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On the stable b-matching polytope

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Abstract

We characterize the bipartite stable b-matching polytope in terms of linear constraints. The stable b-matching polytope is the convex hull of the characteristic vectors of stable b-matchings, that is, of stable assignments of a two-sided multiple partner matching model. Our proof uses the comparability theorem of Roth and Sotomayor [13] and follows a similar line as Rothblum did in [14] for the stable matching polytope.

Keywords: stable matching, polytope, linear description

1 Introduction

In this paper, we consider a well-known generalization of the stable marriage problem of Gale and Shapley [9]. Their stable marriage model consists of finitely many men and women with strict preferences on the possible marriage partners. A *stable marriage scheme* is a matching of the marriage graph so that no man and woman exists that mutually prefer each other to their eventual marriage partner. Gale and Shapley have proved that the so called deferred acceptance algorithm always finds such a marriage scheme for any preference profiles of the agents. In [9], Gale and Shapley also considered the stable admissions problem, where one side of the market is a set of colleges, the other side is a set of students. Here again, each agent has a strict preference order on the acceptable members of the other side of the market, moreover each college has a quota for admissible students. In the *stable admissions problem*, we are looking for a stable market situation, that is, for a set of college-student pairs so that each student is in at most one pair, no college turns up in more pairs than its quota and there exists no college c and student s so that both s and c can improve on their situation if c admits s (and possibly quit other admissions). It turned out

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that a natural modification of the deferred acceptance algorithm always finds a stable assignment.

Here, we consider the stable b-matching problem, a generalization of the stable admissions problem where each agent in both sides of the market has a quota. We shall give a linear description for the bipartite stable b-matching polytope. Formally, a *bipartite preference system* is a pair $(G, \mathcal{O})$ where $G = (U \cup V, E)$ is a finite bipartite graph with bipartition (U, V), and $\mathcal{O} = \{\leq_z : z \in U \cup V\}$ is a family of linear orders, $\leq_z$ being an order on the set $D(z)$ of edges incident with the vertex z. For a quota function $b : U \cup V \to \mathbb{N}$, a *stable b-matching* of bipartite preference model $(G, \mathcal{O})$ is a subset M of the edge set E such that

1. each agent z is incident with at most $b(z)$ edges of M, that is, $d_M(z) \leq b(z)$ for any $z \in U \cup V$, or in other words, M is a *b-matching* and
2. M is *dominating*, i.e. any edge $e \in E$ outside M has an end node z such that z is incident with $b(z)$ edges of M and for any edge m of M incident with z we have $m >_z e$.

(Here, $d_M(z)$ denotes the number of edges of M that is incident with z.) A stable $\mathbf{1}$-matching is called a *stable matching*. We denote by $P^b(G, \mathcal{O})$ the convex hull of characteristic vectors in $\mathbb{R}^E$ of stable b-matchings of bipartite preference system $(G, \mathcal{O})$. It is well-known that in any bipartite preference system there exists a stable b-matching and a standard modification of the deferred acceptance algorithm finds one. Actually, it finds an optimal one, that is, any agent of U gets the best partners he can have in a stable stable b-matching and agents of V receive the worst possible partners or vice versa.

The area of stable matchings has become quite popular after the results of Gale and Shapley. From the different generalizations and approaches, we focus on the ones that connect the area to linear programming. Vande Vate seems to be the first who started this direction by giving a linear description of the convex hull of the characteristic vectors of stable matchings in [16].

Theorem 1.1 (Vande Vate '89 [16]). *Let $(G, \mathcal{O})$ be a bipartite preference system with $|U| = |V|$ and $E = U \times V$. Then*

$$P^{\mathbf{1}}(G, \mathcal{O}) = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ x(D(z)) = 1\ \forall\ z \in U \cup V,\ x(\psi(e)) \leq 1\ \forall\ e \in E\}$$

where $\psi(uv) := \{f \in E : f \leq_u uv \text{ or } f \leq_v uv\}$. □

Rothblum gave a shorter proof of a modified description for a more general problem in [14], and his proof was further simplified by Roth *et al.* in [11].

