is disjoint from $\{e_1, e_2, \ldots, e_l\}$ then M is a stable b-matching of $(G_{i+1}, \mathcal{O}_{i+1}, b)$, hence $M^{(b)}$ is a matching by the induction hypothesis. Otherwise, by the proof of part B of Lemma 4.5, $\{e_1, e_2, \ldots, e_l\} \subseteq M$, and $N := N \cup \{f_1, f_2, \ldots f_k\} \setminus \{e_1, e_2, \ldots e_k\}$ is a stable b-matching of $(G_{i+1}, \mathcal{O}_{i+1}, b)$, hence $N^{(b)}$ is a matching by the induction hypothesis. To get $M^{(b)}$, we only have to switch along alternating cycle $e_1^{(b)}, f_1^{(b)}, e_2^{(b)}, f_2^{(b)}, \ldots, e_l^{(b)}, f_l^{(b)}$, so $M^{(b)}$ is a matching, too. □

Note that in [5, 4] it is indicated how it is possible to construct $(G^{(b)}, \mathcal{O}^{(b)})$ with the help of the Gale-Shapley algorithm (see also [17]). Note also, that from Theorem 5.1 one might come to the false impression that b-splitting gives a way to reduce the SMA to the SR. (Or, as b-splitting preserves bipartiteness, to reduce the bipartite stable b-matching problem to the stable marriage problem.) It is not difficult to construct a bipartite SMA instance $(G, \mathcal{O}, b)$ in such a way that any b-split $(G^{(b)}, \mathcal{O}^{(b)})$ that satisfies Theorem 5.1 has a stable matching $M^{(b)}$ that corresponds to a nonstable b-matching M of $(G, \mathcal{O}, b)$.

6 Conclusion: other extensions

In this work we have reduced the SMA to the SR. So if we have an algorithm that decides for any SR instance whether a stable matching exists and finds one if exists one, then we can use that algorithm to find a stable b-matching if exists for any SMA instance. We can also use our reduction to find a maximum weight stable b-matching as soon as we have an algorithm that finds a maximum weight stable matching for SR instances. That is, if we have a weight function $w : E \to \mathbb{R}$ on the edges then we can ask for a stable (b-)matching M with maximum weight $w(M) := \sum\{w(m) : m \in M\}$. Rothblum in [12] gave a compact characterization of the stable matching polytope for any SM instance, and standard tools of linear programming allow us to find a maximum weight stable matching efficiently. Using our reduction, we can find a maximum weight stable b-matching for any bipartite SMA instance.

There are other known extensions of the notion of a stable matching than the ones that we have studied above. If, instead of linear orders $\prec_v$, we have linear pseudoorders, i.e. indifference is allowed then we can define the notion of (b-)domination similarly as in Sections 2 and 3. This gives rise to the notion of a super stable (b-)matching. In [9], Irving and Manlove extend Irving's algorithm to find a super stable matching if it exists.

The notion of bistability is due to Weems [18] and it describes a *bistable matching* as a stable matching of both SR instances $(G, \mathcal{O})$ and $(G, \mathcal{O}^{-1})$, where $\mathcal{O}^{-1}$ denotes the set of reverse orders of orders of $\mathcal{O}$. Weems in [18] extends the Gale-Shapley algorithm to find a bistable matching for the bipartite case, if one exists. Based on a polyhedral approach, Sethuraman and Teo in [13] suggested an efficient way to find a bistable matching for any SR instance.

In the above work, we reduced some other extensions of the notion of the SR and the SM to the SR or to the SM. It is a natural question to ask whether the decision of the existence of a super stable (b-)matching or a bistable (b-)matching can also be reduced to the SR or to the SM (depending on the bipartiteness of the underlying graph). If one would succeed to do so, then it probably would become easy to handle combined notions like super bistable b-matchings as well.

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