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Combined connectivity augmentation and orientation problems

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Abstract

Two important branches of graph connectivity problems are connectivity augmentation, which consists of augmenting a graph by adding new edges so as to meet a specified target connectivity, and connectivity orientation, where the goal is to find an orientation of an undirected or mixed graph that satisfies some specified edge-connection property. In the present work an attempt is made to link the above two branches, by considering degree-specified and minimum cardinality augmentation of graphs so that the resulting graph admits an orientation satisfying a prescribed edge-connection requirement, such as (k, l)-edge-connectivity. The results are obtained by combining the supermodular polyhedral methods used in connectivity orientation with the splitting off operation, which is a standard tool in solving augmentation problems.

Keywords: Graph orientation; Connectivity augmentation; Supermodularity

1 Introduction

In a connectivity augmentation problem the goal is to augment a graph or digraph by adding a cardinality- or degree-constrained new graph so as to meet a specified target connectivity. Initial deep results of the area are due to Lovász [9] and to Watanabe and Nakamura [14] on augmenting a graph to make it k-edge-connected. Since then, augmentation results for many different connectivity properties of graphs and digraphs have been proved, employing various versions of the splitting off technique, which was

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originally introduced by Lovász [9] and subsequently developed further by Mader [10] and others.

In a connectivity orientation problem one is interested in the existence of an orientation of an undirected graph that satisfies some specified edge-connection properties. For example, classical results of Nash-Williams [11] and of Tutte [13] characterize graphs having k-edge-connected and rooted k-edge-connected orientations, respectively. To formulate a common generalization of their results, we call a digraph $D = (V, A)$ (k, l)-*edge-connected* for non-negative integers $k \geq l$ if there is a node $s \in V$ such that there are k edge-disjoint paths from s to any other node, and there are l edge-disjoint paths to s from any other node. Then (k, k)-edge-connectivity is equivalent to k-edge-connectivity, and $(k, 0)$-edge-connectivity is equivalent to rooted k-edge-connectivity from some node s. Good characterizations of undirected and mixed graphs having a (k, l)-edge-connected orientation were given in [3] and in [5] using submodular flows and related polyhedral methods (the characterizations for undirected graphs are significantly less complicated than those for the more general case of mixed graphs).

In this paper an attempt is made to link these two branches of connectivity problems by studying combined augmentation and orientation problems. For example we characterize undirected and mixed graphs that can be augmented by adding an appropriate degree-specified undirected graph so as to have a (k, l)-edge-connected orientation. Another new result concerns the minimum number of new edges whose addition to an initial undirected graph results in a graph admitting a (k, l)-edge-connected orientation. Our proof methods for these characterizations combine the splitting off technique used in connectivity augmentation with extensions of the supermodular polyhedral techniques used in [5] to solve connectivity orientation problems. Since these methods are constructive from an algorithmic point of view, the proofs give rise to polynomial algorithms for finding a feasible augmentation.

The results are presented in the customary framework for connectivity orientations. We consider graphs that can have loops and multiple edges. Given a graph $G = (V, E)$ and a set function $h : 2^V \to \mathbb{Z}$ (called the *requirement function*), an orientation $\vec{G}$ of G is said to *cover* h if $\varrho_{\vec{G}}(X) \geq h(X)$ for every set $X \subseteq V$, where $\varrho_{\vec{G}}(X)$ denotes the number of edges of the digraph $\vec{G}$ entering the set X. Throughout the paper we assume that $h(\emptyset) = h(V) = 0$. The *h-orientation problem* is to find an orientation of G that covers h. For general h this includes **NP**-complete problems, so special classes of set functions must be considered. A set function h is called *crossing G–supermodular* with respect to a given graph $G = (V, E)$ if

$$h(X) + h(Y) \leq h(X \cap Y) + h(X \cup Y) + d_G(X, Y) \tag{1}$$

for every crossing pair (X, Y), where the sets $X, Y \subseteq V$ are *crossing* if none of $X - Y$, $Y - X$, $X \cap Y$ and $V - (X \cup Y)$ are empty, and $d_G(X, Y)$ is the number of edges in E connecting $X - Y$ and $Y - X$ (for $d_G(X, \overline{X})$ we will also use $d_G(X)$). Note that for any graph G this condition is weaker than crossing supermodularity (it is equivalent if G is the empty graph). For a set $X \subseteq V$, let $i_G(X)$ denote the number of edges $uv \in E$ with $u, v \in X$; then the set function $h + i_G$ is crossing supermodular if and only if h is crossing G–supermodular. As in [5], we restrict our attention to crossing G–supermodular set functions. The *augmentation problem* corresponding to h-orientation is the following: given an undirected graph G, find an undirected graph H (either with specified degrees, or with minimum number of edges), so that $G + H$ has an orientation covering h.

It was shown in [3] that for a graph G and a non-negative crossing G–supermodular requirement function h the h-orientation problem can be solved in polynomial time. In Sections 3 and 4 we solve the corresponding degree-specified and minimum cardinality augmentation problems, respectively. Our methods also provide a solution for the minimum cost augmentation problem for node-induced cost functions.

In Section 5 these results are applied to the augmentation problem where the aim is to obtain a graph admitting a (k, l)-edge-connected orientation; in this case the characterizations can be simplified. The theorems can also be interpreted without referring to orientations. A graph G is called (k, l)*-tree-connected* if each graph obtained by deleting any l edges from G contains k edge-disjoint spanning trees. It is known that if $k \geq l$, then (k, l)-tree-connected graphs are exactly those that have a (k, l)-edge-connected orientation; thus the results imply a solution for the (k, l)-tree-connectivity augmentation problem.