Theorem 1.2 (Rothblum '92 [14]). *Let $(G, \mathcal{O})$ be a bipartite preference system. Then*

$$P^{\mathbf{1}}(G, \mathcal{O}) = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ x(D(z)) \leq 1\ \forall\ z \in U \cup V,\ x(\phi(e)) \geq 1\ \forall\ e \in E\}$$

where $\phi(uv) := \{f \in E : f \geq_u uv \text{ or } f \geq_v uv\}$. □

Based on these results, standard tools of linear programming allow us to find a maximum weight stable matching in polynomial time. Eventually, a linear programming approach has been developed to the theory of stable matchings by Abeledo, Blum, Roth, Rothblum, Sethuraman, Teo and others (see [3, 4, 1, 2, 11, 15]).

But these results handle only the stable matching problem and do not say much about stable b-matchings. The following theorem of Baïou and Balinski [5] is an exception as it gives a linear description of the stable admissions polytope and generalizes Theorem 1.2.

Theorem 1.3 (Baïou and Balinski '99 [5]). *Let $(G, \mathcal{O})$ be a bipartite preference system and $b : U \cup V \to \mathbb{N}$ be a quota function so that $b(u) = 1$ for all nodes u of U. Then*

$$\begin{aligned} P^b(G,\mathcal{O}) = \{x \in \mathbb{R}^E : & x \geq \mathbf{0}, \\ & x(D(u)) \leq 1 \ \forall\ u \in U,\ x(D(v)) \leq b(v) \ \forall\ v \in V, \\ & x(C(v, u_1, u_2, \dots, u_{b(v)})) \geq b(v) \\ & \textit{for all combs } C(v, u_1, u_2, \dots, u_{b(v)})\}\ , \end{aligned}$$

where a comb is defined for $v \in V$ and $vu_1 <_v vu_2 <_v \dots <_v vu_{b(v)}$ as

$$\begin{aligned} C(v, u_1, u_2, \dots, u_{b(v)}) = & \{uv \in E : uv \geq_v u_1 v\} \cup \\ & \cup \{u_i v' \in E : u_i v' \geq_{u_i} u_i v \textit{ for some } i = 1, 2, \dots, b(v)\}\ . \quad \square \end{aligned}$$

Because of the comb constraints, the above characterization can consist of $\Omega(n^B)$ linear inequalities, where n is the number of "colleges" and B is the maximum of all quotas. But in spite of the exponential number of constraints, it is still possible to find an optimum weight stable admission by the ellipsoid method, using the separation algorithm of Baïou and Balinski. Note however, that the main significance of Theorem 1.3 lies in the the description of the polytope itself and not in the fact that we can optimize over stable admissions. This is because already Theorem 1.2 is good enough to find a maximum weight stable admissions scheme by the well known node splitting construction described in Lemma 5.6 in [13]. In the node splitted matching model, Rothblum's description characterizes a stable matching polytope P so that the stable admissions polytope is a projection of P. Moreover, the related LP needs only $O(n + mB)$ constraints, where n is the number of agents, m is the number of possible admissions and B is the maximum of the quotas. Note also that the node splitting construction does not seem to be sufficient to optimize over stable b-matchings with Theorem 1.2.

In [8, 6], Fleiner has described an approach to the theory of stable matchings based on the lattice theoretic fixed point theorem of Tarski. He also proved a generalization of the stable marriage theorem in a matroid model. Further, by using the theory of blocking polyhedra and lattice polyhedra, he gave a linear description of the related matroid-kernel polytope. If this matroid theorem is applied to the special case of the stable b-matching problem then it gives the following linear description of the stable b-matching polytope.

Theorem 1.4 (Fleiner 2000 [8, 7]). *Let $(G, \mathcal{O})$ be a bipartite preference system and $b : U \cup V \to \mathbb{N}$ be a quota function. Then*

$$P^b(G, \mathcal{O}) = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ x(A) \leq 1\ \forall A \in \mathcal{A},\ x(B) \geq 1\ \forall\ B \in \mathcal{B}\}$$

where

$$\begin{aligned} \mathcal{A} &:= \{A \subseteq E : |A \cap M| \leq 1 \text{ for any stable } b\text{-matching } M\} \text{ and} \\ \mathcal{B} &:= \{B \subseteq E : B \cap M \neq \emptyset \text{ for any stable } b\text{-matching } M\}\ . \quad \square \end{aligned}$$

Note that the constraints in Theorem 1.2 are special cases of the ones in Theorem 1.4. However, there are two important differences between Theorem 1.4 and the above earlier results. A shortage of Fleiner's description is that it uses implicit constraints, hence if it is specialized to the stable marriage problem, it might require more constraints than Rothblum's explicit description. (This is why Theorem 1.4 is rather an extension than a generalization of Theorem 1.2.) A positive feature of Fleiner's result is that unlike Baïou and Balinski, both the matrix and the right hand side vector in the description contains only 0 and 1 entries.

