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A proof of Connelly's conjecture on 3-connected generic cycles

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Abstract

A graph $G = (V, E)$ is called a *generic cycle* if $|E| = 2|V| - 2$ and every $X \subset V$ with $2 \leq |X| \leq |V| - 1$ satisfies $i(X) \leq 2|X| - 3$. Here $i(X)$ denotes the number of edges induced by X. The operation *extension* subdivides an edge uw of a graph by a new vertex v and adds a new edge vz for some vertex $z \neq u, w$. R. Connelly conjectured that every 3-connected generic cycle can be obtained from K_4 by a sequence of extensions. We prove this conjecture. As a corollary, we also obtain a special case of a conjecture of Hendrickson on generically globally rigid graphs.

Keywords: graphs; connectivity; rigidity

1 Introduction

Let $G = (V, E)$ be a loopless undirected graph, where V is the set of vertices and E is the set of edges of G. For a subset $X \subseteq V$ let $i(X)$ denote the number of edges induced by X in G. A graph $G = (V, E)$ with $|V| \geq 4$ is called a *generic cycle* if $|E| = 2|V| - 2$ and G satisfies

$$i(X) \leq 2|X| - 3 \text{ for all } X \subset V \text{ with } 2 \leq |X| \leq |V| - 1. \quad (1)$$

It is easy to see by (1) that every generic cycle G is a simple graph (i.e. G has no multiple edges) with minimum degree 3 and with at least four vertices of degree 3.

Generic cycles appear in rigidity problems of graphs. A graph is said to be *generically rigid* in the plane if every embedding of G in the plane with algebraically independent coordinates results in a rigid framework (where vertices of G correspond to

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joints and edges of G correspond to rigid rods). By a celebrated result of Laman [7] a graph $G = (V, E)$ is minimally generically rigid in the plane (or *isostatic*) if and only if $|E| = 2|V| - 3$ and G satisfies (1). Thus the "minimally redundantly rigid" graphs are the generic cycles. Furthermore, the edge set of a generic cycle corresponds to a cycle in a certain rigidity matroid. This motivates the name "generic cycle". (For more details on the rigidity background and the matroid connections see e.g. [4].)

R. Connelly conjectured that 3-connected generic cycles have a simple constructive characterization (see e.g. [4, p.99]). The operation *extension* of a graph $H = (V, E)$ consists of subdividing an edge $uw \in E$ by a new vertex v and adding a new edge vz for some $z \neq u, w$. It is easy to see that an extension of a 3-connected generic cycle is also a 3-connected generic cycle. Connelly conjectured that every 3-connected generic cycle can be obtained from the complete graph K_4 on four vertices (which is the smallest generic cycle) by a sequence of extensions. To prove this conjecture it is enough to show that every 3-connected generic cycle on at least five vertices has a vertex v of degree 3 which can be the last vertex added by such a sequence of extensions, i.e. which can be eliminated from G by the inverse operation of extension. This operation is called *splitting off*: it consists of deleting one of the edges vz incident to v and replacing the remaining two edges vu, vw by a new edge uw (and then deleting v).

Our main result (Theorem 4.4) shows that every 3-connected generic cycle G has a vertex of degree 3 which can be split off in such a way that the resulted graph is also a 3-connected generic cycle. This implies that Connelly's conjecture is true. Note that it is not true that *any* vertex of degree 3 can be split off. For example, there is no splitting off at the topmost vertex of the graph of Figure 1(a) which results in a generic cycle (or preserves 3-connectivity). The graph of Figure 1(c) has no vertex of degree 3 which can be split off preserving the generic cycle property. This shows that the 3-connectivity condition is necessary. By using our new characterization of 3-connected generic cycles we can prove a special case of a conjecture of Hendrickson on *generically globally rigid* graphs (see Section 5 for the definition).

In the rest of this section we mention some related results. Based on earlier work of Henneberg [6] and Laman [7], Tay and Whiteley [9] gave a constructive characterization of isostatic graphs: they showed that every graph $G = (V, E)$ with $|E| = 2|V| - 3$ satisfying (1) can be obtained from an edge K_2 by a sequence of extensions and "vertex attachments". The latter operation adds a new vertex v and two edges vu, vw for some $u, w \in V$, $u \neq w$. To show this they proved that *any* vertex of degree 3 can be split off (and *any* vertex of degree 2 can be deleted) from an isostatic graph on at least three vertices in such a way that the resulted graph is isostatic. Tay [8] extended this result to the family of graphs $G = (V, E)$ satisfying $|E| = k|V| - k - 1$ and $i(X) \leq k|X| - k - 1$ for every $X \subseteq V$ with $|X| \geq 2$, in the following sense. He proved that by using one of two operations (including a more general version of splitting off) G can be "reduced" along *any* vertex of degree at most $2k - 1$ in such a way that the smaller graph also belongs to the family. If $k \geq 3$ then some vertices of degree at most $2k - 1$ may not be splittable. Recently Frank and Szegő [3] proved that *there exists* a vertex of degree at most $2k - 1$ which can be split off. This led to a constructive characterization of this family of graphs.

Constructive characterizations of 3-connected graphs have also been investigated. The most relevant result is due to Barnette and Grünbaum [1]. They showed that every 3-connected graph can be obtained from K_4 by a sequence of extensions and "double extensions". The latter operation consists of subdividing two edges uw and xy by two new vertices v and z, respectively, and adding a new edge vz. It is easy to see that these operations preserve 3-connectedness. Using extensions only is not sufficient: consider any 3-regular 3-connected graph on at least 6 vertices.

