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Non-Separable Detachments of Graphs

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Non-Separable Detachments of Graphs

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Dedicated to the memory of Crispin Nash-Williams.

Abstract

Let $G = (V, E)$ be a graph and $r : V \to Z_+$. An *r-detachment* of G is a graph H obtained by 'splitting' each vertex $v \in V$ into $r(v)$ vertices, called the *pieces* of v in H. Every edge $uv \in E$ corresponds to an edge of H connecting some piece of u to some piece of v. An *r-degree specification* for G is a function f on V, such that, for each vertex $v \in V$, $f(v)$ is a partition of $d(v)$ into $r(v)$ positive integers. An *f-detachment* of G is an r-detachment H in which the degrees in H of the pieces of each $v \in V$ are given by $f(v)$. Crispin Nash-Williams [3] obtained necessary and sufficient conditions for a graph to have a k-edge-connected r-detachment or f-detachment. We solve a problem posed by Nash-Williams in [2] by obtaining analogous results for non-separable detachments of graphs.

1 Introduction

All graphs considered are finite, undirected, and may contain loops and multiple edges. We shall use the term *simple graph* for graphs without loops or multiple edges. Let $G = (V, E)$ be a graph and $r : V \to Z_+$. An *r-detachment* of G is a graph H obtained by 'splitting' each vertex $v \in V$ into $r(v)$ vertices. The vertices $v_1, ..., v_{r(v)}$ obtained by splitting v are called the *pieces* of v in H. Every edge $uv \in E$ corresponds to an edge of H connecting some piece of u to some piece of v. An *r-degree specification* is a function f on V, such that, for each vertex $v \in V$, $f(v)$ is a partition of $d(v)$ into $r(v)$ positive integers. An *f-detachment* of G is an r-detachment in which the degrees of the pieces of each $v \in V$ are given by $f(v)$.

Crispin Nash-Williams [3] obtained the following necessary and sufficient conditions for a graph to have a k-edge-connected r-detachment or f-detachment. For X, Y disjoint subsets of $V(G)$, let $e(X, Y)$ be the number of edges of G from X to Y, $e(X)$ the number of edges between the vertices of X, $b(X)$ the number of components of $G - X$ and $r(X) = \sum_{x \in X} r(x)$. For $v \in V$, we use $d(v)$ to denote the degree of v. Thus $d(v) = e(v, V - v) + 2e(v)$.

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Theorem 1.1 (Nash-Williams). *Let $G = (V, E)$ be a graph and $r : V \to Z_+$. Then G has a connected r-detachment if and only if $e(X) + e(X, V - X) \geq r(X) + b(X) - 1$ for every $X \subseteq V$.*
Furthermore, if G has a connected r-detachment then G has a connected f-detachment for every r-degree specification f.

Theorem 1.2 (Nash-Williams). *Let $G = (V, E)$ be a graph, $r : V \to Z_+$, and $k \geq 2$ be an integer. Then G has a k-edge-connected r-detachment if and only if*
(a) G is k-edge-connected,
(b) $d(v) \geq kr(v)$ for each $v \in V$,
and neither of the following statements is true:
(c) k is odd and G has a cut-vertex v such that $d(v) = 2k$ and $r(v) = 2$,
(d) k is odd, $|V| = 2$, $|E| = 2k$, G is loopless, and $r(v) = 2$ for each vertex $v \in V$.
Furthermore, if G has a k-edge-connected r-detachment then G has a k-edge-connected f-detachment for any r-degree specification f for which each term d_i^v is at least k for every $v \in V$ and every $1 \leq i \leq r(v)$.

Let G be a graph. A vertex v is a *cut-vertex* of G if $|E(G)| \geq 2$ and either v is incident with a loop or $G - v$ has more components than G. A graph is *non-separable* if it is connected and has no cut-vertices. Nash-Williams proposed the following problem in [2, p.145]:

"It might also be worth looking at the question whether one can give necessary and sufficient conditions on a graph G and function $r : V(G) \to Z_+$ for the existence of a non-separable r-detachment of G, i.e. an r-detachment of G which has no cut-vertices - but of course it is not self-evident that a reasonable set of necessary and sufficient conditions for this must even exist."

In this paper we answer this question by showing necessary and sufficient conditions for the existence of a non-separable r-detachment of a graph. We also solve the degree specified version. We shall need the following slight strengthening of Theorem 1.1.

Theorem 1.3. *Let $G = (V, E)$ be a graph, $r : V \to Z_+$ and $V_2 = \{v \in V : r(v) \geq 2\}$. Then G has a connected r-detachment if and only if $e(X) + e(X, V - X) \geq r(X) + b(X) - 1$ for every $X \subseteq V_2$.*

Proof. Necessity follows from Theorem 1.1. To see sufficiency suppose that G does not have an r-detachment. By Theorem 1, $e(X) + e(X, V - X) \leq r(X) + b(X) - 2$ for some $X \subseteq V$. If $x \in X - V_2$, then $r(x) = 1$, and putting $X' = X - x$ we have $e(X') + e(X', V - X') \leq r(X') + b(X') - 2$. Hence we can construct $X'' \subseteq V_2$ with $e(X'') + e(X'', V - X'') \leq r(X'') + b(X'') - 2$. □

2 Main Results

Let G be a graph and $N(G) = \{v \in V : d(v) \geq 4\}$. Given $r : V \to Z_+$, let $N_1(G, r) = \{v \in N(G) : r(v) = 1\}$, and $N_2(G, r) = \{v \in N(G) : r(v) \geq 2\}$.

Theorem 2.1. *Let $G = (V, E)$ be a graph with at least two edges and $r : V \to Z_+$. Then G has a non-separable r-detachment if and only if*
(a) G is 2-edge connected,
(b) $d(v) \geq 2r(v)$ for all $v \in V$,
(c) $e(v) = 0$ for all $v \in N_1(G, r)$, and
(d) $e(X, V-X-y)+e(X) \geq r(X)+b(X+y)-1$ for all $y \in N_1(G, r)$ and $X \subseteq N_2(G, r)$.

The degree specified version is as follows.

Theorem 2.2. *Let $G = (V, E)$ be a graph with at least two edges, $r : V \to Z_+$, and let f be an r-degree specification, where $f(v) = (f_1^v, f_2^v, \ldots, f_{r(v)}^v)$ and $f_1^v \geq f_2^v \geq \ldots \geq f_{r(v)}^v$, for each $v \in V$. Then G has a non-separable f-detachment if and only if*
(a) G is 2-edge connected,
(b) $f_i^v \geq 2$ for all $v \in V$ and all $1 \leq i \leq r(v)$,
(c) $e(X + v, V - X - v) + e(X + v) - f_1^v \geq r(X + v) + b(X + v) - 2$ for all $v \in N(G)$ and $X \subseteq N_2(G, r) - v$.

Note that condition (c) of Theorem 2.2 implies that $e(v) = 0$ for all $v \in N_1(G)$ by taking $X = \emptyset$. We shall need the following lemmas. Since loops create complications in notation, and since we only need the lemmas for loopless graphs, we add the hypothesis to the lemmas that the graphs are loopless. Note however that they may be applied to graphs with loops by subdividing their loops.

Lemma 2.3. *Let G be a 2-edge-connected loopless graph and $v \in N(G)$. Define $r : V \to Z_+$ by $r(v) = 2$ and $r(u) = 1$ for all $u \in V - v$. Then there exists a 2-edge-connected r-detachment H of G such that, at least one of the pieces of v in H, has degree two. Furthermore, if v is a cut-vertex of G, then there exists a 2-edge-connected r-detachment H' of G such that, for the pieces v_1, v_2 of v in H', we have $d_{H'}(v_2) = b(v)$, and neither v_1 nor v_2 is a cut-vertex of H'.*

Proof. The first part of the lemma is easy and well known (it follows for example from Theorem 1.2). To prove the second part, let $C_1, C_2, \ldots, C_b$ be the components of $G - v$, where $b = b(v)$. Let H' be an r-detachment of G such that H' has exactly one edge from v_2 to each component C_i. Since G is 2-edge-connected, there is also at least one edge from v_1 to each C_i. To see that H' is 2-edge-connected, it suffices to show that H' has a cycle containing v_1, v_2. (Since the 2-edge-connectivity of G implies that every cut-edge of H' must separate v_1 and v_2.) We can construct such a cycle by choosing edges v_1x_i, v_2y_i in H' with $x_i, y_i \in C_i$ and an x_iy_i-path in C_i for each $i \in \{1, 2\}$. The fact that neither v_1 nor v_2 is a cut-vertex of H' follows easily from the construction of H'. □

Let G be a graph. A *block* B of G is a non-separable subgraph of G which is maximal with respect to subgraph inclusion. We say that G is a *block* if G is a block of itself (or equivalently, if G is non-separable). A vertex $v \in V(B)$ is an *internal vertex* of B (in G) if v is not a cut-vertex of G. An *end-block* of G is a block which contains at most one cut-vertex of G. Note that if G is separable then G has at least two end-blocks. We say that G is a *uv-block-path* if G is connected with exactly two end-blocks, B_1

and B_2 say, and u, v are internal vertices of B_1 and B_2, respectively, in G. For edges ux, vz in a graph G, we define the graph $G(ux, vz)$ obtained by *switching* ux and vz by putting $G(ux, vz) = G - \{ux, vz\} \cup \{uz, vx\}$. Note that switching preserves the degree sequence of G. We shall use the following lemmas to determine when switching can be used to reduce the number of blocks in a detachment of G.