In [5], submodular flows were used in the solution to the h-orientation problem when h is a crossing G–supermodular set function that can have negative values; this implies for example a characterization for (k, l)-edge-connected orientability of a mixed graph M. In Section 6 we generalize this result by considering the h-orientation problem for set functions which are *positively crossing G–supermodular:* (1) holds for every crossing pair (X, Y) for which $h(X), h(Y) > 0$. The main result is a characterization for the corresponding degree-specified augmentation problem. The proof exploits the TDI-ness of a system closely related to the intersection of two base polyhedra.

2 Preliminaries

A *family of sets* is a collection of subsets of the ground set V, with possible repetition. The *union* of two families $\mathcal{F}_1$ and $\mathcal{F}_2$, denoted by $\mathcal{F}_1 + \mathcal{F}_2$, is the family where the multiplicity of every subset is the sum of its multiplicities in $\mathcal{F}_1$ and $\mathcal{F}_2$. If every member of a family $\mathcal{F}$ is replaced by its complement, the resulting family is denoted by $\mathrm{co}(\mathcal{F})$. For an element $v \in V$, $d_{\mathcal{F}}(v)$ denotes the number of members of $\mathcal{F}$ containing v; $\mathcal{F}$ is *regular* if $d_{\mathcal{F}}(v)$ is the same for every $v \in V$. A family $\mathcal{F}$ is a *composition of* X for $X \subseteq V$ if $\mathcal{F} + \{V - X\}$ is regular. The *covering number* of $\mathcal{F}$ is $\min_{v \in V} d_{\mathcal{F}}(v)$; for example, a partition of a set $X \subset V$ is a composition of X with covering number 0. If $\mathcal{F}$ is a composition of $X \subset V$ for which $\mathrm{co}(\mathcal{F})$ is a partition of $\overline{X}$, then $\mathcal{F}$ is called a *co-partition of* X. A *co-partition* of V (or simply a co-partition) is a family $\mathcal{F}$ for which $\mathrm{co}(\mathcal{F})$ is a partition. A family $\mathcal{F}$ is *cross-free* if it has no two crossing members. Simple examples are partitions and co-partitions; in fact, it is easily seen that these are the only minimal regular cross-free families:

Proposition 2.1. *Every regular cross-free family decomposes into partitions and co-partitions.* □

For a vector $x : V \to \mathbb{R}$ and a set $Y \subseteq V$, we use the notation $x(Y) = \sum_{v \in Y} x(v)$. For $\nu \in \mathbb{R}$, $(\nu)^+$ denotes $\max\{\nu, 0\}$. The *upper truncation* of a set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is

$$p^\wedge(X) := \max \left\{ \sum_{Z \in \mathcal{F}} p(Z) \mid \mathcal{F} \text{ a partition of } X \right\} . \tag{2}$$

If p is intersecting supermodular, then $p^\wedge$ is fully supermodular (see [1]). If p is crossing supermodular, then so is $p^\wedge$. With the set function p we associate the polyhedra

$$C(p) := \{x : V \to \mathbb{R} \mid x(Y) \geq p(Y) \ \forall Y \subseteq V\} , \tag{3}$$

$$B(p) := \{x : V \to \mathbb{R} \mid x(V) = p(V);\ x(Y) \geq p(Y) \ \forall Y \subseteq V\} . \tag{4}$$

Clearly, $C(p) = C(p^\wedge)$. A polyhedron is a *contra-polymatroid* if it equals $C(p)$ for some monotone increasing fully supermodular function p; it is a *base polyhedron* if it corresponds to $B(p)$ for some fully supermodular function p.

The following two theorems are important tools in the upcoming proofs. The first one deals with base polyhedra given by crossing supermodular functions, while the second is a generalization of Mader's directed splitting off theorem.

Theorem 2.2 (S. Fujishige [7]). *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a crossing supermodular function. Then $B(p)$ is non-empty if and only if*

$$\sum_{i=1}^{t} p(X_i) \leq p(V) \ , \ \sum_{i=1}^{t} p(\overline{X_i}) \leq (t-1)p(V)$$

both hold for every partition $\{X_1, \ldots, X_t\}$ of V. Furthermore, if $B(p)$ is non-empty, then it is a base polyhedron, thus its vertices are integral. □

Theorem 2.3 ([4]). *Let p be a positively crossing supermodular set function on V; let m_i, m_o be non-negative integer-valued functions on V for which $m_i(V) = m_o(V)$. There exists a digraph $D = (V, A)$ such that $\varrho_D(v) = m_i(v)$, $\rho_D(V - v) = m_o(v)$ $\forall v \in V$, and $\varrho_D(X) \geq p(X)$ $\forall X \subseteq V$ if and only if*

$$m_i(X) \geq p(X) \text{ for every } X \subseteq V \ ,$$

$$m_o(V - X) \geq p(X) \text{ for every } X \subseteq V \ . \quad \square$$

Let $G = (V, E)$ be a graph. For a family $\mathcal{F}$ of sets and $u, v \in V$, let $\mathcal{F}_{\overline{u}v} := \{X \in \mathcal{F} \mid u \notin X, v \in X\}$. We define

$$e_G(\mathcal{F}) := \sum_{e=uv \in E} \max\{|\mathcal{F}_{\overline{u}v}|, |\mathcal{F}_{\overline{v}u}|\}.$$