In the next section, we generalize Theorem 1.2 to the stable b-matching polytope. To this end, we use the Comparability Theorem of Roth and Sotomayor [13] and then we follow a similar line as the proof of Rothblum in [14]. The Comparability Theorem states that in a fixed bipartite preference system, if two stable b-matchings are different for some agent, then this agent strictly prefers one b-matching to the other.

Theorem 1.5 (Roth and Sotomayor '89 [12]). *Let M and M' be two stable b-matchings for bipartite preference system $(G, \mathcal{O})$, let z be a vertex of graph G and $M_z := M \cap D(z)$ and $M'_z := M' \cap D(z)$. If $M_z \neq M'_z$ then $|M_z| = |M'_z| = b(z)$ and the $b(z)$ $<_z$-best edges of $M_z \cup M'_z$ are either M_z or M'_z.*

Actually, in [12, 13] Roth and Sotomayor proves the above theorem only in case of the stable admissions problem. Gale includes a sketch of the proof of the general theorem in [10]. For sake of self containedness, we give a short proof of this result.

Proof. For any edge m of $M \setminus M'$, there is an end node w of m with domination property 2. Orient m to w. Similarly, we can orient all edges of $M' \setminus M$. Property 2. implies that if an edge of M is oriented to some vertex w then no edge of M' can be oriented to w. This means that connected components of $M \triangledown M'$ are such that any edge of $M \setminus M'$ is oriented to U and all edges of $M' \setminus M$ are directed to V of vice versa. Hence for any vertex z of G where M and M' does not coincide, the least preferred edges are either the edges of $M \setminus M'$ or the edges of $M' \setminus M$. Theorem 1.5 follows. $\square$

2 The stable b-matching polytope

In this section, we formulate and prove our main result. We shall use an immediate corollary of Theorem 1.5 that seems to be unobserved so far.

Corollary 2.1. *Let $(G, \mathcal{O})$ be a bipartite preference system and $b : U \cup V \to \mathbb{N}$ be a quota function. Then for any vertex z of G, there is a partition of $D(z)$ into $b(z)$ parts $D_1(z), D_2(z), \ldots, D_{b(z)}(z)$ so that $|M \cap D_i(z)| \le 1$ for any stable b-matching M and any integer i with $1 \le i \le b(z)$.*

Proof. Fix vertex z. Let $D'(z)$ be the set of those edges of $D(z)$ that can appear in some stable b-matching. Note that it is enough to partition the set of the e_i's into $b(z)$ parts with the required property, as we can put edges of $D(z) \setminus D'(z)$ into any of the parts without violating the Corollary 2.1.

For a stable b-matching M, let $M_z := M \cap D(z)$ denote the set of M-edges incident with z. If M and M' are stable b-matchings and the $<_z$ minimal edge of M_z is the same as the $<_z$-minimal edge of M' then Theorem 1.5 yields that $M_z = M'_z$. Hence there is a linear order $M_z^1 \prec M_z^2 \prec \ldots$ on possible sets M_z so that for $i < j$ the best $b(z)$ edges of $M_z^i \cup M_z^j$ are the edges of M_z^j. By induction on k, we show how to partition $\bigcup_{i=1}^k M_z^i$ into $b(z)$ parts with the required property.