Lemma 2.4. *Let G be a loopless graph and $ux, vz \in E(G)$ such that ux, vz belong to vertex disjoint cycles in G. Suppose that G is either a block or a uv-block-path. Then $G(ux, vz)$ is a block.*

Proof. Choose disjoint cycles C_1, C_2 containing ux, vz, respectively. Then $(C_1 - ux) \cup (C_2 - vz) \cup \{uz, vx\}$ induces a cycle in $G(ux, vz)$ containing u, x, v, w. Since every end-block of $G - \{ux, vz\}$ contains either u, x, v or z as an internal vertex, it follows that $G(ux, vz)$ is a block. □

We shall use the following rather technical lemmas to show that, if a graph G has an f-detachment with a unique cut-vertex $y \in N_1(G)$, then either G has a non-separable f-detachment, or we can find a set $X \subseteq N_2(G)$ such that $r(X) + b(X + y)$ is large.

Let G be a graph, $y \in V(G)$, $r : V \to Z_+$ with $r(y) = 1$, and f be an r-degree specification for G. Let H be an f-detachment of G, $W \subseteq V(H)$. and $u, v \in V(H) - W$. We say that u and v are *W-separated* in H if u and v belong to different components of $H - W$. Define sequences of sets $R_1, R_2, \ldots \subseteq V(G)$, $S_1, S_2, \ldots \subseteq V(H)$, and $W_0 \subseteq W_1 \subseteq \ldots \subseteq V(H)$, recursively, as follows. Let $W_0 = \{y\}$, and, for $i \geq 1$, let

$$R_i = \{v \in V(G) : \text{at least two pieces of } v \text{ are } W_{i-1}\text{-separated in } H\},$$

$$S_i = \{v_j \in V(H) : v_j \text{ is a piece of some } v \in R_i\},$$

and $W_i = S_i \cup W_{i-1}$.

It follows from these definitions that $S_i \cap S_j = \emptyset = R_i \cap R_j$ for $i \neq j$ and $W_i = \{y\} \cup S_1 \cup S_2 \cup \ldots \cup S_i$. Also note that $S_i = \emptyset$ for all $i \geq 1$ if y is not a cut-vertex of H.

Lemma 2.5. *Let H be a connected f-detachment of G. Let Z be a component of $H - W_{i-1}$ for some $i \geq 1$ and $uv, wx \in E(Z)$. Suppose $Z(uv, wx)$ is connected. Then $H' = H(uv, wx)$ is a connected f-detachment of G and $S_m(H') = S_m(H)$ for all $1 \leq m \leq i$.*

Proof. Since H and H' have the same degree sequence, H' is an f-detachment of G. Furthermore H' is connected since H and $Z(uv, wx)$ is connected. We shall show that $S_m(H') = S_m(H)$ for all $1 \leq m \leq i$ by induction on i. If $i = 1$ then $W_0(H') = \{y\} = W_0(H)$. Since $Z(uv, wx)$ is connected, we have $R_1(H) = R_1(H')$ and hence $S_1(H) = S_1(H')$.

Suppose $i \geq 2$. By induction, $S_m(H') = S_m(H)$ for all $1 \leq m \leq i - 1$. Thus $W_{i-1}(H') = W_{i-1}(H)$. Since $Z(uv, wx)$ is connected, we have $R_i(H) = R_i(H')$ and hence $S_i(H) = S_i(H')$. □

Let H be a graph and y be a vertex of H and B be an end-block of H. We say that H is a *block-star centered on* y if every block of H contains y. We say that H is an *extended block-star centered on* y *with distinguished end-block* B if every end-block of H, with the possible exception of B, contains y. Note that every block-star is an extended block-star and every block is a block-star. An edge e of H is a *cut-edge* of H if $H - e$ has more components than H.

Lemma 2.6. *Let G be a loopless graph, $y \in V(G)$, $r : V \to Z_+$ with $r(y) = 1$, and f be an r-degree specification for G such that each term in $f(v)$ is at least two for all $v \in V(G)$. Suppose that G has an f-detachment which is a block-star centered on y, and that H has been chosen amongst all such f-detachments so that $b_H(y)$ is as small as possible. Then each edge of $H - y$ incident to a vertex in S_i is a cut-edge of $H - y$ for all $i \geq 1$.*

Proof. We proceed by contradiction. Suppose the lemma is false. Since the lemma is vacuously true if y is not a cut-vertex of H we have $b_H(y) \geq 2$. Choose an f-detachment K of G such that:

(i) $b_K(y) = b_H(y)$,

(ii) K is an extended block-star centered on y with distinguished end-block B.

(iii) for some edge $v_j x \in E(B - y)$ and $i \geq 1$ we have $v_j \in S_i$ and $v_j x$ is not a cut edge of $K - y$,

(iv) each edge of $K - y$ which is incident to a vertex of S_m is a cut-edge of $K - y$ for all $1 \leq m \leq i - 1$,

(v) subject to (i)-(iv), i is as small as possible.

Note that K exists since if H is a counterexample to the lemma and we choose an edge which is not a cut-edge of $H - y$ and is incident to a vertex of S_i such that i is as small as possible, then H will satisfy (i)-(iv). Our proof technique forces us to work with extended block-stars rather than block-stars because the switching operations we use preserve the property of being an extended block-star, but may not preserve the property of being a block-star.

Since $v_j x$ is not a cut-edge of $K - y$, $v_j x$ is contained in a cycle C of $K - y$. Let $B_1, B_2, \ldots, B_t$ be blocks of K such that $y \in V(B_1)$, $V(B_i) \cap V(B_s) \neq \emptyset$ if and only if $|i - s| \leq 1$, $B_t = B$, and $y \notin V(B_2)$ if $t \geq 2$. Then $v_j x \in E(B_t)$ and $C \subseteq B_t$. Since $v_j \in S_i$, v_j is a piece of some vertex $v \in R_i$. Thus we may choose another piece v_k of v such that v_j and v_k are W_{i-1}-separated in K.

Claim 2.7. $i \geq 2$.

Proof. Suppose $i = 1$. Then v_j and v_k are y-separated in K. Hence v_j and v_k belong to different end-blocks B_t and B_0 of K. Choose an edge $v_k z \in E(B_0)$. Since K has minimum degree at least two, $v_k z$ is contained in a cycle C' in B_0. Since $y \notin V(C)$, C and C' are vertex disjoint. Applying Lemma 2.4 to the block-path

$F = B_0 \cup B_1 \cup \ldots \cup B_t$, we deduce that $F(v_j x, v_k z)$ is a block. Thus $H' = K(v_j x, v_k z)$ is an f-detachment of G which is a block-star centered on y, and $b_{H'}(y) = b_H(y) - 1$. This contradicts the hypothesis that $b_H(y)$ is as small as possible. □

Since $v_j, v_k \notin S_{i-1}$ they are not W_{i-2}-separated in K. Thus they both belong to the same component Z of $K - W_{i-2}$. In particular v_j, v_k are not y-separated in K so $v_k \in V(B_s)$ for some s, $1 \leq s \leq t$. By (iv), $V(C) \cap W_{i-2} = \emptyset$, and hence $C \subseteq Z$. Let P' be a path from v_k to C in Z. We may extend P' around C if necessary to obtain a $v_k v_j$-path P in Z which avoids the edge $v_j x$. Let $v_k z$ be the edge of P incident with v_k. Since v_j, v_k are W_{i-1} separated but not W_{i-2}-separated, we can choose $u \in V(P) \cap S_{i-1}$.

Let $K' = K(v_j x, v_k z)$. We shall show that K' contradicts the above choice of K.

Claim 2.8. *(a) K' is a connected f-detachment of G and $S_j(K') = S_j(K)$ for all $1 \leq m \leq i - 1$.*
(b) u and $v_j z$ are contained in a common cycle of $K' - y$.