Note that $e_G(\mathcal{F})$ is the maximum of $\sum_{X \in \mathcal{F}} \varrho_{\vec{G}}(X)$, taken over all possible orientations $\vec{G}$ of G (for regular families this sum is the same for any orientation). For partitions it equals the number of *cross-edges* (edges whose two endpoints are in different members of the partition). More generally, if $\mathcal{F}$ is a regular family with covering number α, then $e_G(\mathcal{F}) = \frac{1}{2} \sum_{X \in \mathcal{F}} d_G(X)$, hence

$$e_G(\mathcal{F}) = \alpha |E| - \sum_{X \in \mathcal{F}} i_G(X) . \tag{5}$$

3 Degree-specified augmentation

The main result of this section is a theorem on the degree-specified augmentation problem concerning h-orientation for nonnegative crossing supermodular requirement functions. The special case when $m \equiv 0$, that is, the degree specification is 0 on every node, corresponds to the orientation theorem in [3], while a $2k$-edge-connectivity augmentation theorem (otherwise a simple consequence of the splitting off theorem of Lovász) is obtained if the value of the requirement function is k on every proper subset of V. The characterizations given by the theorem are good in the sense that they provide an easily verifiable certificate if the augmentation is impossible. Moreover, the proof is constructive and gives rise to a polynomial algorithm, since it involves polyhedral and splitting off problems that can be solved in polynomial time.

Theorem 3.1. *Let $G = (V, E)$ be a graph, $h : 2^V \to \mathbb{Z}_+$ a non-negative crossing G–supermodular set function on V, and $m : V \to \mathbb{Z}_+$ a degree specification with $m(V)$ even. There exists an undirected graph $H = (V, F)$ such that $G+H$ has an orientation covering h and $d_H(v) = m(v)$ for all $v \in V$ if and only if the following hold for every partition $\mathcal{F}$ of V:*

$$\frac{m(V)}{2} \geq \sum_{X\in\mathcal{F}} h(X) - e_G(\mathcal{F}) \;, \tag{6}$$

$$\min_{X\in\mathcal{F}} m(\overline{X}) \geq \sum_{X\in\mathcal{F}} h(X) - e_G(\mathcal{F}) \;, \tag{7}$$

$$\frac{m(V)}{2} \geq \sum_{X\in co(\mathcal{F})} h(X) - e_G(co(\mathcal{F})) \;, \tag{8}$$

$$\min_{X\in\mathcal{F}} m(\overline{X}) \geq \sum_{X\in co(\mathcal{F})} h(X) - e_G(co(\mathcal{F})) \;. \tag{9}$$

Proof. To see the necessity of these conditions, observe that $m(V)/2$ is the number of new edges, while $\sum_{X\in\mathcal{F}} h(X) - e_G(\mathcal{F})$ measures the deficiency of a partition $\mathcal{F}$, i.e. the difference between the total requirement of the partition and the portion of this requirement that is satisfied by an arbitrary orientation of G. Hence (6) simply requires that the deficiency of a partition should not exceed the number of new edges. The necessity of (7) is also straightforward since each new cross-edge must have an endnode in $\overline{X}$, so the number of new cross-edges, which should be at least the deficiency of $\mathcal{F}$, is at most $m(\overline{X})$. (Note that if $m \equiv 0$, then (6) and (7) are equivalent.) The necessity of (8) and (9) can be seen analogously.

To prove sufficiency, we add a new node z to the set of nodes, and for every $v \in V$ we add $m(v)$ parallel edges between v and z; the resulting graph is denoted by $G' = (V', E')$. The following extension of the set function h is considered:

$$h'(z) = h'(V) := \frac{m(V)}{2} \;,$$
$$h'(X+z) = h'(X) := h(X) \qquad \text{if } \emptyset \neq X \subset V \;.$$

The proof consists of finding first an orientation of G' that covers h', and splitting then off the directed edges at z so that the resulting digraph on the ground set V covers h. To find an orientation covering h', we resort to a well-known lemma on the in-degrees of orientations (see e.g. [8]):

Lemma 3.2. *For a given vector $x' : V' \to \mathbb{Z}_+$, there is an orientation $\vec{G'}$ of G' such that $\varrho_{\vec{G'}}(v) = x'(v)$ for every $v \in V'$ if and only if $x'(V') = |E'|$ and $x'(Y) \geq i_{G'}(Y)$ for every $Y \subseteq V'$.* □

Lemma 3.2 and the non-negativity of h imply that if we can find a vector $x' : V' \to \mathbb{Z}_+$ that satisfies $x'(V') = |E'|$ and

$$x'(Y) \geq h'(Y) + i_{G'}(Y) \text{ for every } Y \subseteq V', \tag{10}$$

then there is an orientation $\vec{G'}$ of G' such that $\varrho_{\vec{G'}}(v) = x'(v)$ for every $v \in V'$, and such that $\vec{G'}$ covers h', since $\varrho_{\vec{G'}}(Y) = x'(Y) - i_{G'}(Y) \geq h'(Y)$. A vector x' satisfying (10) is called *feasible*. By definition $h'(z) = h'(V) = m(V)/2$, hence $x'(z)$ must be equal to $m(V)/2$; let $x : V \to \mathbb{Z}_+$ denote the restriction of x' to V. It easily follows from the definition of h' that the vector x' is feasible if and only if x is an element of the polyhedron $B(p_m)$ (defined in (4)) associated with the set function

$$p_m(X) := h(X) + i_G(X) + \left(m(X) - \frac{m(V)}{2}\right)^+ \qquad (X \subseteq V)\ .$$