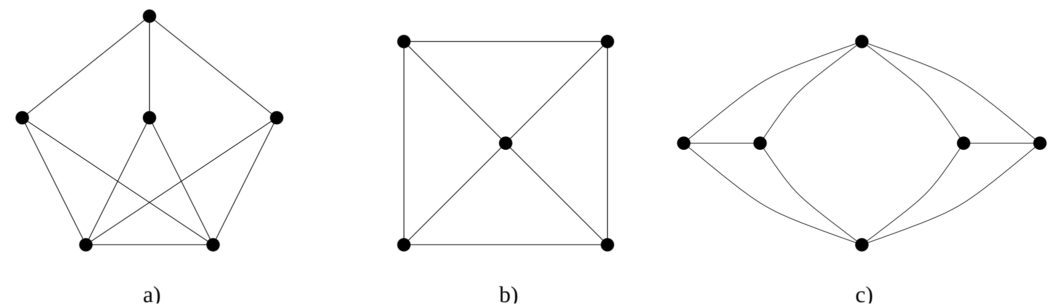

Figure 1: Generic cycles

2 Properties of generic cycles and fragments of 2-connected graphs

In this section we prove several basic properties of generic cycles and 2-connected graphs. We start with some definitions. Given a graph $G = (V, E)$ and two disjoint subsets $X, Y \subset V$, we use $d(X, Y)$ to denote the number of edges from X to Y. We define $d(X) := d(X, V - X)$. The *degree* of a vertex v is denoted by $d(v)$. Let $V_3 := \{v \in V : d(v) = 3\}$ denote the set of degree 3 vertices of G. For convenience, vertices of degree 3 are called *nodes*. The subgraph induced by some $X \subseteq V$ is denoted by $G[X]$. We call $G[V_3]$ the *subgraph of nodes of* G. A node of G with degree at most one (exactly two, exactly three) in the subgraph of nodes of G is called a *leaf node* (*series node*, *branching node*, respectively). A *wheel* $W_n = (V, E)$ is a graph on $n \geq 4$ vertices which has a vertex z which is adjacent to all the other vertices and for which $W_n[V - z]$ is a cycle. Thus the subgraph of nodes of a wheel W_n with $n \geq 5$ is a cycle.

Lemma 2.1. *If $G = (V, E)$ is a generic cycle then either G is a wheel or $G[V_3]$ is a forest.*

Proof. Suppose that the subgraph of nodes of G contains a cycle and choose a shortest (diagonal free) cycle C of $G[V_3]$. Since G is not a cycle, $\bar{C} := V - V(C) \neq \emptyset$. Since each vertex of C is a node and C has no diagonals, $|\bar{C}| = 1$ implies that G is a wheel. Hence we may assume that $|\bar{C}| \geq 2$. In this case $i(\bar{C}) = 2|V| - 2 - i(C) - d(C, \bar{C}) = 2|V| - 2 - |C| - |C| = 2(|V| - |C|) - 2 = 2|\bar{C}| - 2$, contradicting (1). $\square$

We shall frequently use the following equality, which is easy to check by counting the contribution of an edge of $G = (V, E)$ to the two sides: for every pair $X, Y \subseteq V$ we have

$$i(X) + i(Y) + d(X - Y, Y - X) = i(X \cap Y) + i(X \cup Y). \tag{2}$$

A set $X \subset V$ with $|X| \geq 2$ is called *critical* in a generic cycle $G = (V, E)$ if $i(X) = 2|X| - 3$ holds. In the rest of this subsection let $G = (V, E)$ be a given generic cycle.

Lemma 2.2. *Let $X, Y \subset V$ be critical sets with $|X \cap Y| \geq 2$ and $|X \cup Y| \leq |V| - 1$. Then $X \cap Y$ and $X \cup Y$ are both critical, and $d(X - Y, Y - X) = 0$.*

Proof. By (1) and (2) we get $2|X| - 3 + 2|Y| - 3 + d(X - Y, Y - X) = i(X) + i(Y) + d(X - Y, Y - X) = i(X \cap Y) + i(X \cup Y) \leq 2|X \cap Y| - 3 + 2|X \cup Y| - 3 = 2|X| - 3 + 2|Y| - 3$. Thus equality holds everywhere, and hence $X \cap Y$ and $X \cup Y$ are critical, and $d(X - Y, Y - X) = 0$. □

A graph $H = (V, E)$ is 2-*connected* (3-*connected*) if it has at least three (resp. four) vertices and $H - X$ is connected for any $X \subset V$ with $|X| \leq 1$ ($|X| \leq 2$, respectively). A pair $u, v \in V$ is a *cutpair* in a 2-connected graph H if $H - \{u, v\}$ is disconnected.

Lemma 2.3. *(a) For every $\emptyset \neq X \subset V$ we have $d(X) \geq 3$ and if $d(X) = 3$ holds then either $|X| = 1$ or $|V - X| = 1$;*
(b) If $X \subset V$ is critical with $|X| \geq 3$ then $G[X]$ is 2-connected;
(c) G is 2-connected, and for any cutpair a, b and for any bipartition A, B of $G - \{a, b\}$ with $d(A, B) = 0$ we have that $ab \notin E$, $A + \{a, b\}$ and $B + \{a, b\}$ are both critical, and $d(a), d(b) \geq 4$.

Proof. To prove (a) first consider a bipartition $X \cup Y = V$, $X \cap Y = \emptyset$ of V with $|X|, |Y| \geq 2$. By (1) we obtain $|E| = i(X) + i(Y) + d(X) \leq 2|X| - 3 + 2|Y| - 3 + d(X) = 2|V| - 6 + d(X) = |E| - 4 + d(X)$. This implies $d(X) \geq 4$. Furthermore, (1) implies that each vertex of G has degree at least 3. This proves (a).

To verify (b) consider a critical set X with $|X| \geq 3$ and suppose that for some $v \in X$ the graph $G[X - v]$ can be partitioned into two non-empty sets A, B such that there are no edges from A to B in $G[X - v]$. Then (1) gives $2|X| - 3 = i(X) = i(A + v) + i(B + v) \leq 2(|A| + 1) - 3 + 2(|B| + 1) - 3 = 2(|A| + |B| + 1) - 4 = 2|X| - 4$, a contradiction.

It is easy to see that G is 2-connected (by using an argument similar to that of the proof of (b)). To prove (c) suppose that a, b is a cutpair in G and A, B is a bipartition of $G - \{a, b\}$ with $d(A, B) = 0$. By (1) and (2), and since there is no edge from A to B, this gives $2(|A| + |B| + 2) - 2 = 2(|A| + 2) - 3 + 2(|B| + 2) - 3 \geq i(A + \{a, b\}) + i(B + \{a, b\}) = i(V) + i(\{a, b\}) = 2|V| - 2 + i(\{a, b\}) = 2(|A| + |B| + 2) - 2 + i(\{a, b\})$. Thus equality holds everywhere. Hence $A + \{a, b\}$ and $B + \{a, b\}$ are both critical and $ab \notin E$. It follows from (b) that $G[A + \{a, b\}]$ and $G[B + \{a, b\}]$ are both 2-connected. Hence a and b have at least two neighbours in each of these subgraphs. Since $ab \notin E$, this implies $d(a), d(b) \geq 4$. □

Lemma 2.4. *Let $X \subset V$ be a critical set with $|V - X| \geq 2$. Then $V - X$ contains at least two nodes.*

Proof. Let X be a critical set and let $Y := V - X$. Clearly, $2i(Y) = \sum_{v \in Y} d_{G[Y]}(v) = \sum_{v \in Y} d(v) - d(Y)$. For a contradiction suppose that $d(v) \geq 4$ for at least $|Y| - 1$ vertices of Y. This implies $\sum_{v \in Y} d(v) \geq 4|Y| - 1$. Using this inequality and Lemma 2.3(a) we can count as follows. $2i(Y) + 2d(Y) = \sum_{v \in Y} d(v) + d(Y) \geq 4|Y| - 1 + 4$. Since the left hand side is even, this implies $2i(Y) + 2d(Y) \geq 4|Y| + 4$, and hence $i(Y) + d(Y) \geq 2|Y| + 2$.