Proof. (a) follows from Lemma 2.5 since $P[z, v_j] \cup (C - v_j x) \cup \{z v_k\}$ is a connected subgraph of $Z(v_j x, v_k z)$.
(b) follows since $P[z, v_j] \cup \{v_j z\}$ is a cycle in $K' - y$ containing u and $v_j z$. □

Let $F_1 = B_1 \cup B_2 \cup \ldots \cup B_t$. Since $u \in S_{i-1}$, (iv) implies that each edge of $K - y$ incident with u is a cut-edge of $K - y$. Thus there is exactly one edge from u to each component of $F_1 - \{y, u\}$. Let X be the component of $F_1 - \{y, u\}$ which contains v_j. Since u has a unique neighbour in X and u lies on the $v_j v_k$-path P in $F_1 - y$, $v_k \notin V(X)$. Furthermore $C \subseteq B_t \subseteq X$. Let F_2 be the graph obtained from F_1 by adding a new edge yu and put $F_3 = F_2 - X$.

Claim 2.9. *F_3 is a block.*

Proof. Since $F_1 - y$ is connected, $F_3 - y$ is connected. Thus if F_3 were not a block then we could choose an end-block B^* of F_3 such that $y \notin V(B^*)$. Then B^* would be an end-block of K which did not contain y and was distinct from $B = B_t$, since $B_t \subseteq X$. This would contradict (ii). □

Since F_3 is a block, we can choose a cycle C' in F_3 which contains $v_k z$. Then C' is vertex disjoint from C since $C \subseteq X$. Furthermore, F_2 is either a block or a block-path with distinct end-blocks F_3 and B_t. Since $v_k z \in E(F_3) \cap E(C')$ and $v_j x \in E(B_t) \cap E(C)$, Lemma 2.4 implies that $F_2(v_j x, v_k z)$ is a block. Thus $F_1(v_j x, v_k z) = F_2(v_j x, v_k z) - yu$ is either a block or a yu-block-path. Combining this with Claim 2.8, we deduce:

Claim 2.10. *$K' = K(v_j x, v_k z)$ is an extended block-star, centered on y, with distinguished end-block B', where B' is the block of $F_1(v_j x, v_k z)$ which contains u. Furthermore $b_{K'}(y) = b_K(y)$, u is an internal vertex of K', $u \in S_{i-1}(K')$ and u is contained in a cycle of $K' - y$ which is contained in B'.*

Choose $u'x' \in E(B' - y)$ such that $u'x'$ is not a cut-edge of $K' - y$, $u' \in S_p(K')$ for some $p \geq 1$ and p is as small as possible. Then $p \leq i - 1$ since $u \in S_{i-1}(K') \cap V(B')$. To show that K' contradicts our choice of K to minimise i, it only remains to show that (iv) holds for K':

Claim 2.11. *Each edge of $K' - y$ which is incident to a vertex of $S_m(K')$ is a cut-edge of $K' - y$ for all $1 \leq m \leq p - 1$.*

Proof. Suppose the claim is false and let C^* be a cycle of $K' - y$ which contains a vertex of $S_m(K')$ for some m, $1 \leq m \leq p - 1$. The minimality of P implies that C^* contains no edge of B'. Since u is an internal vertex of K' contained in B', and u and $v_j z$ are contained in a cycle of $K' - y$ by Claim 2.8(b), we may deduce that $v_j z \in E(B')$ and hence $v_j z \notin E(C^*)$. Since v_k and x belong to different components of $K - \{y, u\}$, every path from v_k to x in $K - y$ contains u. Thus every cycle of $K' - y$ which contains $v_k x$ also contains u and hence is contained in B'. Thus $v_k x \notin E(C^*)$. Since $v_k x, v_j z \notin E(C^*)$ we have $C^* \subseteq K - y$. Since $S_m(K') = S_m(K)$ by Claim 2.8(a), the existence of C^* now contradicts condition (iv) in the choice of K. □

This completes the proof of the Lemma. □

To simplify notation, we shall first prove Theorem 2.1 for the special case when G is loopless and $N(G)$ is an independent set of vertices in G. The general case follows easily from this special case by the simple procedure of subdividing every edge of G and extending r by putting $r(v) = 1$ for each subdivision vertex v. Thus we shall prove:

Theorem 2.12. *Let $G = (V, E)$ be a loopless graph with at least two edges and $r : V \to Z_+$. Suppose that $N(G)$ is an independent set of vertices in G. Then G has a non-separable r-detachment if and only if*
(a) G is 2-edge connected,
(b) $d(v) \geq 2r(v)$ for all $v \in V$, and
(c) $e(X, V - X) \geq r(X) + b(X + y) - 1$ for all $y \in N_1(G, r)$ and $X \subseteq N_2(G, r)$.

Proof. We first prove necessity. Suppose H is a non-separable r-detachment of G. Then H is 2-edge-connected and since 'detaching' vertices cannot increase edge-connectivity, (a) holds. Since H has minimum degree at least two we also have (b). Condition (c) follows from the easy part of Theorem 1.1, since $H - y$ is a connected $r|_{V-y}$-detachment of $G - y$.

We next prove sufficiency. We proceed by contradiction. Suppose that the theorem is false and choose a counterexample (G, r) such that

$$\gamma(G, r) := |N_1(G, r)| + \sum_{v \in N(G)} (d(v) - 3)$$

is as small as possible, and, subject to this condition, $|V(G)|$ is as small as possible. If $N(G) = \emptyset$ then G has maximum degree at most three and, by (b), $r(v) = 1$ for all $v \in V(G)$. Using (a) we deduce that G is a non-separable r-detachment of itself. Hence we may suppose that $N(G) \neq \emptyset$ and hence $\gamma(G, r) \geq 1$.

Claim 2.13. *Suppose that $U \subset V$ such that $e(U, V-U) = 2$. Then either $|U| = 1$ or $|V-U| = 1$.*

Proof. Suppose $|U| \geq 2$ and $|V-U| \geq 2$. Let $U_1 = U$ and $U_2 = V-U$. For $i \in \{1,2\}$, let G_i be the graph obtained from G by contracting U_i to a single vertex u_i of degree two. Define $r_i : V(G_i) \to Z_+$ by putting $r_i(u_i) = 1$ and $r_i(v) = r(v)$ for $v \in V(G_i) - u_i$. Then $\gamma(G_i, r_i) \leq \gamma(G, r)$ and $|V(G_i)| < |V(G)|$. Since contraction preserves edge-connectivity, G_i satisfies (a). Clearly (G_i, r_i) also satisfies (b). Suppose (G_i, r_i) does not satisfy (c). Then $e_{G_i}(X, V(G_i) - X) \leq r_i(X) + b_{G_i}(X+y) - 2$ for some $y \in N_1(G_i, r_i)$ and $X \subseteq N_2(G_i, r_i)$. Since $u_i \notin N(G_i)$, u_i belongs to some component of $G_i - (X+y)$, and $X + y \subseteq V(G)$. Thus X, y contradict the fact that (c) holds for G. Hence (c) holds for (G_i, r_i) and, by induction, G_i has a non-separable r_i-detachment for $i = 1, 2$. Since $e(U, V-U) = 2$, this implies that G has a non-separable r-detachment. □

Claim 2.14. $N_1(G, r) \neq \emptyset$.

Proof. Suppose $N_1(G, r) = \emptyset$. Choose $v \in N_2(G, r)$ such that $r(v)$ is as large as possible and $d(v)$ is as small as possible. Define $r_v : V(G) \to Z_+$ by $r_v(v) = 2$ and $r_v(u) = 1$ for all $u \in V - v$. By Lemma 2.3, we can construct a 2-edge-connected r_v-detachment H of G such that, for the pieces v_1, v_2 of v in H, we have $d_H(v_2) = 2$. Define $r' : V(H) \to Z_+$ by $r'(v_1) = r(v) - 1$, $r'(v_2) = 1$, and $r'(u) = r(u)$ for all $u \in V(H) - \{v_1, v_2\}$. Then $\gamma(H, r') < \gamma(G, r)$. By construction (H, r') satisfies (a) and (b). If (H, r') also satisfies (c), then, by induction H has a non-separable r'-detachment H'. Clearly, H' is the required r-detachment of G. Hence

$$e_H(X, V(H) - X) \leq r'(X) + b_H(X+y) - 2 \tag{1}$$

for some $y \in N_1(H, r')$ and $X \subseteq N_2(H, r')$. Since $N_1(G, r) = \emptyset$ and $d_H(v_2) = 2$, we must have $y = v_1$, and $r'(v_1) = 1$. Thus $r(v) = 2$. The choice of v now implies

$$r(u) = 2 \text{ for all } u \in N(G). \tag{2}$$

In particular $r'(X) = r(X) = 2|X|$. The choice of v also implies that $d_H(u) = d_G(u) \geq d_G(v) = d_H(v_1) + 2$ for all $u \in N(G) - v$. Thus $e_H(X, V(H) - X) \geq |X|(d_H(v_1) + 2)$. Since H is 2-edge-connected, each component of $H - (X + v_1)$ has at least two edges to $X + v_1$. Thus $2b_H(X + v_1) \leq e_H(X, V(H) - X) + d_H(v_1)$. Substituting these inequalities into (1), we deduce that $d_H(v_1)(|X| - 1) \leq 2(|X| - 2)$. Hence $X = \emptyset$. Now (1) implies that $b_H(v_1) \geq 2$. Thus v_1 is a cut-vertex of H and hence v is a cut-vertex of G.