As M_z^1 is a b-matching, we can partition M_z^1 into $b(z)$ (possibly empty) parts $D_1^1(z), D_2^1(z), \ldots, D_{b(z)}^1(z)$, so that each part contains at most one edge. So any stable b-matching intersects any $D_i^1(z)$ in at most one edge. Assume that we have a partition $D_1^k(z), D_2^k(z), \ldots, D_{b(z)}^k(z)$ of $\bigcup_{i=1}^k M_z^i$ with this property. To construct partition $D_1^{k+1}(z), D_2^{k+1}(z), \ldots, D_{b(z)}^{k+1}(z)$ of $\bigcup_{i=1}^{k+1} M_z^i$, we keep the old parts $D_i^k(z)$ and assign the new edges of $M_z^{k+1} \setminus \bigcup_{i=1}^k M_z^i$ into parts so that the required property is preserved.

By Theorem 1.5, $|M_z^{k+1} \setminus M_z^k| = |M_z^k \setminus M_z^{k+1}|$, moreover no edge of $M_z^{k+1} \setminus M_z^k$ is present in M_z^i for $i \le k$. So we can distribute the edges of $M_z^{k+1} \setminus M_z^k$ into the parts of the edges of $M_z^k \setminus M_z^{k+1}$ in such a way that we put exactly one edge to each of the parts. By this, we have partitioned $\bigcup_{i=1}^{k+1} M_z^i$ into $b(z)$ parts $D_1^{k+1}(z), D_2^{k+1}(z), \ldots, D_{b(z)}^{k+1}(z)$ so that any M_z^l intersects any part $D_j^{k+1}(z)$ in at most one edge for $l \le k+1$ and $1 \le j \le b(z)$. Also by Theorem 1.5, if $l > k+1$ then $M_z^l \cap \bigcup_{i=1}^{k+1} M_z^i = M_z^{k+1} \cap \bigcup_{i=1}^{k+1} M_z^i$, hence the new partition satisfies the induction hypothesis. $\square$

Although Corollary 2.1 claims only the existence of the partitions of $D(z)$, by using the deferred acceptance algorithm, we can efficiently construct them. The above proof shows that if we construct all the M_z^i's then finding a feasible partition is straightforward. By the deferred acceptance algorithm, we can find a z-worst stable b-matching M^1 so that $M^1 \cap D(z) = M_z^1$. If $|M_z^1| < b(z)$ then we have found all M_v^i's by Theorem 1.5. Otherwise assume that we have found $M_z^1, M_z^2 \ldots M_z^k$. Let e^k be the least preferred edge of M_z^k. Delete e^k and all edges that are less preferred than e^k by z. Construct another z-worst stable b-matching M^{k+1} in the reduced bipartite preference system. If $|M^{k+1} \cap D(z)| = b(z)$, then M^{k+1} is a stable b-matching of the original preference system, as well. That is, $M^{k+1} \cap D(z) = M_z^{k+1}$.

If $M_z^1, M_z^2 \ldots M_z^k$ did not exhaust all the M_z^i's then there is a stable b-matching M^{k+1} of the original preference system that is also a stable b-matching in the reduced one so that $M_z^{k+1} = M^{k+1} \cap D(z)$. This means by Theorem 1.5 that any stable b-matching in the reduced system contains $b(z)$ edges of $D(z)$. Hence, if $|M^{k+1} \cap D(z)| < b(z)$, then we see that we have found all the M_z^i's. This argument shows that at most

$|D(z)|$ executions of the deferred acceptance algorithm finds all M_z^i's, and a partition described in Corollary 2.1.

Theorem 2.2. *Let $(G, \mathcal{O})$ be a bipartite preference system and $b : U \cup V \to \mathbb{N}$ be a quota function. Then for any vertex z of G there is a partition of $D(z)$ into parts $D_1(z), D_2(z), \dots, D_{b(z)}(z)$ such that*

$$\begin{aligned} P^b(G, \mathcal{O}) = \{x \in \mathbb{R}^E : & x \geq \mathbf{0}, \\ & x(D_i(z)) \leq 1 \ \forall\ z \in U \cup V, 1 \leq i \leq b(z), \\ & x(\phi_{i,j}(uv)) \geq 1 \ \forall uv \in E, 1 \leq i \leq b(u), 1 \leq j \leq b(v)\}\ , \end{aligned} \tag{1}$$

where

$$\phi_{i,j}(uv) := \{uv\} \cup \{uv' : uv' >_u uv, v' \in D_i(u)\} \cup \{u'v : u'v >_v uv, u' \in D_j(v)\}\ .$$

Note that Theorem 2.2 is a strengthening of Theorem 1.4 and a genuine generalization of Theorem 1.2. Although Theorem 2.2 is not a generalization of Theorem 1.3, it describes a more general polytope by only $O(mB^2)$ constraints. (Here, m is the number of edges of G, and B is the maximum value of b).