Therefore, since X is critical, (1) gives $|E| = i(X) + i(Y) + d(Y) \geq 2|X| - 3 + 2|Y| + 2 = 2|V| - 1$, a contradiction. This shows that Y contains at least two nodes. □

2.1 Fragments and ends of 2-connected graphs

Let $G = (V, E)$ be a graph. For some $X \subseteq V$ let $N(X)$ denote the set of *neighbours* of X (that is, $N(X) := \{v \in V - X : uv \in E \text{ for some } u \in X\}$). A set $X \subset V$ is called a *fragment* in a 2-connected graph G if $|N(X)| = 2$ and $V - X - N(X) \neq \emptyset$. An inclusionwise minimal fragment is an *end*. The proofs of the following simple lemmas are omitted.

Lemma 2.5. *Let G be a 2-connected graph with at least one cutpair. Then (a) there exist two ends A, B with $A \subseteq V - B - N(B)$ and $B \subseteq V - A - N(A)$; (b) for every end X the subgraph $G[X]$ is connected.*

Lemma 2.6. *Let A be an end in a 2-connected graph G and suppose that $|N(Y) \cap A| = 1$ for some $Y \subset A$. Then $N(A) \subset N(Y)$.*

Lemma 2.7. *Let A be an end with $|A| \geq 2$ in a 2-connected graph G and let $N(A) = \{x, y\}$. Then $G[A \cup N(A)] + xy$ is 3-connected.*

3 Finding admissible nodes

Recall that splitting off a node v with $N(v) = \{u, w, z\}$ means deleting one of the edges incident to v, say vz, and replacing the remaining two edges vu, vw by a new edge uw (and deleting v as well). To specify the split we perform at v we say that this split is made *on the pair* uv, wv. Let G_v denote the graph obtained from G by splitting off node v. Since each node can be split off in three different ways, G_v depends on the split as well. When we write G_v later on then either it will be clear which split is meant or it will be irrelevant. The pair uv, wv (and the corresponding splitting) is called *admissible* if splitting off v on the pair uv, wv results in a generic cycle G_v. We call a node v *admissible* if there is an admissible splitting at v. Otherwise v is *non-admissible*. In this section we show that every 3-connected generic cycle has an admissible node (in fact, either it has at least four admissible nodes or it has three pairwise non-adjacent admissible nodes).

Lemma 3.1. *Let v be a node of a generic cycle $G = (V, E)$ with neighbour set $\{u, w, z\}$. Then v cannot be split off on the pair uv, wv if and only if there is a critical set X in G with $u, w \in X$ and $v, z \notin X$.*

Proof. First suppose that X is a critical set in G with $u, w \in X$ and $v, z \notin X$. Then by splitting off the pair uv, wv (and hence adding a new edge uw) increases $i(X)$ by one. Since $z \notin X$, X contradicts (1) in G_v. Thus v cannot be split on uv, wv.

Conversely, suppose that $Y \subset V(G_v) = V - v$ violates (1) in G_v after splitting off v on the pair uv, wv. Then $i_{G_v}(Y) \geq 2|Y| - 2$. Since $i_{G_v}(Y) \leq i(Y) + 1$, it follows that Y is critical in G and $u, w \in Y$. If $z \in Y$ then $i(Y + v) = i(Y) + 3 = 2|Y| - 3 + 3 = 2|Y + v| - 2$, contradicting (1). Thus $z \notin Y$. $\square$

If v is a node with $N(v) = \{u, w, z\}$ and X is a critical set with $u, w \in X$ and $v, z \notin X$ then we call X a *v-critical set on u and w*. If $d(z) = 3$ then it is obvious that splitting off v on uv, wv is non-admissible, since such a split would make $d_{G_v}(z) = 2$. (This observation shows that all branching nodes are non-admissible.) In this case $V - \{v, z\}$ is a "trivial" v-critical set on u and w. "Non-trivial" critical sets will be of particular interest: if X is a v-critical set on u and w for some node v with $N(v) = \{u, w, z\}$, and $d(z) \geq 4$, then X is called *node-critical*.

Lemma 3.2. *Let $G = (V, E)$ be a 3-connected generic cycle with $|V| \geq 5$ and suppose that v is a non-admissible leaf node of G. Then there exist two v-critical sets X, Y such that $|X \cap Y| \geq 2$ and $X \cup Y = V - v$. Moreover, if v is adjacent to a node z, then X and Y can be chosen to satisfy $z \in X \cap Y$ as well.*

Proof. Let $N(v) = \{x, y, z\}$. Since v is non-admissible, Lemma 3.1 implies that there exist three v-critical sets X, Y, Z on y and z, x and z, x and y, respectively. Suppose that no two of these sets intersect each other in at least two vertices. Let m denote the number of those edges in $G[X \cup Y \cup Z]$ which do not belong to the edge set of $G[X]$, $G[Y]$, or $G[Z]$. Then $2(|X \cup Y \cup Z|) - 3 = 2(|X| + |Y| + |Z| - 3) - 3 \geq i(X \cup Y \cup Z) = i(X) + i(Y) + i(Z) + m = 2|X| - 3 + 2|Y| - 3 + 2|Z| - 3 + m = 2(|X| + |Y| + |Z| - 3) - 3 + m = 2(|X \cup Y \cup Z|) - 3 + m$. Thus equality holds everywhere, and hence $X \cup Y \cup Z$ is critical and $m = 0$. Since $d(v, X \cup Y \cup Z) = 3$, this implies that $X \cup Y \cup Z = V - v$ (otherwise $(X \cup Y \cup Z) + v$ violates (1)). Hence, since $|V| \geq 5$, at least one of the three critical sets X, Y, Z (say, X) satisfies $|X| \geq 3$. But we have $m = 0$, and hence y, z is a cutpair in G, contradicting the fact that G is 3-connected. This contradiction shows that we have two sets (say, X and Y) with $|X \cap Y| \geq 2$. Hence $X \cup Y$ is also critical by Lemma 2.2 and so $X \cup Y = V - v$ follows, since $d(v, X \cup Y) = 3$.