By Lemma 2.3, we can construct a 2-edge-connected r_v-detachment H' of G such that, for the pieces v_1', v_2' of v in H', we have $d_{H'}(v_2) = b_G(v)$, and neither v_1' nor v_2' is a cut-vertex of H'. Defining r' as above (i.e. $r'(v_1') = r(v) - 1 = 1$, $r'(v_2') = 1$, and $r'(u) = r(u)$ for all $u \in V(H') - \{v_1', v_2'\}$) we have $\gamma(H', r') < \gamma(G, r)$, and (H', r') satisfies (a) and (b). Thus we may again deduce that (H', r') fails to satisfy (c). Hence $e_{H'}(X, V(H') - X) \leq r'(X) + b_{H'}(X + y) - 2$ for some $y \in N_1(H', r')$ and

$X \subseteq N_2(H', r')$. Since $N_1(G, r) = \emptyset$ we must have $y = v_i'$ for some $i \in \{1, 2\}$. Using (2), we may now deduce as above that $X = \emptyset$ and hence v_i' is a cut-vertex of H'. This contradicts the choice of H'. □

Choose $y \in N_1(G, r)$ such that $d(y)$ is as large as possible. Define $r_y : V(G) \to Z_+$ by $r_y(y) = 2$ and $r_y(u) = r(u)$ for all $u \in V - y$. Clearly $\gamma(G, r_y) < \gamma(G, r)$, and (G, r_y) satisfies (a) and (b).

Claim 2.15. *(G, r_y) satisfies (c).*

Proof. Suppose we have $e_G(X, V - X) \leq r_y(X) + b_G(X + y') - 2$ for some $y' \in N_1(G, r_y)$ and $X \subseteq N_2(G, r_y)$. Since (c) holds for (G, r) we must have $y \in X$ and

$$e_G(X, V - X) = r(X) + b_G(X + y') - 1. \tag{3}$$

Let $C_1, C_2, \ldots, C_b$ be the components of $G - (X + y')$. Since (c) holds for (G, r), we may apply Theorem 1.3 to deduce that $G - y'$ has a connected r'-detachment H, where $r' = r|_{V-y'}$. Let X^* be the set of all pieces of vertices of X in H. Since (b) holds for G, Theorem 1.1 implies that H may be constructed to have the additional property that $d_H(x_i) \geq 2$ for all $x_i \in X^*$. Let H' be the detachment of $G - y'$ obtained from H by 're-attaching' all the pieces of v, for each $v \in V - X - y'$. Thus H' is a connected r''-detachment of $G - y'$, where $r''(v) = r(v)$ for $v \in X$ and $r''(v) = 1$ for $v \in V - X - y'$. Using the fact that equality holds in (3), we have exactly $r(X) + b - 1$ edges in H' joining the vertices in X^* and the components $C_1, C_2, \ldots, C_b$. Since H' is connected and $|X^*| = r(X)$, the graph T obtained from H' by contracting each component C_i to a single vertex c_i, is a tree. Since $d_T(x_i) \geq 2$ for all $x_i \in X^*$, no vertex of X^* is an end-vertex of T. Since $r''(y) = 1$, we can label the unique piece of y in H as y. We then have $y \in X^*$ and $d_T(y) = d_G(y)$. Thus T has at least $d_G(y)$ end-vertices, all of which belong to $\{c_1, c_2, \ldots, c_b\}$. Furthermore, if T has exactly $d_G(y)$ end-vertices, then all vertices of T other than y have degree one or two. Let $S = \{C_i : d_T(c_i) = 1, 1 \leq i \leq b\}$. Then $e_{H'}(C_i, V(H') - C_i) = 1$ for all $C_i \in S$. Since G is 2-edge-connected and $r''(v) = 1$ for all $v \in C_i$, there is at least one edge in G from C_i to y' for each $C_i \in S$. Since $|S| \geq d_G(y)$, it follows that $d_G(y') \geq d_G(y)$. It now follows from the initial choice of y that we must have $d_G(y') = |S| = d_G(y)$, that there is exactly one edge in G from y' to each $C_i \in S$ and to no other vertices of G, and that all vertices of T other than y have degree one or two. Again, since $r''(v) = 1$ for all $v \in C_i$, we have $e_G(C_i, V(G) - C_i) = 2$ for all $1 \leq i \leq b$. Using Claim 2.13, we deduce that $|V(C_i)| = 1$ for all $1 \leq i \leq b$. Thus $V(C_i) = \{c_i\}$, $H' = H$ and since G is loopless, $d_G(c_i) = 2$. By (b), $r(c_i) = 1 = r''(c_i)$ for all $1 \leq i \leq b$ and $H = T$. Let G' be the graph obtained from H by adding y' and the edge $y'c_i$ for each $C_i \in S$. (Thus G' is obtained by adding an edge from y' to each end-vertex of T.) Then G' is the required non-separable r-detachment of G. □

Since (c) holds for (G, r_y), we may apply induction to deduce that G has a non-separable r_y-detachment. It follows that G has an r-detachment H such that H is a block-star centered on y. We may suppose that H has been chosen such that the number of blocks of H is as small as possible. Let f be the r-degree specification for

G given by H. Since G has no non-separable f-detachment, $b_H(y) \geq 2$. For $i \geq 0$, let S_i and W_i be the subsets of $V(H)$ defined as for Lemma 2.6. Since the sets S_i are pairwise disjoint and H is finite, we may choose i such that $S_{i+1} = \emptyset$. Let $X' = W_i - y$ and $X = \{x \in V(G) :$ some piece of x in H belongs to $X'\}$. By Lemma 2.6, every edge $x_1 v \in E(H - y)$ with $x_1 \in X'$ is a cut-edge in $H - y$. Thus the graph F we get from $H - y$ by contracting each component of $H - X' - y$ to a single vertex is a forest with $b_H(y)$ components and $b_H(X' + y) + |X'|$ vertices. Using the facts that $X + y \subseteq N(G)$, and $N(G)$ is an independent set of vertices in G, we deduce that $|E(F)| = e_H(X', V(H) - X')$. Thus

$$e_H(X', V(H) - X') = b_H(X' + y) + |X'| - b_H(y). \tag{4}$$

We have $e_H(X', V(H) - X') = e_G(X, V(G) - X)$, $|X'| = r(X)$ and $b_H(y) \geq 2$. Furthermore, for each $v \in V(G) - X - y$, all pieces of v in H belong to the same component of $H - X' - y$, since $S_{i+1} = \emptyset$. Thus $b_G(X + y) = b_H(X' + y)$. Substituting into (4) we obtain $e_G(X, V(G) - X) \leq r(X) + b_G(X + y) - 2$. This contradicts hypothesis (c) of the theorem and completes our proof. □

Proof of Theorem 2.1. Let G' be obtained from G by subdividing every edge of G. Then G' is loopless, $N(G') = N(G)$ and $N(G')$ is independent in G'. Extend r to r' by putting $r'(v) = r(v)$ for all $v \in V(G)$ and $r'(v) = 1$ for all $v \in V(G') - V(G)$. Then $N_1(G', r') = N_1(G, r)$ and $N_2(G', r') = N_2(G, r)$. We shall show that conditions (a), (b), (c) and (d) of Theorem 2.1 hold for (G, r) if and only if conditions (a), (b), and (c) of Theorem 2.12 hold for (G', r'). Clearly Theorem 2.1 (a) and (b) hold for (G, r) if and only if Theorem 2.12 (a) and (b) hold for (G', r'). Furthermore for $y \in N_1(G, r) = N_1(G', r')$ and $X \subseteq N_2(G, r) = N_2(G', r')$, we have $r(X) = r'(X)$, and $e_G(X, V - X - y) + e_G(X) - b_G(X + y) - e_G(y) = e_{G'}(X, V - X) - b_{G'}(X + y)$. If Theorem 2.1 (c) and (d) hold for (G, r), then $e_G(y) = 0$ and the above equalities imply that Theorem 2.12 (c) holds for (G', r'). Suppose, on the other hand, that Theorem 2.12 (c) holds for (G', r'). Taking $X = \emptyset$ we have $b_{G'}(y) \leq 1$ for all $y \in N_1(G'r')$ and hence $e_G(y) = 0$ for all $y \in N_1(G, r)$. Thus Theorem 2.1 (c) holds for (G, r). The above equalities now imply that Theorem 2.1 (d) also holds for (G, r). □

We shall next prove Theorem 2.2. Given a graph $G = (V, E)$ and $X \subseteq V$, let $\Gamma(X)$ be the set of vertices of $V - X$ which are adjacent to vertices in X and put $\gamma(X) = |\Gamma(X)|$. We shall use the following operation to adjust the degree sequence in a detachment of a graph. For vertices x, y, z of G with $xz \in E(G)$, we define the graph $G(xz \to yz)$ obtained by *flipping* xz to yz by putting $G(xz \to yz) = G - xz + yz$. The following lemma characterises when we may flip edges in a non-separable graph and preserve non-separability.