Claim 3.3. *The set function p_m is crossing supermodular.*

Proof. The G–supermodularity of h implies that $h + i_G$ is crossing supermodular. Let $m^*(X) := (m(X) - m(V)/2)^+$; we show that this set function is fully supermodular. Indeed, if $m^*(Y) = 0$, then $m^*(X) + m^*(Y) = m^*(X) \leq m^*(X \cup Y) = m^*(X \cap Y) + m^*(X \cup Y)$. If $m^*(X), m^*(Y) > 0$, then $m^*(X) + m^*(Y) = m(X \cap Y) + m(X \cup Y) - m(V) \leq m^*(X \cap Y) + m^*(X \cup Y)$. The sum of a crossing supermodular and a fully supermodular function is crossing supermodular. □

Claim 3.4. *Suppose that (6)–(9) are true. Then $B(p_m)$ is non-empty.*

Proof. By Theorem 2.2 it suffices to show that $\sum_{X \in \mathcal{F}} p_m(X) \leq |E| + m(V)/2$ and $\sum_{X \in \mathrm{co}(\mathcal{F})} p_m(X) \leq (t-1)(|E| + m(V)/2)$ for every partition $\mathcal{F}$ with t members. Observe that a partition has at most one member X with $m(X) > m(V)/2$. If there is no such member, then (6) and the identity (5) imply that $\sum_{X \in \mathcal{F}} p_m(X) \leq |E| + m(V)/2$; if there is one such member, then (7) and (5) imply the same. Similarly, a co-partition has at most one member X with $m(X) < m(V)/2$, so (8) or (9) (depending on the existence of such a member) and (5) for the co-partition $\mathrm{co}(\mathcal{F})$ imply $\sum_{X \in \mathrm{co}(\mathcal{F})} p_m(X) \leq (t-1)(|E| + m(V)/2)$. □

By Theorem 2.2, $B(p_m)$ is a base polyhedron with integral vertices, and for such a vertex x the corresponding vector $x' : V' \to \mathbb{Z}_+$ is feasible. By Lemma 3.2, G' has an orientation $\vec{G'} = (V', \vec{E'})$ with in-degree vector x', and the feasibility of x' implies that $\vec{G'}$ covers h'.

Let $m_i(v)$ be the multiplicity of the edge zv in $\vec{G'}$, $m_o(v)$ be the multiplicity of the edge vz in $\vec{G'}$, and let $\vec{G}$ denote the digraph obtained from $\vec{G'}$ by deleting the node z. Then $m_i(X) \geq h(X) - \varrho_{\vec{G}}(X)$ and $m_o(V - X) \geq h(X) - \varrho_{\vec{G}}(X)$ for every $X \subseteq V$, since $\vec{G'}$ covers h'. By the crossing G–supermodularity of h, the set function $p(X) := h(X) - \varrho_{\vec{G}}(X)$ is crossing supermodular. Applied on these values, Theorem 2.3 asserts that there exists a digraph D with underlying undirected graph H, such that H satisfies the degree specifications, and $\vec{G} + D$ covers h. Since $\vec{G} + D$ is an orientation of $G + H$, this proves Theorem 3.1. □

If the requirement function is monotone decreasing (that is, $h(X) \geq h(Y)$ if $X \subseteq Y$), or symmetric, then the conditions of Theorem 3.1 can be simplified.

Corollary 3.5. *Let $G = (V, E)$ be a graph, $h : 2^V \to \mathbb{Z}_+$ a non-negative, monotone decreasing crossing G–supermodular set function on V, and $m : V \to \mathbb{Z}_+$ a degree specification with $m(V)$ even. There exists an undirected graph $H = (V, F)$ such that $G + H$ has an orientation covering h and $d_H(v) = m(v)$ for all $v \in V$ if and only if (6) and (7) hold for every partition $\mathcal{F}$ of V.*

Proof. The co-partition type constraints (8) and (9) are unnecessary in this case, since $\sum_{X \in \mathcal{F}} h(X) \geq \sum_{X \in \text{co}(\mathcal{F})} h(X)$ and $e_G(\mathcal{F}) = e_G(\text{co}(\mathcal{F}))$ for every partition $\mathcal{F}$. □

Corollary 3.6. *Let $G = (V, E)$ be a graph, $h : 2^V \to \mathbb{Z}_+$ a non-negative, symmetric crossing G–supermodular set function on V, and $m : V \to \mathbb{Z}_+$ a degree specification with $m(V)$ even. There exists an undirected graph $H = (V, F)$ such that $G + H$ has an orientation covering h and $d_H(v) = m(v)$ for all $v \in V$ if and only if $m(X) \geq 2h(X) - d_G(X)$ for every $X \subseteq V$.*

Proof. The co-partition type constraints are redundant for the same reason as in Corollary 3.5. Let $\{X_1, \ldots, X_t\}$ be a partition such that $m(X_1) \geq m(X_i)$ $(i = 2, \ldots, t)$. If $m(X_i) \geq 2h(X_i) - d_G(X_i)$ for every i, then by adding up these inequalities we obtain (6); by adding $m(\overline{X_1}) \geq 2h(X_1) - d_G(X_1)$ to the inequalities featuring the rest of the partition members, we obtain (7). □

4 Minimum cardinality augmentation

Theorem 3.1 characterized degree specifications that are 'good' in the sense that a corresponding feasible augmentation exists. In this section we derive a min-max theorem for minimum cardinality augmentation, by analyzing the properties of these good degree specifications. We show that they are the integral vectors (with even co-ordinate sum) of a contra-polymatroid.

Theorem 4.1. *Let $G = (V, E)$ be a graph, and $h : 2^V \to \mathbb{Z}_+$ a non-negative crossing G–supermodular set function. There is an undirected graph $H = (V, F)$ with γ edges such that $G + H$ has an orientation covering h if and only if*

$$\gamma \geq \sum_{X \in \mathcal{F}} h(X) - e_G(\mathcal{F}) \tag{11}$$

holds for every partition and co-partition $\mathcal{F}$ of V, and

$$2\gamma \geq \sum_{Z \in \mathcal{F}} h(Z) - e_G(\mathcal{F}) \tag{12}$$

holds for every cross-free regular family $\mathcal{F}$ that for some $X \subset V$ decomposes into a partition of X and a co-partition of $\overline{X}$.