Proof of Theorem 2.2. Choose partitions of $D(z)$ for each vertex z of G as in Corollary 2.1. By this choice, the characteristic vector of any stable b-matching M will satisfy the right hand side of (1): a characteristic vector is nonnegative; M contains at most one edge of $D_i(z)$; and any edge e either belongs to M or it has an end vertex z so that for $1 \leq k \leq b(z)$ each $D_k(z)$ will contain an edge m of M with $e <_z m$. Hence the polyhedron described on the right hand side of (1) contains $P^b(G, \mathcal{O})$.

To justify the opposite containment, we shall decompose a vector x satisfying the right hand side of (1) into a convex combination of characteristic vectors of stable b-matchings. To do this, we need the following lemma.

Lemma 2.3. *Let x be a vector satisfying the right hand side of (1) and $uv \in D_i(u) \cap D_j(v)$. Then edge uv is the most preferred edge in $D_i(u) \cap \operatorname{supp}(x)$ if and only if uv is the least preferred edge of $D_j(v) \cap \operatorname{supp}(x)$.*

Proof. From $x(\phi_{i.j}(uv)) \geq 1$ and $x(D_j(v)) \leq 1$ it follows that if uv is the most preferred edge of $D_i(u) \cap \operatorname{supp}(x)$ then uv is the least preferred edge of $D_j(v) \cap \operatorname{supp}(x)$. This means that $\operatorname{supp}(x)$ intersects at least as many $D_j(v)$'s for $v \in V$ as many $D_i(u)$'s for $u \in U$. But the same argument holds if we exchange the role of U and V, thus $\operatorname{supp}(x)$ intersects exactly as many $D_j(v)$'s as many $D_i(u)$'s. So the set of most preferred edges of $D_i(u) \cap \operatorname{supp}(x)$ for $u \in U$ and $1 \leq i \leq b(u)$ is the same as the set of least preferred edges of $D_j(v) \cap \operatorname{supp}(x)$ for $v \in V$ and $1 \leq j \leq b(v)$. □

Let x be a vector satisfying the right hand side of (1) and let M consist of the most preferred edges of sets $D_i(u) \cap \operatorname{supp}(x)$ for $u \in U$ and $1 \leq i \leq b(u)$. Denote amount $\min\{x(m) : m \in M\}$ by δ. As $x - \delta\chi^M$ has a strictly smaller support than x has, to finish the proof by induction on $|\operatorname{supp}(x)|$, it is enough to show that M is a stable b-matching and that $x' := \frac{1}{1-\delta}(x - \delta\chi^M)$ satisfies the constraints in the right hand side of (1).

First we prove that M is a stable b-matching. By Lemma 2.3, M contains at most one edge from each $D_k(z)$ for $z \in U \cap V$ and $1 \leq k \leq b(z)$, hence M is indeed a b-matching.

For domination, fix edge uv. If property 2. does not hold for uv then there must be an integer i so that $1 \leq i \leq b(u)$ and there is no edge m of $M \cap D_i(u)$ with $m \geq_u uv$. If $x(uv) > 0$ then choose j so that $uv \in D_j(v)$. From $x(\phi_{i,j}(uv)) \geq 1$ and $x(D_j(v)) \leq 1$, it follows that uv is the least preferred edge by v from $D_j(v) \cap \text{supp}(x)$. Then by Lemma 2.3, uv is selected into M, a contradiction. Otherwise $x(uv) = 0$. For any $1 \leq j \leq b(v)$, we have $x(\phi_{i,j}(uv)) \geq 1$ and $x(D_j(v)) \leq 1$. This implies that $x(\{e \in D_j(v) : e >_v uv\}) = 1$ so set $D_j(v) \cap \text{supp}(x)$ is not empty. Let m_j be the least preferred edge by v of $D_j(v) \cap \text{supp}(x)$. As $m_j >_v uv$ for all j, property 2. holds again for uv.