To see the second part of the statement of the lemma suppose that z is a node. Then the edges xz, yz cannot be both present in G because then x, y would be a cutpair. Thus we may assume, without loss of generality, that $yz \notin E$. Then for the v-critical set X on y and z we must have $|X| \geq 3$. By Lemma 2.3(b) $G[X]$ is 2-connected and hence z has two neighbours in X. If z has no neighbours in Y then $xz \notin Y$, $|Y| \geq 3$, and z is an isolated vertex in $G[Y]$. This would contradict Lemma 2.3(b). Hence z has a neighbour in Y and this implies that $|X \cap Y| \geq 2$. By Lemma

2.2 this gives that $X \cup Y$ is also critical, and since $d(v, X \cup Y) \geq 3$, we must have $X \cup Y = V - v$, as required. $\square$

It is easy to see that the wheels are all 3-connected generic cycles. It is also easy to see that each node v of a wheel W_n, $n \geq 5$, is admissible (and the unique admissible splitting at v preserves 3-connectivity as well). Thus in the following four lemmas we shall assume that the given generic cycle is not a wheel.

Lemma 3.3. *Let $G = (V, E)$ be a 3-connected generic cycle and let v be a node with $N(v) = \{x, y, z\}$ and $d(z) \geq 4$. Suppose that X is a v-critical set on x, y and suppose that either (a) there is a non-admissible series node $u \in V - X - v$ with precisely one neighbour w in X, and w is a node, or (b) there is a non-admissible leaf node $t \in V - X - v$. Then there is a node-critical set X' in G with $|X'| > |X|$ and $(X \cap V_3) \subseteq (X' \cap V_3)$.*

Proof. (a) Let $u \in V - X - v$ be a non-admissible series node with $N(u) = \{w, p, n\}$. By our assumption $N(u) \cap X = \{w\}$ and $d(w) = 3$. Since u is a series node, we can assume that $d(p) = 3$ and $d(n) \geq 4$. Since u is non-admissible, there exists a u-critical set Y on w and p by Lemma 3.1. Since $G[V_3]$ has no cycles, we have $pw \notin E$ and hence $|Y| \geq 3$. Thus $G[Y]$ is 2-connected by Lemma 2.3(b) and hence Y contains two neighbours of w. Since $G[X]$ is connected, at least one of these neighbours of w must be in X. Thus $|X \cap Y| \geq 2$. Furthermore, $X' := X \cup Y \subseteq V - u - n$. By Lemma 2.2 it follows that X' is a u-critical set on w and p. Thus, since $d(n) \geq 4$ and $p \notin X$, the set X' is a node-critical set with $|X'| > |X|$ and $(X \cap V_3) \subseteq (X' \cap V_3)$, as required.

(b) Since t is a non-admissible leaf node, Lemma 3.2 implies that there exist two t-critical sets Y_1 and Y_2 with $Y_1 \cup Y_2 = V - t$, $|Y_1|, |Y_2| \geq 3$, and if t has a neighbour r which is a node then we can assume $r \in Y_1 \cap Y_2$. Note that Y_1 and Y_2 are node-critical. If $|X| = 2$ then X induces the edge xy. Since $x, y \in Y_1 \cup Y_2$ and $d(Y_1 - Y_2, Y_2 - Y_1) = 0$ by Lemma 2.2, in this case either $X \subset Y_1$ or $X \subset Y_2$ holds, which proves part (b) of the lemma by choosing $X' = Y_1$ or $X' = Y_2$. Thus we may assume $|X| \geq 3$. Since $Y_1 \cup Y_2 = V - t$, $t \notin X$, and $|X| \geq 3$, we have $|X \cap Y_1| \geq 2$ or $|X \cap Y_2| \geq 2$. Let us assume, without loss of generality, that $|X \cap Y_1| \geq 2$ holds.

We must have $d(t, X) \leq 2$, since $d(t, X) = 3$ would imply that $X + t$ violates (1). If $d(t, X) = 2$ then $X + t$ is also critical and by choosing $X' = X + t$ the lemma follows. Thus we may assume that $d(t, X) \leq 1$ (and hence $|N(t) \cap X| \leq 1$). First suppose that $N(t) \cap X = \emptyset$. Since $|X \cap Y_1| \geq 2$, it follows by Lemma 2.2 that $X \cup Y_1$ is a t-critical set. Therefore the lemma follows by choosing $X' = X \cup Y_1$.

Now suppose that $|N(t) \cap X| = 1$ and let $N(t) \cap X = \{s\}$. If $s \in Y_1$ then $N(t) - (X \cup Y_1) \neq \emptyset$ and hence $X \cup Y_1$ is a node-critical set. Thus we are done as above, by choosing $X' = X \cup Y_1$. If $d(s) = 3$ then $s \in Y_1 \cap Y_2$, thus we may assume that $d(s) \geq 4$ and $s \notin Y_1$. Since $Y_1 \cup Y_2 = V - t$, this gives $s \in Y_2$. Hence if $|X \cap Y_2| \geq 2$ then we are done by choosing the node-critical set $X' = X \cup Y_2$. If $|X \cap Y_2| = 1$ then $|X \cap Y_1| = |X| - 1$. Since $s \notin Y_1$, we have $(N(t) - s) \subseteq Y_1 - X$ and hence $|Y_1 - X| \geq 2$. This shows that $|Y_1| > |X|$. Since $X - s \subset Y_1$ and $d(s) \geq 4$, we have $(X \cap V_3) \subseteq (Y_1 \cap V_3)$ as well. Thus $X' = Y_1$ is a proper choice in this case. This proves the lemma. $\square$

Every generic cycle G has at least four nodes. Hence if G is not a wheel then the subgraph of nodes of G is a forest on at least four nodes. It is easy to check that $(*)$ a forest on at least four nodes satisfies at least one of the following: (i) it has at least four leaf nodes (recall that a leaf node has degree at most one); (ii) it has three pairwise non-adjacent leaf nodes; (iii) it is a path on at least four nodes. We shall use this simple observation in the next three lemmas.