Proof. In both types of conditions, $\sum_{X \in \mathcal{F}} h(X) - e_G(\mathcal{F})$ measures the difference between the total requirement of the family $\mathcal{F}$ and the sum of the in-degrees of its members for an arbitrary orientation of G. Now necessity follows easily by observing that each of the γ oriented new edges can cover at most one member of a (sub)partition or a (sub)-copartition.

Sufficiency will be proved by showing that if (11) and (12) hold, then there exists a vector $m : V \to \mathbb{Z}_+$ with $m(V) = 2\gamma$ satisfying (6)–(9); thus by Theorem 3.1 we can find a feasible augmentation with degree-specification m. The essential result in the proof is that the polyhedron

$$C := \{m : V \to \mathbb{Z}_+ \mid \ m \text{ satisfies (6)–(9)}\}$$

is a contra-polymatroid. In order to show this, we first transform (6)–(9), which are conditions on partitions and co-partitions, into requirements for the subsets of V. Define the set functions

$$p_1(X) := h(X) + i_G(X) \ ,$$
$$p_2(X) := h(\overline{X}) + i_G(\overline{X}) - |E| \ .$$

By the crossing G–supermodularity of h, the set functions p_1 and p_2 are crossing supermodular, therefore the set functions $p_1^\wedge$ and $p_2^\wedge$ (as defined in (2)) are also crossing supermodular. By the identity (5), a non-negative vector m satisfies (6)–(9) if and only if the following hold:

$$\begin{aligned} m(V) &\geq 2 \max_{X \subset V} (p_1^\wedge(X) + p_2(X)) \ , \\ m(X) &\geq p_1^\wedge(X) + p_2(X) \text{ for every } X \subset V \ , \\ m(V) &\geq 2 \max_{X \subset V} (p_1(X) + p_2^\wedge(X)) \ , \\ m(X) &\geq p_1(X) + p_2^\wedge(X) \text{ for every } X \subset V \ . \end{aligned}$$

We define a new set function

$$p(X) \ := \ \max\{p_1^\wedge(X) + p_2(X),\ p_1(X) + p_2^\wedge(X),\ 0\} \quad (X \subset V) \ , \tag{13}$$
$$p(V) \ := \ 2 \max_{X \subset V} p(X) \ . \tag{14}$$

Thus the conditions (6)–(9) are 'coded into' p, i.e. the polyhedron C can be characterized as

$$C = \{m : V \to \mathbb{Z} \mid m(X) \geq p(X) \ \forall X \subseteq V\} \ .$$

To prove that C is a contra-polymatroid, we will show that the set function $p^\wedge$ is fully supermodular. First we establish some other properties of $p^\wedge$:

Proposition 4.2. *For every proper subset X of V, the value of $p^\wedge(X)$ is*

$$p^\wedge(X) = \max_{X' \subseteq X} (p_1^\wedge(X') + p_2^\wedge(X')) \ . \tag{15}$$

Proof. By the definitions of p and the upper truncation, the value of $p^\wedge(X)$ is attained by taking two appropriate partitions of some $X' \subseteq X$, and adding up p_1 on the members of the first one, plus p_2 on the members of the second one. Thus $p^\wedge$ is less than or equal to the maximum on the right side of (15). For the other inequality, suppose indirectly that there exists $X' \subseteq X$ and partitions $\mathcal{F}_1$ and $\mathcal{F}_2$ of X' such that

$$p^\wedge(X) < \sum_{Z \in \mathcal{F}_1} p_1(Z) + \sum_{Z \in \mathcal{F}_2} p_2(Z) \ .$$

Repeat the following step as many times as possible:

- If $Z_1 \in \mathcal{F}_1$ and $Z_2 \in \mathcal{F}_2$ are crossing, then replace Z_1 in $\mathcal{F}_1$ by $Z_1 - Z_2$, and replace Z_2 in $\mathcal{F}_2$ by $Z_2 - Z_1$.

Observe that the resulting families are partitions of a decreasing sequence of proper subsets of X', so the procedure terminates after a finite number of steps. Furthermore, Z_1 and $\overline{Z_2}$ are crossing, so $h(Z_1) + h(\overline{Z_2}) \leq h(Z_1 \cap \overline{Z_2}) + h(Z_1 \cup \overline{Z_2}) + d_G(Z_1, \overline{Z_2})$, which implies that $p_1(Z_1) + p_2(Z_2) \leq p_1(Z_1 - Z_2) + p_2(Z_2 - Z_1)$. Let $\mathcal{F}_1'$ and $\mathcal{F}_2'$ denote the families obtained at the end of the procedure; then $\mathcal{F}_1'$ and $\mathcal{F}_2'$ are partitions of some $X'' \subseteq X'$, and $p^\wedge(X) < \sum_{Z \in \mathcal{F}_1'} p_1(Z) + \sum_{Z \in \mathcal{F}_2'} p_2(Z)$. Moreover, $\mathcal{F}_1' + \mathcal{F}_2'$ is cross-free, which means that there is a partition $X_1, \ldots, X_t$ of X'', such that for every i either $\mathcal{F}_1'$ contains X_i and $\mathcal{F}_2'$ contains a partition of X_i, or vice versa. But then $\sum_{Z \in \mathcal{F}_1'} p_1(Z) + \sum_{Z \in \mathcal{F}_2'} p_2(Z) \leq p^\wedge(X'') \leq p^\wedge(X)$, a contradiction. □

Proposition 4.3. *The set function p satisfies*

$$p(X) + p(Y) \leq p^\wedge(X \cap Y) + p^\wedge(X \cup Y) \tag{16}$$

for every pair (X, Y).