Lemma 2.16. *Let $G = (V, E)$ be a non-separable graph and let $x, y \in V$ be distinct vertices of G. Let $xz_1, xz_2, \ldots, xz_t$ be distinct edges of $G - y$ with $t \geq 3$. Then $G(xz_i \to yz_i)$ is separable for all $1 \leq i \leq t$ if and only if there exist distinct components $C_1, C_2, \ldots, C_t$ of $G - \{x, y\}$ with $z_i \in V(C_i)$ and $e(x, C_i) = 1$ for all $1 \leq i \leq t$.*

Proof. Sufficiency is easy to see. To prove necessity first note that $G(xz_i \to yz_i)$ has no loops for all $1 \leq i \leq t$ since G is non-separable and $z_i \neq y$. Suppose that $G(xz \to yz)$ has a cut-vertex for all $1 \leq i \leq t$. It is easy to see that flipping an edge xz_i to yz_i creates a cut-vertex if and only if there is a set $W \subset V - x$ in G with $e(W, x) = 1$, $\gamma(W) = 2$, and $z_i \in W$, $y \in W \cup \Gamma(W)$. We call W a *certificate* for z_i.

Let us choose a minimal family $\mathcal{F} = \{W_1, ..., W_m\}$ which contains a certificate for z_i for all $1 \leq i \leq t$. Since $e(W_i, x) = 1$ for $1 \leq i \leq m$ and by the minimality of $\mathcal{F}$ we have $t = m$ and $\Gamma(x) \cap W_i \cap W_j = \emptyset$ for all $1 \leq i < j \leq t$. Furthermore, since $e(W_i, x) = 1$ and G is non-separable, each W_i induces a connected subgraph of G.

First we show that $W_i \cap W_j = \emptyset$ for $1 \leq i < j \leq t$. Suppose that $W_i \cap W_j \neq \emptyset$ holds for two distinct $W_i, W_j \in \mathcal{F}$. Since $e(W_i, x) = e(W_j, x) = 1$ and $d(x) \geq 3$, it follows that $Z := V - (W_i \cup W_j) - \{x\}$ is non-empty. The subgraphs $G[W_i]$ and $G[W_j]$ are connected, hence we have that both $\Gamma(W_i) \cap (W_j - W_i)$ and $\Gamma(W_j) \cap (W_i - W_j)$ are non-empty. Since $\gamma(W_i) = \gamma(W_j) = 2$ and $x \in \Gamma(W_i) \cap \Gamma(W_j)$, this implies that $e(W_i \cup W_j, Z) = 0$. Hence x is a cut-vertex in G, a contradiction.

Now suppose that $y \in W_i$ holds for some $1 \leq i \leq t$. Since $W_i \cap W_j = \emptyset$, we must have $y \in \Gamma(W_j)$ for all $W_j \in \mathcal{F} - W_i$. Since $x \in \Gamma(W_i)$ and $t \geq 3$, this implies $\gamma(W_i) \geq 3$, a contradiction. Thus $y \in \Gamma(W_i)$ for all $1 \leq i \leq t$. This, and the facts that $\gamma(W_j) = 2$ and $x \in \Gamma(W_j)$ for all $1 \leq j \leq t$, imply that $C_i = G[W_i]$, $1 \leq i \leq t$, are the required components of $G - \{x, y\}$. $\square$

Corollary 2.17. *Let $t \geq 3$ be an integer. Let $G = (V, E)$ be a non-separable graph, $x, y \in V$ and $xz_i \in E(G - y)$ for $1 \leq i \leq t$. If $t \geq d(y) - e(\{x, y\}) + 1$, then $G(xz_i \to yz_i)$ is non-separable for some $1 \leq i \leq t$.*

Proof. Suppose that for all $1 \leq i \leq t$ the graph $G(xz_i \to yz_i)$ is separable. We may apply Lemma 2.16 and deduce that $b(\{x, y\}) \geq t$. Since $t \geq d(y) - e(\{x, y\}) + 1$, it follows that $e(C, y) = 0$ for some component C of $G - \{x, y\}$. Thus x is a cut-vertex in G, a contradiction. $\square$

Corollary 2.18. *Let $G = (V, E)$ be a non-separable graph and let $x, y, w \in V$ be distinct vertices of G such that $d(x) \geq 3$ and $xy, xw \notin E$. Then there exists a $z \in \Gamma(x)$ such that either $G(xz \to yz)$ or $G(xz \to wz)$ is non-separable.*

Proof. Suppose that for all $z \in \Gamma(x)$ the graph $G(xz \to yz)$ is separable. By Lemma 2.16 we have $b(\{x, y\}) = d(x)$. Let $C_1, C_2, \ldots, C_{d(x)}$ be the components of $G - \{x, y\}$, where $w \in V(C_1)$. Then each neighbour of x other than the unique neighbour in C_1 belongs to the same component of $G - \{x, w\}$. Thus Lemma 2.16 implies that $G(xz \to wz)$ is non-separable for some $z \in \Gamma(x)$. $\square$

Lemma 2.19. *Let $G = (V, E)$ be a non-separable graph and suppose that $b(\{x, y\}) = d(x) \geq 3$ for some pair $x, y \in V$. Let w be a vertex in some component C of $G - \{x, y\}$ with $e(w, y) = e(w, x) = 0$ and let $z \in \Gamma(x) - C$. Then either $G(xz \to wz)(wz' \to yz')$ is non-separable for some $z' \in \Gamma(w)$ or every edge incident to w in G is a cut-edge in C.*

Proof. Let the components of $G-\{x,y\}$ be $C_1, C_2, ..., C_{d(x)}$ and, without loss of generality, suppose that $z \in V(C_1)$. Then $C_1 \neq C$. We first observe that $H := G(xz \to wz)$ is non-separable. This follows from the fact that, since $d(x) \geq 3$, there is a cycle containing x and z in H. Thus we need to show that either there is flip from w to y in H which creates no cut-vertices or C has the required property. Suppose that $H(wz' \to yz')$ has a cut-vertex for every $z' \in \Gamma(w) \cap V(C)$. Using the facts that $d_H(w) \geq 3$, $e_H(w,y) = 0$, and y is a cut-vertex in $H(wz \to yz)$, we may apply Lemma 2.16 to deduce that $b_H(\{w,y\}) = d_H(w) = d_G(w) + 1$. This implies that every edge incident to w in H is a cut-edge in $H - y$. Since $z \notin V(C)$ and H is obtained from G by flipping xz to wz, and $y \notin V(C)$, we may deduce that every edge incident to w in G is a cut-edge in C. This proves the lemma. □

We next apply the above results to obtain some preliminary results on f-detachments.

Lemma 2.20. *Let $G = (V,E)$ be a loopless graph and $r : V(G) \to Z_+$. Suppose that G has a non-separable r-detachment H. Let f be the r-degree specification for G given by $f_1^v = d(v)$ if $r(v) = 1$; and $f(v) = (f_1^v, f_2^v, \ldots, f_{r(v)}^v)$ where $f_1^v = \lceil (d(v) - 2r(v) + 4)/2 \rceil$, $f_2^v = \lfloor (d(v) - 2r(v) + 4)/2 \rfloor$, and $f_i^v = 2$ for all $v \in N_2(G,r)$ and $3 \leq i \leq r(v)$. Then G has a non-separable f-detachment.*

Proof. For each $v \in V$, let $v_1, v_2, \ldots, v_{r(v)}$ be the pieces of v in H, where $d_H(v_1) \geq d_H(v_2) \geq \ldots \geq d_H(v_{r(v)})$. Note that the pieces of v are independent in H since G is loopless.

Suppose $d_H(v_i) \geq 3$ for some $3 \leq i \leq r(v)$. Using Corollary 2.18, we can construct a new non-separable r-detachment by flipping an edge $v_i z$ to either $v_1 z$ or $v_2 z$ for some $z \in \Gamma_H(v_i)$. Applying this operation iteratively, and relabelling v_1 and v_2 if necessary, we may construct a non-separable r-detachment H' of G on the same vertex set as H with $d_{H'}(v_1) \geq d_{H'}(v_2)$ and $d_{H'}(v_i) = 2$ for all $v \in N_2(G,r)$ and $i \leq 3 \leq r(v)$.