Lemma 3.4. *Let $G = (V, E)$ be a generic cycle. Let $\mathcal{X} = \{X \subset V : X$ is a node-critical set in $G\}$. If $\mathcal{X} = \emptyset$ then either G has four admissible nodes or G has three pairwise non-adjacent admissible nodes.*

Proof. If G has a non-admissible leaf node or series node then G has a node-critical set by Lemma 3.1. Thus every leaf or series node is admissible. This proves the lemma by $(*)$. □

Lemma 3.5. *Let G be a 3-connected generic cycle and suppose that v is an admissible node in G. Let $\mathcal{Y} = \{Y \subset V : v \in Y,\ Y$ is a node-critical set in $G\}$. If $\mathcal{Y} = \emptyset$ then either G has four admissible nodes or G has three pairwise non-adjacent admissible nodes or G has two adjacent admissible nodes.*

Proof. Let $w \neq v$ be a leaf node and suppose that w is non-admissible. Then by Lemma 3.2 there exist two node-critical sets X, Y with $X \cup Y = V - w$, contradicting the fact that v is in no node-critical set. Thus every leaf is admissible. This implies the lemma by $(*)$ unless $G[V_3]$ is a path P on at least four nodes. If v is not a leaf node and v is not adjacent to the two leaf nodes of P then G has three pairwise non-adjacent admissible nodes. If v is adjacent to a leaf then G has two adjacent admissible nodes. Finally, if v is a leaf node, then let vt and tq be the first two edges on P. We claim that t is admissible. Otherwise by Lemma 3.1 there is a t-critical set on v and q, contradicting the fact that $\mathcal{Y}$ is empty. Thus G has two adjacent admissible nodes. □

Lemma 3.6. *Let G be a 3-connected generic cycle and suppose that v and w are adjacent admissible nodes in G. Let $\mathcal{Z} = \{Z \subset V : v, w \in Z,\ Z$ is a node-critical set in $G\}$. If $\mathcal{Z} = \emptyset$ then either G has four admissible nodes or G has three pairwise non-adjacent admissible nodes.*

Proof. Let $u \neq v, w$ be a leaf node. If u is non-admissible then by Lemma 3.2 there exist two u-critical sets X, Y with $|X \cap Y| \geq 2$ and $X \cup Y = V - u$. By Lemma 2.2 we have $d(X - Y, Y - X) = 0$. Thus either $\{v, w\} \subset X_1$ or $\{v, w\} \subset X_2$ (or both), contradicting $\mathcal{Z} = \emptyset$. Thus every leaf is admissible. Hence $(*)$ implies that either G has four admissible nodes or G has three pairwise non-adjacent admissible nodes or $G[V_3]$ is a path P on at least four nodes. Consider the last case. If v and w are both inner nodes on P then G has four admissible nodes. Otherwise suppose, without loss of generality, that v is a leaf on P and let the first three edges on P be vw, wt, tq. We claim that splitting off t on the edges wt, tq is admissible. If not, then there exists a t-critical set X on w and q by Lemma 3.1. Since $wq \notin E$, we have $|X| \geq 3$. Thus by Lemma 2.3(b) $G[X]$ is 2-connected. This implies that both neighbours of w other than t must be in X and hence $v \in X$ follows. This contradicts the fact that $\mathcal{Z} = \emptyset$. Thus t is admissible and hence G has at least four admissible nodes. $\square$

The next result on the number of admissible nodes is the main result of this section.

Theorem 3.7. *Let $G = (V, E)$ be a 3-connected generic cycle with $|V| \geq 5$. Then either G has four admissible nodes or G has three pairwise non-adjacent admissible nodes.*

Proof. The theorem trivially holds if G is a wheel, so suppose that G is not a wheel. Hence the subgraph of nodes of G is a forest by Lemma 2.1. Let $\mathcal{X} = \{X \subset V : X \text{ is a node-critical set in } G\}$. If $\mathcal{X} = \emptyset$ then we are done by Lemma 3.4. Otherwise let X be a maximum size member of $\mathcal{X}$. Since X is node-critical, there exists a node v and $t \in N(v)$ such that X is a v-critical set, $d(t) \geq 4$, and $t \notin X$. Clearly, $X + v$ is also critical and $|V - X - v| \geq 2$. By applying Lemma 2.4 to $X + v$ we obtain that $V - X - v$ contains at least two nodes. The maximality of $|X|$ implies that if z is a branching node in $V - X - v$ then z has at most one neighbour in X (otherwise either $X + z$ would be a larger node-critical set or $X + z$ would contradict (1)). Therefore the leaves of the subforest $G[V_3 \cap (V - X - v)]$ cannot be branching nodes in G. This subforest has at least two nodes and hence it has at least two leaves. This implies that $V - X - v$ contains two non-branching nodes u, w. If u (or w) is a series node then it has precisely one neighbour in X by the maximality of X and by (1), and this neighbour must be a node, since u is a leaf in the subforest. By Lemma 3.3 and by the maximality of $|X|$ the nodes u, w are admissible.

Now let us define $\mathcal{Y} = \{Y \subset V : u \in Y,\ Y \text{ is a node-critical set in } G\}$. If $\mathcal{Y} = \emptyset$ then either Lemma 3.5 completes the proof of the theorem or G has two adjacent admissible nodes. First suppose that G has no adjacent admissible nodes. Then $\mathcal{Y} \neq \emptyset$ and we can choose a maximum size member Y of $\mathcal{Y}$. Since Y is node-critical, there exists a node v' and $t' \in N(v')$ such that Y is v'-critical, $d(t') \geq 4$, and $t' \notin Y$. By using a similar argument we applied to $X \in \mathcal{X}$ above, we obtain that there exist two admissible nodes u', w' in $V - Y - v'$. Since we are in the case when G has no adjacent admissible nodes, and since $u \in Y$, it follows that G has three pairwise non-adjacent admissible nodes, as required.

Now suppose that G has two adjacent admissible nodes s, r. Let $\mathcal{Z} = \{Z \subset V : s, r \in Z,\ Z \text{ is a node-critical set in } G\}$. If $\mathcal{Z} = \emptyset$ then the theorem follows by Lemma 3.6. Otherwise let Z be a maximum size member of $\mathcal{Z}$. Since Z is node-critical, there exists a node v'' and $t \in N(v'')$ such that Z is v''-critical, $d(t'') \geq 4$ and $t'' \notin Z$. By using a similar argument we applied to $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ above we obtain that there exist two admissible nodes in $V - Z - v''$. Since $s, r \in Z$ are admissible, this shows that G has at least four admissible nodes. This proves the theorem. $\square$

The theorem is best possible in the sense that there exist 3-connected generic cycles containing precisely four admissible nodes but no three pairwise non-adjacent admissible nodes (see Figure 1(b)) and there exist 3-connected generic cycles containing precisely three non-adjacent admissible nodes but no four admissible nodes (see Figure 1(a)). Furthermore, the graph of Figure 1(c) is a generic cycle with no admissible nodes. This shows that the assumption on 3-connectivity is essential.