Proof. The inequality is obvious if one of $p(X)$ and $p(Y)$ is zero, or X and Y are not intersecting. If $X \cup Y = V$, then $p(X) + p(Y) \leq 2\max\{p(X), p(Y)\} \leq p(V) = p(X \cup Y) \leq p^\wedge(X \cap Y) + p^\wedge(X \cup Y)$.

By Proposition 4.2 it suffices to prove that if $p(X), p(Y) > 0$ and X and Y are crossing, then

$$p(X) + p(Y) \leq p_1^\wedge(X \cap Y) + p_2^\wedge(X \cap Y) + p_1^\wedge(X \cup Y) + p_2^\wedge(X \cup Y) \ .$$

Using the definition of p and the crossing supermodularity of $p_1^\wedge$ and $p_2^\wedge$,

$$\begin{aligned} p(X)+p(Y) &\le p_1^\wedge(X)+p_2^\wedge(X)+p_1^\wedge(Y)+p_2^\wedge(Y) \\ &\le p_1^\wedge(X\cap Y)+p_2^\wedge(X\cap Y)+p_1^\wedge(X\cup Y)+p_2^\wedge(X\cup Y)\ . \end{aligned}$$

□

This property turns out to be sufficient for the supermodularity of $p^\wedge$:

Lemma 4.4. *If a set function p (with $p(\emptyset)=0$) satisfies* (16) *for every pair* (X,Y)*, then $p^\wedge$ is fully supermodular.*

Proof. For a set $X\subseteq V$, let $\mathcal{F}_X$ denote a partition of X for which $p^\wedge(X)=\sum_{Z\in\mathcal{F}_X}p(Z)$. Let $X,Y\subseteq V$ be an arbitrary pair. Starting from the family $\mathcal{F}=\mathcal{F}_X+\mathcal{F}_Y$, repeat the following operation as many times as possible:

- If there is an intersecting pair Z_1 and Z_2 in the family, remove both of them, and add the sets of $\mathcal{F}_{Z_1\cap Z_2}$ and of $\mathcal{F}_{Z_1\cup Z_2}$ to the family.

The operation doesn't change $d_\mathcal{F}$, and doesn't decrease $\sum_{Z\in\mathcal{F}}p(Z)$, since p has the property (16). Since the operation either increases the cardinality of the family, or increases $\sum_{Z\in\mathcal{F}}|Z|^2$ without changing the cardinality, after a finite number of steps we get a laminar family $\mathcal{F}'$ for which $\sum_{Z\in\mathcal{F}'}p(Z)\ge\sum_{Z\in\mathcal{F}}p(Z)$. Such a family decomposes into a partition of $X\cap Y$ and a partition of $X\cup Y$, hence $p^\wedge(X)+p^\wedge(Y)\le p^\wedge(X\cap Y)+p^\wedge(X\cup Y)$. □

Lemma 4.4 and Proposition 4.3 imply that $p^\wedge$ is fully supermodular, and it is obviously monotone increasing, hence C is a contra-polymatroid defined by $p^\wedge$. It is known that in this case the minimum cardinality of an integral element of the contra-polymatroid C is $p^\wedge(V)$. Thus, for a fixed γ, there exists an integral element m of C with $m(V)=2\gamma$ if and only if $p^\wedge(V)\le 2\gamma$. This inequality follows from conditions (11) and (12): if $p^\wedge(V)=p(V)$, then it corresponds to (11); if the value of $p^\wedge(V)$ is attained on a partition $\mathcal{F}^*$, then it follows from (12), X being the union of the members of $\mathcal{F}^*$ where p is positive. This concludes the proof of Theorem 4.1. □

Remark. The following example shows that (11) itself is not sufficient in Theorem 4.1. Let $V=\{v_1,v_2,v_3,v_4\}$, $E=\{v_1v_2,v_1v_3,v_1v_4\}$. Let $h=1$ on the sets $\{v_2\},\{v_3\},\{v_4\}$ and on their complement; let $h=0$ on all other sets. We need at least 2 new edges for a feasible orientation (two edges suffice, since after adding v_2v_3 and v_3v_4 the graph has a strong orientation) but (11) requires only $\gamma\ge 1$, since the only deficient partitions are $\{\{v_i\},V-v_i\}$ $(i=2,3,4)$.

Remark. A cost function $c:E\to\mathbb{R}_+$ is called *node induced* if $c(uv)=c'(u)+c'(v)$ where $c':V\to\mathbb{R}_+$ is a linear cost function on the nodes. To solve the minimum cost augmentation problem for node induced cost functions, one can find a minimum cost element with even co-ordinate sum of the contra-polymatroid C according to the cost function c', using the greedy algorithm. Then this vector can be used as a degree specification to find a minimum cost augmentation.

For general edge costs the problem is **NP**-complete: let G be the empty graph, and let $c(e) = 1$ on the edges of a fixed graph G^*, $c(e) = 2$ on the other edges. Let $h(X) = 1$ if $X \neq \emptyset, V$; thus h is crossing supermodular. Now the minimum cost of the augmentation is $|V|$ if and only if G^* contains a Hamiltonian cycle.