Suppose $d_{H'}(v_1) \geq d_{H'}(v_2) + 2$. By Corollary 2.17, we can construct a new non-separable r-detachment by flipping an edge $v_1 z$ to $v_2 z$ for some $z \in \Gamma_H(v_1)$. Applying this operation iteratively to H' we may construct the required non-separable f-detachment G. □

Let S be the set of all sequences of integers of length r in decreasing order of magnitude. Let $\geq_r$ be the lexicographic ordering on S, (hence $f = (f_1, f_2, \ldots, f_r) \geq_r g = (g_1, g_2, \ldots, g_r)$ if and only if either $f = g$, or, for some $1 \leq i \leq r$, $f_1 = g_1, f_2 = g_2, \ldots, f_{i-1} = g_{i-1}, f_i > g_i$).

Lemma 2.21. *Let $G = (V,E)$ be a loopless graph, $r : V(G) \to Z_+$, and f and g be two r-degree specifications for G. Suppose that G has a non-separable f-detachment, $f(v) \geq_{r(v)} g(v)$ and $g_i^v \geq 2$ for all $v \in V$ and $1 \leq i \leq r(v)$. Then G has a non-separable g-detachment.*

Proof. We assume that, for each $v \in V$ the sequences $f(v)$ and $g(v)$ occur in decreasing order of magnitude. Note that $\sum_{i=1}^r (v) f_i^v = d_G(v) = \sum_{i=1}^r (v) g_i^v$ for all $v \in V$. Given an integer z let $z^* = z$ if $z \geq 0$, and otherwise let $z^* = 0$. We may suppose that f has been chosen to satisfy the hypotheses of the lemma and such that:

(i) $\theta_1(f) = \sum_{v\in V}\sum_{3\le i\le r(v)}(f_i^v - g_i^v)^*$ is as small as possible;

(ii) subject to (i), $\theta_2(f) = \sum_{v\in V}\sum_{3\le i\le r(v)}(g_i^v - f_i^v)^*$ is as small as possible;

(iii) subject to (i) and (ii), $\theta_3(f) = \sum_{v\in V}(f_1^v - g_1^v)$ is as small as possible.

Let H be a non-separable f-detachment of G. Let $v_1, v_2, \ldots, v_{r(v)}$ be the pieces of v in H, where $d_H(v_i) = f_i^v$ for all $v \in V$ and all $1 \le i \le r(v)$.

Suppose $\theta_1(f) \ge 1$. Then we may choose $v \in V$ such that $f_i^v \ge g_i^v + 1$ for some $3 \le i \le r(v)$. Then $f_i^v \ge 3$ and by Corollary 2.18 there exists a $z \in \Gamma_H(v_i)$ such that either $H(v_iz \to v_1z)$ or $H(v_iz \to v_2z)$ is non-separable. In both cases, the resulting non-separable graph H' is an f'-detachment of G for some r-degree specification f' with $f'(u) \ge_{r(u)} g(u)$ for all $u \in V$ and $\theta_1(f') = \theta_1(f) - 1$. This contradicts the choice of f. Thus $\theta_1(f) = 0$ and $f_i^v \le g_i^v$ for all $v \in V$ and all $3 \le i \le r(v)$.

Suppose $\theta_2(f) \ge 1$. Then we may choose $v \in V$ such that $g_i^v \ge f_i^v + 1$ for some $3 \le i \le r(v)$. We first consider the case when $f_1^v \ge g_1^v + 1$. Then $f_1^v > f_i^v$ and by Corollary 2.17 there exists a $z \in \Gamma_H(v_1)$ such that $H(v_1z \to v_iz)$ is non-separable. The resulting non-separable graph H' is an f'-detachment of G for some r-degree specification f' with $f'(u) \ge_{r(u)} g(u)$ for all $u \in V$, $\theta_1(f') = 0$, and $\theta_2(f') = \theta_2(f) - 1$. (Note that $f'(v) \ge_{r(v)} g(v)$ since $\theta_1(f) = 0$ and hence either $f_1^v - 1 > g_1^v$, or else $f_1^v - 1 = g_1^v$ and $f_2^v \ge g_2^v$ with equality only if $f'(v) = g(v)$.) This contradicts the choice of f and hence $f_1^v \le g_1^v$. Since $f(v) \ge_{r(v)} g(v)$, $\theta_1(v) = 0$ and $\theta_2(v) \ge 1$, we must have $f_1^v = g_1^v$ and $f_2^v \ge g_2^v + 1$. We can now obtain a contradiction as above by using Corollary 2.17 to show that there exists a $z \in \Gamma_H(v_2)$ such that $H(v_2z \to v_iz)$ is non-separable. Thus we must have $\theta_2(f) = 0 = \theta_1(f)$ and hence $f_i^v = g_i^v$ for all $v \in V$ and all $3 \le i \le r(v)$.

Suppose $\theta_3(f) \ge 1$. Then we may choose $v \in V$ such that $f_1^v \ge g_1^v + 1$. Then $f_1^v > g_2^v > f_2^v$ and by Corollary 2.17 there exists a $z \in \Gamma_H(v_1)$ such that $H(v_1z \to v_2z)$ is non-separable. The resulting graph H' is a non-separable f'-detachment of G for some r-degree specification f' with $f'(u) \ge_{r(u)} g(u)$ for all $u \in V$, $\theta_1(f') = 0 = \theta_2(f')$, and $\theta_3(f') = \theta_3(f) - 1$. This contradicts the choice of f and hence $\theta_3(f) = 0$. Thus $f = g$ and the lemma is trivially true. □

We shall need one more lemma which is an extension of Lemma 2.3.

Lemma 2.22. *Let G be a 2-edge-connected loopless graph, $v \in N(G)$, and $r : V \to Z_+$ be such that $r(v) \ge 2$ and $r(u) = 1$ for all $u \in V - v$. Suppose that v is a cut-vertex of G and that $b(v) \ge r(v)$. Then there exists a 2-edge-connected r-detachment H of G such that, for the pieces $v_1, v_2, \ldots, v_{r(v)}$ of v in H, we have $d_H(v_2) = b_G(v) - 2r(v) + 4$, $d_H(v_i) = 2$ for $3 \le i \le r(v)$ and neither v_1 nor v_2 is a cut-vertex of H.*

Proof. We use induction on $r(v)$. If $r(v) = 2$ then the lemma follows from Lemma 2.3. Hence suppose $r(v) \ge 3$ and choose $x, y \in \Gamma(v)$ belonging to different components of $G - v$. Let G' be the graph obtained from G by detaching v into two vertices v' and v'', where $d_{G'}(v'') = 2$ and $\Gamma_{G'}(v'') = \{x, y\}$. It can be seen that G' is 2-edge connected and $b_{G'}(v') = b_G(v) - 1$. The lemma now follows by applying induction to (G', v', r') where $r'(v') = r(v) - 1$. $\square$

As for Theorem 2.1, we first prove Theorem 2.2 in the case when G is loopless and $N(G)$ is independent.

Theorem 2.23. *Let $G = (V, E)$ be a loopless graph with at least two edges, $r : V(G) \to Z_+$, and let f be an r-degree specification, where $f(v) = (f_1^v, f_2^v, \ldots, f_{r(v)}^v)$ and $f_1^v \geq f_2^v \geq \ldots \geq f_{r(v)}^v$ for all $v \in V$. Suppose that $N(G)$ is independent. Then G has a non-separable f-detachment if and only if*
(a) G is 2-edge connected,
(b) $f_i^v \geq 2$ for all $v \in V$ and all $1 \leq i \leq r(v)$,
(c) $e(X + v, V - X - v) - f_1^v \geq r(X + v) + b(X + v) - 2$ for all $v \in N(G)$ and all $X \subseteq N_2(G, r) - v$.

Proof. We first prove necessity. Suppose that G has a non-separable f-detachment H. It is easy to see that conditions (a) and (b) must hold for G. Choose $v \in N(G)$ and $X \subseteq N_2(G, r) - v$. Let $C_1, C_2, \ldots, C_b$ be the components of $G - (X + v)$, where $b = b(X + v)$, and let C_i' be the subgraph of H induced by the pieces of all vertices of C_i. Let v_1 be the piece of v in H of degree f_1^v. Let S be the set of all pieces of vertices of X in H and all pieces of v other than v_1. Then $|S| = r(X + v) - 1$. Since G is loopless and $N(G)$ is independent, we have $e(X+v, V-X-v) - f_1^v$ edges in H joining the subgraphs $C_1', C_2', \ldots, C_b'$ and vertices in S. Since H is non-separable $H - v_1$ is connected and hence we must have $e(X + v, V - X - v) - f_1^v \geq b + r(X + v) - 2$. Thus (c) holds for G.