3.1 A characterization of generic cycles

In this subsection we give a constructive characterization of generic cycles based on the fact that every 3-connected generic cycle has an admissible node. We note that we shall not use this characterization in the proof of our main result in the next section (but we shall rely on some of the lemmas given in this subsection).

Let $H = (V, E)$ be a graph and suppose that for two sets $X, Y \subset V$ with $|X|, |Y| \geq 3$ and $X \cup Y = V$ we have $X \cap Y = \{a, b\}$, $d(X - Y, Y - X) = 0$, and $ab \notin E$. A 2-*separation of* H along the cutpair a, b (or along the pair X, Y) results in two graphs $H[X] + ab$ and $H[Y] + ab$. The inverse operation of 2-separation is 2-sum: given two graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ with two designated edges $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$, the 2-*sum* of H_1 and H_2 (along the edge pair u_1v_1, u_2v_2), denoted by $H_1 \oplus H_2$, is the graph obtained from $H_1 - u_1v_1$ and $H_2 - u_2v_2$ by identifying u_1 with u_2 and v_1 with v_2. The following lemma can be verified by simple calculations, using inequality (1).

Lemma 3.8. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be generic cycles and let $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$. Then the 2-sum $G_1 \oplus G_2$ along the edge pair u_1v_1, u_2v_2 is a generic cycle.*

The next lemma is also easy to prove, using Lemma 2.3(c) and (1).

Lemma 3.9. *Let $G = (V, E)$ be a generic cycle and let G' and G'' be the graphs obtained from G by a 2-separation. Then G' and G'' are both generic cycles.*

Let $G = (V, E)$ be a graph and let $uw \in E$. Recall that an *extension* of G along uw is obtained from G by subdividing the edge uv by a new vertex v (i.e. replacing the edge uw by a path uvw) and adding a new edge vz for some $z \in V - \{u, v\}$. The next lemma is easy to prove.

Lemma 3.10. *Let G be a generic cycle (a 3-connected graph) and let G' be obtained from G by an extension. Then G' is a generic cycle (a 3-connected graph, respectively).*

The following theorem shows that we can construct any generic cycle G by using a sequence of 2-sum and extension operations, starting from a collection of disjoint K_4's. Note that the 2-sum (extension) operation is always performed on two distinct connected components (within a connected component, respectively). In the next section we shall prove that if G is 3-connected then it is sufficient to use extensions only.

Theorem 3.11. *$G = (V, E)$ is a generic cycle if and only if G is a connected graph obtained from disjoint copies of K_4's by taking 2-sums and applying extensions.*

Proof. By Lemma 3.8 and Lemma 3.10 it follows that a connected graph built up from disjoint copies of K_4's by 2-sums and extensions is a generic cycle. To prove the other direction we need to show that if G is a generic cycle then the inverse operation of either the 2-sum or the extension can be applied to G in such a way that the resulting graphs (or graph) are (is) generic cycle(s). The inverse operations are 2-separation and splitting off, respectively. If G has a cutpair then Lemma 2.3(c) and Lemma 3.9 show that we can apply 2-separation. If G is 3-connected then either $G = K_4$ or $|V| \geq 5$ and Theorem 3.7 shows that we can apply splitting off. This proves the theorem. $\square$

4 Finding a feasible node

We call a node v of a 3-connected generic cycle G *feasible* if there is an admissible splitting at v for which G_v is 3-connected. In this section we prove Connelly's conjecture by showing that every 3-connected generic cycle G on at least five vertices has a feasible node. In the next three lemmas we describe three special configurations and prove that if one of these configurations is present in a 3-connected generic cycle $G = (V, E)$ then G has a feasible node.

Lemma 4.1. *Let v and w be two adjacent nodes and suppose that $x \in N(v) \cap N(w)$. Then v is feasible.*

Proof. Let $N(v) = \{w, x, y\}$. Since G is 3-connected, $wy \notin E$. We claim that splitting off v on the pair vw, vy is admissible. If this is not the case, then by Lemma 3.1 there exists a v-critical set X on w and y with $x \notin X$. Since $wy \notin X$, $|X| \geq 3$ must hold. By Lemma 2.3(b) this implies that $G[X]$ is 2-connected and hence w has at least two neighbours in X. This is a contradiction, since w is a node which is adjacent to x and v, and $x, v \notin X$. Thus v is admissible.

Next we show that the graph G_v obtained by this admissible splitting at v is 3-connected. If this is not the case then G_v has a cutpair a, b. Let A and B be two components of $G_v - \{a, b\}$. Since G is 3-connected, v has neighbours $a' \in A$ and $b' \in B$ in G. Since $wy, wx \in E(G_v)$ and a' and b' are separated in G_v, we can assume that $a' = x$, $b' = y$, and $w \in \{a, b\}$. Since $d_{G_v}(w) = 3$, this contradicts Lemma 2.3(c). Thus v is feasible. $\square$

Lemma 4.2. *Let v be a node in G with $i(N(v) + v) = 5$. Then v is feasible.*

Proof. Let $N(v) = \{x, y, z\}$ and suppose that $xz, yz \in E$. First we prove that splitting off v on the pair vx, vy is admissible. If this is not the case, then by Lemma 3.1 there exists a v-critical set X on x and y with $z \notin X$. By (1) we must have $d(z, X) = 2$ and hence $X + z$ is also critical. If $X + z \neq V - v$ then $X + z + v$ contradicts (1), since $d(v, X + z) \geq 3$. If $X + z = V - v$ then either $d(z) = 3$, in which case x, y is a cutpair, contradicting the 3-connectivity of G, or $d(z) \geq 4$, in which case $i(X + v + z) \geq i(X) + 6 = 2|X| - 3 + 6 = 2(|X| + 2) - 1$, contradicting (1). This shows that v is admissible.