5 (k,l)-edge-connected orientations

In the introduction we defined (k,l)-edge-connectivity for non-negative integers $k \geq l$, and mentioned that the (k,l)-edge-connectivity orientation problem is a common generalization of k-edge-connectivity orientation (when $l = k$) and rooted k-edge-connectivity orientation (when $l = 0$). Recently, it was shown in [6] that the case $l = k - 1$ plays an important role in orientation problems with both connectivity and parity constraints. As for the corresponding augmentation problems, both the degree-specified and the minimum cardinality augmentation of a graph to have a k-edge-connected orientation are already solved, but the minimum cost augmentation is **NP**-complete even for $k = 1$. On the other hand, for rooted k-edge-connected orientations, the minimum cost augmentation is known to be solvable by matroid techniques, while no solution has been proposed so far for degree-specified augmentation.

To show how the results of the previous section can be used to solve degree-specified and minimum cardinality augmentation of a graph so that the new graph has a (k,l)-edge-connected orientation, fix a node $s \in V$, and introduce the following family of set functions:

$$h_{kl}(X) := \begin{cases} k & \text{if s} \notin X \ , \\ l & \text{if s} \in X \ . \end{cases} \tag{17}$$

Menger's theorem implies that an orientation is (k,l)-edge-connected from root s if and only if it covers h_{kl}. The set function h_{kl} is crossing G–supermodular for any G. Note that if a digraph is (k,l)-edge-connected from root s, and for some $s' \in V - s$ we take k edge-disjoint paths from s to s' and reverse the orientation of the edges on $k - l$ of them, then we get a digraph that is (k,l)-edge-connected from root s'. Thus the root can be selected arbitrarily in orientation problems.

Theorem 5.1. *Let $G = (V,E)$ be a graph, $m : V \to \mathbb{Z}_+$ a degree specification with $m(V)$ even, and $k \geq l$ non-negative integers. There exists an undirected graph $H = (V,F)$ such that $G + H$ has a (k,l)-edge-connected orientation and $d_H(v) = m(v)$ for all $v \in V$ if and only if the following hold for every partition $\mathcal{F} = \{X_1, \dots, X_t\}$ of V:*

$$\frac{m(V)}{2} \geq (t-1)k + l - e_G(\mathcal{F}) \ , \tag{18}$$

$$\min_i m(\overline{X_i}) \geq (t-1)k + l - e_G(\mathcal{F}) \ . \tag{19}$$

Proof. Since the set function h_{kl} defined in (17) is monotone decreasing, the claim follows from Corollary 3.5. $\square$

Theorem 5.2. *Let $G = (V,E)$ be a graph, and $k \geq l$ non-negative integers. There is a graph H with γ edges such that $G + H$ has a (k,l)-edge-connected orientation if and only if the following two conditions are met:*

1. *$\gamma \geq (t-1)k + l - e_G(\mathcal{F})$ for every partition $\mathcal{F}$ with t members.*

2. *$2\gamma \geq t_1 k + t_2 l - e_G(\mathcal{F})$ for every family $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ where $\mathcal{F}_1$ is a partition of some X with t_1 members, $\mathcal{F}_2$ is a co-partition of $\overline{X}$ with t_2 members, and every member of $\mathcal{F}_2$ is the complement of the union of some members of $\mathcal{F}_1$.*

Proof. As in the proof of Theorem 5.1, we demand that $G+H$ should have an orientation covering h_{kl}. Going back to the proof of Theorem 4.1, the set function p defined in (13) can be defined in this case as

$$p(X) := \begin{cases} \left(p_1^{\wedge}(X) + p_2(X)\right)^+ & \text{if } X \subset V \ , \\ 2\max_{Y \subset V}\left(p_1^{\wedge}(Y) + p_2(Y)\right) & \text{if } X = V \ . \end{cases} \tag{20}$$

As it was proved in Theorem 4.1, a feasible augmentation with γ edges exists if and only if $p^{\wedge}(V) \leq 2\gamma$; by the above characterization of p, this is equivalent to the conditions of the theorem. □

Remark. The graph on Figure 1 shows that the second condition in Theorem 5.2 cannot be simplified. We need to add at least 2 edges to the graph to have a $(3,1)$-edge-connected orientation from root s, but the simplest evidence for this is the family indicated on the figure (consisting of the round sets and the complements of the square sets), whose deficiency is 3, while a new edge can enter at most 2 sets. The figure on the right shows that the addition of 2 edges is sufficient (to see that the digraph is $(3,1)$-edge-connected, observe that it contains 3 edge-disjoint out-arborescences from s, and also an in-arborescence to s).

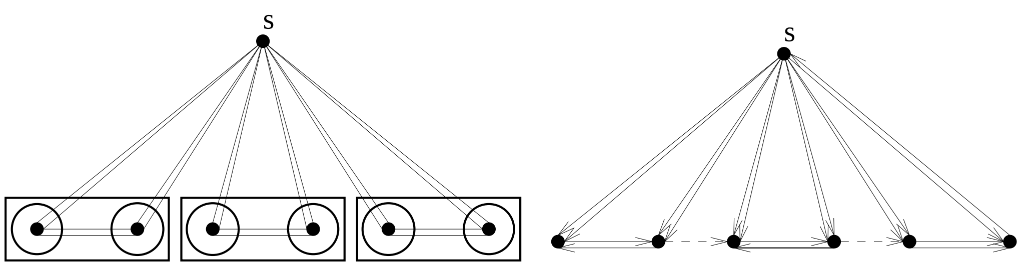

Figure 1

There are other equivalent characterizations of graphs that have a (k,l)-edge-connected orientation. For given non-negative integers k and l, a graph $G=(V,E)$ is called *(k,l)-tree-connected* if any graph obtained by deleting l edges from G contains k edge-disjoint spanning trees; it is called *(k,l)-partition-connected* if $e_G(\mathcal{F}) \geq k(t-1)+l$ for every partition $\mathcal{F}$ with t members. Tutte [13] proved that a graph is $(k,0)$-tree-connected if and only if it is $(k,0)$-partition-connected. This immediately implies that a graph is (k,l)-tree-connected if and only if it is (k,l)-partition-connected.