We next prove sufficiency. We proceed by contradiction. Suppose that (G, r, f) satisfies (a), (b) and (c) and that G does not have a non-separable f-detachment. We first use Theorem 2.12 to show that G has a non-separable r-detachment. Since (G, r) satisfies (a) and (b), it also satisfies Theorem 2.12 (a) and (b). Furthemore, for $y \in N_1(G, r)$ and $X \subseteq N_2(G, r)$ we have $f_1^y = d(y)$, $r(X + y) = r(X) + 1$, and, since G is loopless and $N(G)$ is independent, $e(X + y, V - X - y) = e(X, V - X) + d(y)$. Since (G, r) satisfies (c), it follows that (G, r) also satisfies Theorem 2.12 (c). Hence G has a non-separable r-detachment.

Applying Lemma 2.20, G has a non-separable h-detachment, H for some r-degree specification h satisfying $h_1^v \geq h_2^v \geq h_3^v = \ldots = h_{r(v)}^v = 2$ for all $v \in V$. We may suppose that H has been chosen such that $\theta(h) = \sum_{v \in V} h_1^v$ is as large as possible. If $h_1^v \geq f_1^v$ for all $v \in V$ then $h(v) \geq_{r(v)} f(v)$ for all $v \in V$ and by Lemma 2.21, G has a non-separable f-detachment. Hence we may suppose that $h_1^v \leq f_1^v - 1$ for some $v \in V$. Necessarily we must have $v \in N_2(G, r)$ and $h_2^v \geq f_2^v + 1 \geq 3$.

Let $v_1, v_2, \ldots, v_{r(v)}$ be the pieces of v in H, where $d_H(v_i) = h_i^v$ for $1 \leq i \leq r(v)$. If $H' = H(v_2 z \to v_1 z)$ is non-separable for some $z \in \Gamma_H(v_2)$, then H' is a non-separable h'-detachment of G with $\theta(h') > \theta(h)$. This contradicts the choice of H. Applying Lemma 2.16, we deduce that $b_H(\{v_1, v_2\}) = d_H(v_2) = h_2^v$.

Claim 2.24. $b_H(\{v_1, v_2, \ldots, v_{r(v)}\}) = h_2^v + r(v) - 2$.

Proof. We have already shown that $b_H(\{v_1, v_2\}) = d_H(v_2) = h_2^v$ and hence the claim holds for $r(v) = 2$. Suppose $r(v) \geq 3$ and choose i, $3 \leq i \leq r(v)$. Let z be a neighbour of v_2 in H such that z, v_i belong to different components of $H - \{v_1, v_2\}$. If $H'' = H(v_2 z \to v_i z)(v_i z' \to v_1 z')$ is non-separable for some $z' \in \Gamma_H(v_i)$, then H'' is a non-separable h''-detachment of G with $\theta(h'') > \theta(h)$. This contradicts the choice of H. Applying Lemma 2.19, we deduce that each edge of H incident to v_i is a cut-edge of $H - \{v_1, v_2\}$ for all $3 \leq i \leq r(v)$. Since $h_i^v = 2$ for $3 \leq i \leq r(v)$, it follows that $H - \{v_1, v_2, \ldots, v_{r(v)}\}$ has $h_2^v + r(v) - 2$ components. □

Let L be the graph obtained from H by contracting $\{v_1, v_2 \ldots, v_{r(v)}\}$ back to the single vertex v. Then L is an ℓ-detachment of G for the r'-degree specification for G defined by $r'(v) = 1$, $\ell(v) = d_G(v)$; and $r'(u) = r(u)$ and $\ell(u) = h(u)$ for $u \in V - v$. Furthermore L is a block-star centered on v and $b_L(v) = h_2^v + r(v) - 2$.

Claim 2.25. *Let L' be an ℓ-detachment of G such that L' is a block-star centered on v. Then $b_{L'}(v) \geq b_L(v)$.*

Proof. Suppose $b_{L'}(v) \leq b_L(v) - 1$. By Lemma 2.22, we can detach v in L' into $r(v)$ pieces $v_1', v_2', \ldots v_{r(v)}'$ such that, in the resulting graph H^* we have $d_{H^*}(v_2') = b_{L'}(v) - 2r(v) + 4$, $d_{H^*}(v_i') = 2$ for $3 \leq i \leq r(v)$, $d_{H^*}(v_1') > h_1^v$, and neither v_1' nor v_2' is a cut-vertex of H^*. Using Claim 2.24, we have

$$d_{H^*}(v_2') = b_L(v) - 2r(v) + 4 = h_2^v - r(v) + 2 \leq h_2^v.$$

Since H^* is 2-edge-connected and $d_{H^*}(v_i') = 2$ for $3 \leq i \leq r(v)$, v_i' is not a cut-vertex of H^* for $1 \leq i \leq r(v)$. Thus, if H^* had a cut-vertex x, then x would belong to some component of $L' - v$, and x would be a cut-vertex of L' distinct from v. Hence H^* is a non-separable h^*-detachment of G for some r-degree specification h^* for G satisfying $\theta(h^*) > \theta(h)$. This contradicts the choice of H. □

It follows from Claim 2.25 that we may apply Lemma 2.6 to (G, L, ℓ). Let S_i and W_i be the subsets of $V(L)$ defined as in Lemma 2.6 (but with respect to v). Since the sets S_i are pairwise disjoint and L is finite, we may choose i such that $S_{i+1} = \emptyset$. Let $X' = (W_i - v)$ and $X = \{x \in V(G) :$ some piece of x in L belongs to $X'\}$.

By Lemma 2.6, every edge $x_1 u$ of $L - v$ with $x_1 \in X'$ is a cut-edge in $L - v$. Thus the graph we get from $L - v$ by contracting each component of $L - X' - v$ to a single vertex is a forest F with $b_L(v)$ components and $|X'| + b_L(X' + v)$ vertices. Using the facts that $X + v \subseteq N(G)$, and $N(G)$ is an independent set of vertices in G, we deduce that F has $e_L(X', V(L) - X')$ edges. Thus

$$e_L(X', V(L) - X') = b_L(X' + v) + |X'| - b_L(v). \tag{5}$$

We have $e_L(X', V(L) - X') = e_G(X + v, V(G) - X - v) - d_L(v)$, $|X'| = r(X)$, $b_L(v) = h_2^v + (r(v) - 2 = d_G(v) - h_1^v - (r(v) - 2)$, and $d_L(v) = d_G(v)$. Furthermore, for each $u \in V(G) - X - v$, all pieces of u in L belong to the same component of $L - X' - v$, since $S_{i+1} = \emptyset$. Thus $b_G(X + v) = b_L(X' + v)$. Substituting into (5) we obtain $e_G(X + v, V(G) - X - v) = r(X + v) + b_G(X + v) + h_1^v - 2$. Since $h_1^v \leq f_1^v - 1$, this contradicts the fact that G satisfies (c). □

Proof of Theorem 2.2. Let $\hat{G} = (\hat{V}, \hat{E})$ be obtained from G by subdividing every edge of G. Then $\hat{G}$ is loopless, $N(\hat{G}) = N(G)$ and $N(\hat{G})$ is independent. Extend r to $\hat{r}$ and f to $\hat{f}$ by putting $\hat{r}(v) = r(v)$ and $\hat{f}(v) = f(v)$ for all $v \in V(G)$; $\hat{r}(v) = 1$ and $\hat{f}(v) = 2$ for all $v \in \hat{V} - V$. Then $N_1(\hat{G}, \hat{r}) = N_1(G, r)$, $N_2(\hat{G}, \hat{r}) = N_2(G, r)$. We shall show that conditions (a), (b), and (c) of Theorem 2.2 hold for (G, r, f) if and only if conditions (a), (b), and (c) of Theorem 2.23 hold for $(\hat{G}, \hat{r}, \hat{f})$. Clearly Theorem 2.2 (a) and (b) hold for (G, r, f) if and only if Theorem 2.23 (a) and (b) hold for $(\hat{G}, \hat{r}, \hat{f})$. Furthermore for $v \in N(G) = N(\hat{G})$ and $X \subseteq N_2(G, r) = N_2(\hat{G}, \hat{r})$, we have $f_1^v = \hat{f}_1^v$, $r(X) = \hat{r}(X)$, and $e_G(X+v, V-X-v) + e_G(X+v) - b_G(X+v) = e_{\hat{G}}(X+v, \hat{V}-X-v) - b_{\hat{G}}(X+v)$. Thus Theorem 2.2 (c) holds for (G, r, f), if and only if Theorem 2.23 (c) holds for $(\hat{G}, \hat{r}, \hat{f})$. □

We close this section by noting that our proofs of Theorems 2.1 and 2.2 are constructive and give rise to polynomial algorithms which either construct the specified detachment or construct a certificate that shows it does not exist.