Next we show that the graph G_v obtained by this admissible split at v is 3-connected. Suppose that G_v has a cutpair a, b and let A, B be two components of $G_v - \{a, b\}$. Since a, b is not a cutpair in G, v has two neighbours $a' \in A$ and $b' \in B$ in G, which are separated in G_v. This contradicts the fact that x, y, z are pairwise adjacent in G_v. Thus v is feasible. □

Lemma 4.3. *Let a and b be two nodes of G with $N(a) = N(b)$ such that there exists a node $v \in N(b)$. Then b is feasible.*

Proof. Let $N(a) = N(b) = \{v, x, y\}$. It is easy to see that $|V| = 5$ implies $G = W_5$ and hence each node of G is feasible. Thus we may assume that $|V| \geq 6$. Since G is 3-connected, $vx, vy \notin E$. Let $z = N(v) - \{a, b\}$. First we prove that splitting off b on the edge pair bx, bv is admissible. If this is not the case, then by Lemma 3.1 there exists a b-critical set X on x and v. Since $vx \notin E$, we have $|X| \geq 3$, and hence $G[X]$ is 2-connected. This implies that both vertices in $N(v) - b$ are in X. Thus $a, z \in X$ holds and, since $y \notin X$, a has degree two in $G[X]$. Therefore $X - a$ is also critical. Since $|X - a| \geq 3$, $G[X - a]$ is also 2-connected. This contradicts the fact that v has only one neighbour (namely, z) in $X - a$. This proves that splitting off b on bx, bv is admissible.

To see that this split preserves 3-connectivity suppose that r, s is a cutpair in G_b. Then, since G is 3-connected, it follows that b has at least one neighbour in G in each component of $G_v - \{r, s\}$. Since $N(a) = N(b)$, this implies that $a \in \{r, s\}$. Since $d_{G_b}(a) = 3$ and G_b is a generic cycle, this contradicts Lemma 2.3(c). Thus b is feasible. □

We call the configurations of Lemma 4.1 (Lemma 4.2, Lemma 4.3) a *good triangle* (a *dense node*, a *good pair*, respectively).

Theorem 4.4. *Let $G = (V, E)$ be a 3-connected generic cycle with $|V| \geq 5$. Then G has at least two feasible nodes.*

Proof. We can assume that G is not a wheel. Let us fix a vertex $c \in V$ arbitrarily. We shall prove that G has a feasible node $v' \neq c$. This will imply the existence of at least two feasible nodes. By Theorem 3.7 G has an admissible node in $V - c$. For a contradiction suppose that no admissible node in $V - c$ is feasible in G. Then any admissible splitting at any admissible node $w \neq c$ results in a generic cycle G_w which is not 3-connected. Hence G_w has at least one cutpair, implying that it has two disjoint fragments, and so it has a fragment not containing c. For every admissible node $w \neq c$ let us fix Y_w to be a minimum cardinality fragment not containing c among all fragments of all possible generic cycles G_w we can obtain from G by an admissible splitting at w. Note that $|Y_w| \geq 2$. Let us choose an admissible node $v \neq c$ in G for which $|Y_v|$ is as small as possible and let G_v be the generic cycle which contains Y_v. Let $N_{G_v}(Y_v) = \{x, y\}$. By Lemma 3.9, a 2-separation of G_v along the pair $(Y_v + x + y), (V - Y_v)$ results in two generic cycles G_Y and $G_{\bar{Y}}$. Let G_Y be the generic cycle containing Y_v.

Claim 4.5. *If $|V(G_Y)| = 4$ then G has a feasible node $v' \neq c$.*

Proof. We shall prove that if $|V(G_Y)| = 4$ then either G has a good triangle or G has a dense node or G has a good pair. The claim will then follow from Lemma 4.1, Lemma 4.2, or Lemma 4.3 (by noting that the corresponding feasible node is in Y_v). Let $V(G_Y) = \{a, b, x, y\}$, where $Y_v = \{a, b\}$ and $N_{G_Y}(Y_v) = \{x, y\}$. Since G is 3-connected, we have $|N_G(v) \cap Y_v| \geq 1$ and $|N_G(v) \cap V(G_Y)| \leq 2$. First suppose that v has precisely one neighbour in $V(G_Y)$ in G. We can assume that this neighbour is a. Then b is a dense node in G. Next suppose that $|N_G(v) \cap V(G_Y)| = 2$. Then, without loss of generality, either $\{x, a\} = N_G(v) \cap V(G_Y)$ or $\{a, b\} = N_G(v) \cap V(G_Y)$. In the former case either b is a dense node in G or a is a node in G and a, b and y form a good triangle in G. In the latter case a, b is a good pair in G. □

By Claim 4.5 we can assume that $|V(G_Y)| \geq 5$. Furthermore, since Y_v is an end in G_v by the minimality of $|Y_v|$, Lemma 2.7 implies that G_Y is 3-connected. Hence by Theorem 3.7 either G_Y has four admissible nodes or G_Y has three pairwise non-adjacent admissible nodes. Therefore, since $xy \in E(G_Y)$, we can choose an admissible node t in G_Y for which

$t \notin \{x, y\}$ and either $t \notin N_G(v)$, or there is a $u \in V(G_Y)$ with $N_G(v) \cap Y_v = \{t, u\}$, $tu \in E(G_Y)$, and t, u are nodes in G_Y as well as in G.

To see that such a node t exists in G_Y note that if $N_G(v) \cap Y_v = \{r, s\}$ then G_v arises from G by splitting off v on the pair vr, vs, hence $rs \in E(G_Y)$, and r (resp. s) is a node in G_Y if and only if r (s) is a node in G.

Claim 4.6. *t is an admissible node in G.*

Proof. First suppose that $t \notin N_G(v)$. Let G'_t be the graph obtained from G_Y by an admissible splitting at t on the pair tg, th for some $g, h \in N_{G_Y}(t)$. By the choice of t and since $t \notin N_G(v)$, we have that t is a node in G and $N_G(t) = N_{G_Y}(t)$. Hence splitting off t on the pair tg, th is possible in G as well. Let G_t denote the graph obtained from G by splitting off t on the edges tg, th. Observe that G_t can be obtained from the generic cycle G'_t by a 2-sum operation at $\{x, y\}$ and by an extension (which adds v). Thus G_t is a generic cycle by Lemma 3.8 and Lemma 3.10.