It remains to check that x' satisfies the constraints of (1). By our choice, $x \geq \mathbf{0}$ trivially holds. As we have chosen one edge from each nonempty $D_k(z) \cap \text{supp}(x)$ for all vertices z of G, condition $x'(D_i(z)) \leq 1$ holds for all vertices z. For the third type constraint, pick an edge uv of G and indices i, j with $1 \leq i \leq b(u)$ and $1 \leq j \leq b(v)$. If $u'v <_v uv$ for the $<_v$-worst edge $u'v$ of $\text{supp}(x) \cap D_j(v)$ then

$$x'(\phi_{i,j}(uv)) \geq \frac{1}{1-\delta}(x(\phi_{i,j}(uv)) - \delta) \geq \frac{1-\delta}{1-\delta} = 1$$

holds. Otherwise let $u'v \in D_k(u')$. By Lemma 2.3, $u'v$ is the $<_{u'}$-best edge of $\text{supp}(x) \cap D_k(u')$, so

$$\begin{aligned} x'(\phi_{i,j}(uv)) \geq & \frac{1}{1-\delta}(x(D_j(v) \cap \{e \in E : e \geq_v uv\}) - \delta) = \\ = & \frac{1}{1-\delta}(x(\phi_{i,k}(m)) - \delta) \geq \frac{1-\delta}{1-\delta} = 1 \ . \end{aligned}$$

□

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References

[1] Hernán Abeledo and Yosef Blum. Stable matchings and linear programming. *Linear Algebra Appl.*, 245:321–333, 1996.

[2] Hernán G. Abeledo, Yosef Blum, and Uriel G. Rothblum. Canonical monotone decompositions of fractional stable matchings. *Internat. J. Game Theory*, 25(2):161–176, 1996.

[3] Hernán G. Abeledo and Uriel G. Rothblum. Stable matchings and linear inequalities. *Discrete Appl. Math.*, 54(1):1–27, 1994.

[4] Hernán G. Abeledo and Uriel G. Rothblum. Courtship and linear programming. *Linear Algebra Appl.*, 216:111–124, 1995.

[5] Mourad Baïou and Michel Balinski. The stable admissions polytope. *Math. Program.*, 87(3, Ser. A):427–439, 2000.

[6] Tamás Fleiner. A fixed-point approach to stable matchings and some applications. Technical report, EGRES report TR-2001-1, March 2001. `http://www.cs.elte.hu/egres`.

[7] Tamás Fleiner. A matroid generalization of the stable matching polytope. In Bert Gerards and Karen Aardal, editors, *Proceedings of the 8th International Conference (IPCO VIII) held in Utrecht, June 13–15, 2001*, pages 105–114, Berlin, 2001. Springer-Verlag.

[8] Tamás Fleiner. Stable and crossing structures, August, 2000. PhD dissertation, `http://www.renyi.hu/~fleiner`.

[9] D. Gale and L.S. Shapley. College admissions and stability of marriage. *Amer. Math. Monthly*, 69(1):9–15, 1962.

[10] David Gale. The two-sided matching problem. Origin, development and current issues. *Int. Game Theory Rev.*, 3(2-3):237–252, 2001. Special issue on operations research and game theory with economic and industrial applications (Chennai, 2000).

[11] Alvin E. Roth, Uriel G. Rothblum, and John H. Vande Vate. Stable matchings, optimal assignments, and linear programming. *Math. Oper. Res.*, 18(4):803–828, 1993.

[12] Alvin E. Roth and Marilda Sotomayor. The college admissions problem revisited. *Econometrica*, 57(3):559–570, 1989.

[13] Alvin E. Roth and Marilda A. Oliveira Sotomayor. *Two-sided matching*. Cambridge University Press, Cambridge, 1990. A study in game-theoretic modeling and analysis, With a foreword by Robert Aumann.

[14] Uriel G. Rothblum. Characterization of stable matchings as extreme points of a polytope. *Math. Programming*, 54(1, Ser. A):57–67, 1992.

[15] Chung-Piaw Teo and Jay Sethuraman. The geometry of fractional stable matchings and its applications. *Math. Oper. Res.*, 23(4):874–891, 1998.

[16] John H. Vande Vate. Linear programming brings marital bliss. *Oper. Res. Lett.*, 8(3):147–153, 1989.