3 Some Corollaries and Open Problems

Our first corollary extends Euler's Theorem.

Corollary 3.1. *Let $G = (V, E)$ be a 2-edge-connected graph and $r : V \to Z_+$ such that $d(v) \geq 2r(v)$ for all $v \in V$ and $r(v) \geq 2$ for all $v \in N(G)$. Let f be an r-degree specification for G such that $f(v) = (f_1^v, f_2^v, \ldots, f_{r(v)}^v)$ and $2 \leq f_i^v \leq \lceil d(v)/2 \rceil - r(v) + 2$ for all $v \in V$ and all $1 \leq i \leq r(v)$. Then G has a nonseparable f-detachment.*

Proof. Theorem 2.1 implies that G has a non-separable r-detachment (conditions (b) and (c) of Theorem 2.1 hold vacuously for G since $N_2(G) = \emptyset$). The existence of a non-separable f detachment now follows from Lemma 2.20 and 2.21. It can also be derived from Theorem 2.2. □

The more difficult direction of Euler's theorem follows from Corollary 3.1 by taking $r(v) = d(v)/2$ in a graph in which all vertices have even degree. Our next corollary is a result of Hakimi [1] which characterises the degree sequences of non-separable graphs.

Corollary 3.2. *Let $d_1 \geq d_2 \geq \ldots \geq d_n \geq 2$ be integers with $n \geq 2$. Then there exists a non-separable graph with degree sequence $(d_1, d_2, \ldots, d_n)$ if and only if*
(a) $d_1 + d_2 + \ldots + d_n$ is even, and
(b) $d_1 \leq d_2 + d_3 + \ldots + d_n - 2n + 4$.

Proof. We first prove necessity. Suppose there exists a non-separable graph H with this degree sequence and let $v_i \in V(H)$ have degree d_i for $1 \leq i \leq n$. Clearly (a) holds. Since H is non-separable, $H - v_1$ is connected. Thus $|E(H - v_1)| \geq n - 2$. Hence $d_1 = e(V - v_1, v_1) \leq d_2 + d_3 + \ldots + d_n - 2n + 4$.

Sufficiency follows by applying Theorem 2.2 to the graph G consisting of a single vertex v incident to $(d_1 + d_2 + \ldots + d_n)/2$ loops, by setting $r(v) = n$ and $f(v) = (d_1, d_2, \ldots, d_n)$. □

Our next result considers the case when we only want to detach one vertex in a graph. The special case when $d(v)$ is even and $r(v) = d(v)/2$ gives a 'splitting off' result for non-separable graphs.

Corollary 3.3. *Let $G = (V, E)$ be a graph, $u \in V$, and $r : V \to Z_+$ such that $r(u) = m \geq 2$ and $r(v) = 1$ for all $v \in V - u$. Let f be an r-degree specification for G where $f(u) = (f_1, f_2, \ldots, f_m)$, $f_1 \geq f_2 \geq \ldots \geq f_m \geq 2$, and $f(v) = d(v)$ for $v \in V - u$. Then G has a non-separable f-detachment if and only if*
(a) G is 2-edge-connected,
(b) $e(v) = 0$ and $b(v) = 1$ for all $v \in V - u$,
(c) $f_2 + f_3 + \ldots + f_m \geq b(u) + e(u) + m - 2$, and
(d) $e(u, V - v - u) + e(u) \geq m + b(u, v) - 1$ for all $v \in V - u$.

Proof. The necessity of conditions (a)-(d) is easy to see. To prove sufficiency, we suppose that G satisfies (a)-(d) and use Theorem 2.2 to deduce that G has a non-separable f detachment. It is easy to see that conditions (a) and (b) of Theorem 2.2 hold for G. To see that condition (c) of Theorem 2.2 holds, let $v \in N(G)$ and $X \subseteq N_2(G) - v$. Since $N_2(G) = \{u\}$ we have $(v, X) \in \{(v, \emptyset), (u, \emptyset), (v, \{u\})\}$. Condition (c) of Theorem 2.2 holds for each of these three alternatives since conditions (b), (c), and (d) of the corollary hold for G. Note that when $(v, X) = (v, \{u\})$ we have

$$\begin{aligned} e(X + v, V - X - v) + e(X + v) - f_1^v &= e(\{u, v\}, V - \{u, v\}) + e(\{u, v\}) - d(v) \\ &= e(u, V - v - u) + e(u), \end{aligned}$$

since $e(v) = 0$ by condition (b) of the corollary. □

We next consider non-separable simple detachments.

Corollary 3.4. *Let $G = (V, E)$ be a graph and $r : V \to Z_+$. Then G has a non-separable simple r-detachment if and only if*
(a) G is 2-edge connected,
(b) $d(v) \geq 2r(v)$ for all $v \in V$,
(c) $e(X, V - X - y) + e(X) \geq r(X) + b(X + y) - 1$ for all $y \in N_1(G, r)$ and $X \subseteq N_2(G, r)$, and
(d) $e(u) \leq r(u)(r(u) - 1)/2$ and $e(u, v) \leq r(u)r(v)$ for all $u, v \in V$.

Proof. Necessity of (a),(b),(c) follows from Theorem 2.1 while necessity of (d) is obvious. To see sufficiency we use Theorem 2.1 to deduce that G has a non-seperable r-detachment H. We may assume that H has as few parallel edges as possible. Suppose that $e_H(u_1, v_1) \geq 2$ for two vertices u_1 and v_1 of H. Let u_1 and v_1 be pieces in H of the vertices u and v, respectively, of G, (allowing the possibility that $u = v$). Then (d) implies that there exist distinct pieces u_i of u and v_j of v in H such that $e_H(u_i, v_j) = 0$. Then $H - u_1v_1 + u_iv_j$ has one less parallel edge than H. □

It is an open, and perhaps difficult, problem to characterise when a graph has a non-separable simple detachment for some given degree specification.

A graph $G = (V, E)$ is said to be *k-connected* if $|V| \geq k + 1$ and $G - U$ is connected for all $U \subseteq V(G)$ with $|U| \leq k - 1$. Thus, if $|V| \geq 3$, then G is non-separable if and only if G is 2-connected and loopless. Our next result characterises when a graph has a 2-connected r-detachment.

Corollary 3.5. *Let $G = (V, E)$ be a graph and $r : V \to Z_+$ such that $r(V) \geq 3$. Then G has a 2-connected r-detachment if and only if*
(a) G is 2-edge connected,
(b) $d(v) \geq 2r(v)$ for all $v \in V$,
(c) $e(X, V-X-y)+e(X) \geq r(X)+b(X+y)-1$ for all $y \in N_1(G, r)$ and $X \subseteq N_2(G, r)$.

Proof. This follows easily by applying Theorem 2.1 to (G', r) where G' is the graph obtained from G by deleting all loops incident to vertices in $N_1(G, r)$. □

By Theorem 1.3, condition (c) of Corollary 3.5 is equivalent to the statement "$G-y$ has a connected $r|_{V-y}$-detachment for all $y \in N(G)$ with $r(y) = 1$". It is conceivable that Corollary 3.5 extends to k-connectivity as follows.

Conjecture 3.6. *Let $k \geq 2$ be an integer, $G = (V, E)$ be a graph, and $r : V \to Z_+$ such that $r(V) \geq k + 1$. Then G has a k-connected r-detachment if and only if*
(a) G is k-edge connected,
(b) $d(v) \geq kr(v)$ for all $v \in V$,
(c) $G-y$ has a $(k-r(y))$-connected $r|_{V-y}$-detachment for all $y \in V$ with $r(y) \leq k-1$.

Using Theorem 1.1, it can be seen that the truth of this conjecture for j-connected detachments for all $2 \leq j \leq k$ would be equivalent to the truth of the following conjecture.

Conjecture 3.7. *Let k be a positive integer, $G = (V, E)$ be a graph, and $r : V \to Z_+$ such that $r(V) \geq k + 1$. Then G has a k-connected r-detachment if and only if*
(a) $G - Y$ is $(k - r(Y))$-edge connected for all $Y \subseteq V$ with $r(Y) \leq k - 2$,
(b) $d(v)-e(v, Y) \geq (k-r(Y))r(v)$ for all $v \in V$ and all $Y \subseteq V - v$ with $r(Y) \leq k-2$,
(c) $e(X, V - X - Y) + e(X) \geq r(X) + b(X \cup Y) - 1$ for all $Y \subseteq V$ with $r(Y) \leq k - 1$ and all $X \subseteq V - Y$.

Conjecture 3.7 is true for $k = 1, 2$ by Theorem 1.1, and Corollary 3.5, respectively.

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