Next suppose that $t \in N_G(v)$. By the choice of t this implies that v has two neighbours t, u in Y_v in G and G_v is obtained from G by splitting off v on the pair vt, vu, hence $tu \in E(G_Y)$, and t and u are nodes in G_Y as well as in G. Let $N_{G_Y}(t) = \{u, z, p\}$. Note that since u is a node in G_Y, it follows that splitting off t in G_Y on the pair tz, tp is not admissible. Hence, since t is admissible in G_Y, we can assume that splitting off t on the edges tu, tz is admissible in G_Y. Let G'_t be the generic cycle obtained from G_Y by splitting off t on the pair tu, tz. Let G_t denote the graph obtained from G by splitting off t on the edges tv, tz. Observe that G_t can be obtained from G'_t by a 2-sum at $\{x, y\}$ and by an extension (which adds v). Hence by Lemma 3.8 and Lemma 3.10 it follows that G_t is a generic cycle. Thus t is an admissible node in G. □

By Claim 4.6 t is an admissible node in G. Let G_t be the graph obtained from G by an admissible splitting at t.

Claim 4.7. $|Y_t| < |Y_v|$.

Proof. Since $N_{G_v}(Y_v) = \{x, y\}$ is a cutpair in G_v, it follows that $S := \{x, y, v\}$ is a separating set in G, $N_G(Y_v) = S$, and $Z := G - S - Y_v$ is non-empty. By our assumption G_t has at least one cutpair and hence, by Lemma 2.5, there exist two ends A, B in G_t with $A \subseteq G_t - B - N_{G_t}(B)$ and $B \subseteq G_t - A - N_{G_t}(A)$. Note that $|A|, |B| \geq 2$, since G_t is a generic cycle and hence it has minimum degree 3. Since $t \in Y_v$ and $|Y_v| \geq 2$, the set S is a separating set in G_t as well.

First consider the case when either $A \cap S = \emptyset$ or $B \cap S = \emptyset$ holds. By symmetry we can assume $A \cap S = \emptyset$. Since A induces a connected graph in G_t by Lemma 2.5, and since S is a separating set in G_t, either $A \subseteq Z$ or $A \subseteq Y_v - t$ holds. If $A \subseteq Z$ then we have $t \notin N_G(A)$, since $t \in Y_v$. Thus $|N_G(A)| = |N_{G_t}(A)| = 2$ follows, contradicting the fact that G is 3-connected. If $A \subseteq Y_v - t$ then, since $c \notin A$, $|Y_t| \leq |A| < |Y_v|$ follows.

Now consider the case when $A \cap S \neq \emptyset \neq B \cap S$. Since $A \cap B = \emptyset$, we can assume without loss of generality that $|A \cap S| = 1$. Let $A \cap S = \{q\}$, where $q \in \{v, x, y\}$. The first subcase we deal with is when $A_1 := A \cap Y_v$ and $A_2 := A \cap Z$ are both non-empty. In this case q is a cutvertex in $G_t[A]$. By Lemma 2.6 this implies $N_{G_t}(A) \subset N_{G_t}(A_1) \cap N_{G_t}(A_2)$. Since S separates A_1 and A_2, we have $N_{G_t}(A_1) \cap N_{G_t}(A_2) \subseteq S$. Moreover, we have $|N_{G_t}(A)| = 2$ and $q \in A$. This gives $N_{G_t}(A) = S - q$. Since $B \cap (A \cup N_{G_t}(A)) = \emptyset$, this implies that $B \cap S = \emptyset$, which contradicts our assumption.

The second subcase is when either $A \cap (Y_v - t) = \emptyset$ or $A \cap Z = \emptyset$. First suppose that $A \cap (Y_v - t) = \emptyset$. If $q \notin N_{G_t}(Y_v - t)$ then $N_{G_t}(Y_v - t) = S - q$ and hence $Y_v - t$ is a fragment of G_t not containing c. Thus $|Y_t| \leq |Y_v - t| < |Y_v|$. If there exists an edge from q to $Y_v - t$ in G_t then, since $d_{G_t}(Y_v - t, A - q) = 0$, we have $|N_{G_t}(A - q)| \leq |N_{G_t}(A)| = 2$. Thus $A - q$ is a fragment of G_t, which contradicts the minimality of A. Now suppose that $A \cap Z = \emptyset$. Since G is 3-connected, each vertex $q' \in S$ has a neighbour in Z in G. Since $t \notin S$, each $q' \in S$ has a neighbour in Z in G_t as well. This implies that $|N_{G_t}(A - q)| \leq |N_{G_t}(A)| = 2$, and hence $A - q$ is a fragment in G_t, contradicting the minimality of A. This proves the claim. □

Claim 4.7 contradicts the choice of v. This shows that there exists a feasible node $v' \neq c$ in G, and hence the proof of the theorem is complete. □

Theorem 4.4 implies the following constructive characterization of 3-connected generic cycles (which was conjectured by Connelly).

Theorem 4.8. *$G = (V, E)$ is a 3-connected generic cycle if and only if G can be built up from K_4 by a sequence of extensions.*

5 Generically globally rigid graphs

In this section we apply Theorem 4.8 to solve a special case of a conjecture of Hendrickson on generically globally rigid (or uniquely realizable) graphs in two dimensions. Suppose we choose generic coordinates for the vertices of graph $G = (V, E)$ in the plane. Let us call this a *realization* of G. These coordinates determine the length of every edge of G. If G has no other realization with these edge lengths, up to congruence of the whole plane, then this realization is *unique*. A graph G is called *generically* *globally rigid* (or *uniquely realizable*) if any realization of G is unique. Hendrickson [5] proved that if G is generically globally rigid in the plane then G is redundantly rigid (i.e. $G - e$ is rigid for every $e \in E$) and 3-connected. He conjectured that these two conditions are sufficient as well.

By Laman's theorem [7] it can be seen that a redundantly rigid graph $G = (V, E)$ has $|E| \geq 2|V| - 2$, and $|E| = 2|V| - 2$ holds if and only if G is a generic cycle. Thus to prove Hendrickson's conjecture in the special case when the graph has the least possible number of edges is equivalent to proving that 3-connected generic cycles are generically globally rigid. Furthermore, a result of Connelly [2] implies that if G can be obtained from K_4 by a sequence of extensions then G is globally generically rigid. By Theorem 4.8 every 3-connected generic cycle can be obtained this way. This leads to the following characterization.

Theorem 5.1. *Let $G = (V, E)$ be a graph with $|E| = 2|V| - 2$. Then G is generically globally rigid in the plane if and only if G is redundantly rigid and 3-connected.*

Note that the property of being redundantly rigid and 3-connected can be checked in polynomial time by purely combinatorial methods.

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