Simple calculation shows that for $k \leq l$, a graph G is (k,l)-tree-connected if and only if it is $(k+l)$-edge-connected. Thus the (k,l)-tree-connectivity augmentation problem is interesting only for $k \geq l$, and, by the following proposition, this is exactly what was solved in Theorems 5.1 and 5.2:

Proposition 5.3. *For $k \geq l$, a graph $G = (V, E)$ is (k,l)-tree-connected if and only if it has a (k,l)-edge-connected orientation.*

Proof. It follows from the orientation theorem in [3] (or Theorem 3.1) that for $k \geq l$, a graph has a (k,l)-edge-connected orientation if and only if it is (k,l)-partition-connected. $\square$

Note that Theorem 5.1 has some interest even in the very special case when $G = \emptyset$. A result of Edmonds [2] states that a degree-sequence $m_1, \dots, m_n$ is realizable by a k-edge-connected graph if and only if $\sum_{i=1}^{n} m_i$ is even, and $m_i \geq k$ for every i. Theorem 5.1 implies the following similar result: a degree-sequence $m_1, \dots, m_n$ is realizable by a (k,l)-tree-connected graph if and only if $\sum_{i=1}^{n} m_i$ is at least $2k(n-1) + 2l$, it is even, and $m_i \geq k + l$ for every i.

When $l = 0$, this implies the following tiny result (which is not difficult to prove directly either): If $G = (V, E)$ is a k-edge-connected graph with at least $k(n-1)$ edges, then there is a graph $G' = (V, E')$ containing k edge-disjoint spanning trees with $d_G(v) = d_{G'}(v)\ \forall v \in V$.

Remark. The problems discussed in this section are in some sense about packing trees; one may ask whether a similar augmentation result can be obtained related to covering with trees. This question is not considered here in detail; we remark only that the most basic problem, i.e. the augmentation of a graph such that the resulting graph can still be covered by k forests, is solvable rather easily. The maximum cardinality (or, more generally, maximum weight) augmentation is a standard matroid problem, while the following is true on degree-specified augmentation:

Theorem 5.4. *Let $G = (V, E)$ be a graph, $m : V \to \mathbb{Z}_+$ a degree specification with $m(V)$ even, and k a positive integer. There exists an undirected graph $H = (V, F)$ such that $G + H$ can be covered by k forests and $d_H(v) = m(v)$ for every $v \in V$ if and only if*

$$\left(m(X) - \frac{m(V)}{2} \right)^{+} \leq k(|X| - 1) - i_G(X) \text{ for every } \emptyset \neq X \subseteq V. \tag{21}$$

Proof. We prove the theorem by induction on $m(V)$. By a well-known theorem of Nash-Williams [12], a graph can be covered by k forests if and only if $i_G(X) \leq k(|X| - 1)$ for every non-empty subset X of V; hence we can assume that $m(V) \geq 2$. Let $v \in V$ be an arbitrary node with $m(v) > 0$.

A set X is called *tight* if (21) holds with equality. Let $\mathcal{F}_1$ be the family that consists of the tight sets X for which $m(X) \leq m(V)/2$ and $v \in X$, and let $\mathcal{F}_2$ be the family of tight sets X for which $m(X) \geq m(V)/2$ and $v \notin X$. The union of two members of $\mathcal{F}_1$ is also in $\mathcal{F}_1$, since otherwise the intersection would violate (21); similarly, the intersection of two sets in $\mathcal{F}_2$ is in $\mathcal{F}_2$, since otherwise their union would violate (21). Let X_1 be the maximal member of $\mathcal{F}_1$, and X_2 the minimal member of $\mathcal{F}_2$. Then $v \in X_1 - X_2$ and $m(X_1) \leq m(X_2)$, so there is a node $u \in X_2 - X_1$ with $m(u) > 0$.

Let m' be defined by decreasing $m(u)$ and $m(v)$ by 1, and G' defined by adding an edge uv to G. The node u was chosen such that no member of $\mathcal{F}_1$ contains both u and

v, and every member of $\mathcal{F}_2$ contains u. From this it is easy to see that (21) holds for m' and G', therefore G' can be augmented by adding a graph H' with degree-specification m' such that $G' + H'$ can be covered by k forests. This means that $H' + \{uv\}$ is a good augmenting graph for G. $\square$

6 Positively crossing G–supermodular set functions

Let $M = (V; E, A)$ be a mixed graph, where E is the set of undirected edges and A is the set of directed edges. Then the task of finding a (k, l)-edge-connected orientation of M for a fixed root s is equivalent to finding an orientation of the edges in E that covers the set function $(h_{kl} - \varrho_A)^+$, where h_{kl} is defined in (17). This requirement function is not crossing G–supermodular anymore, but it is positively crossing G–supermodular for any G. This motivates the study of the h-orientation problem for positively crossing G–supermodular set functions, and the corresponding augmentation problems. In [5], the h-orientation problem was solved for crossing G–supermodular h with possible negative values, which includes the mixed graph problem mentioned above (for such an h, $(h)^+$ is positively crossing G–supermodular).

The characterizations in this section involve set families more complicated than partitions and co-partitions. It is known that every cross-free family $\mathcal{F}$ has a *tree-representation* (T, φ), where $T = (W, B)$ is a directed tree, and $\varphi : V \to W$ is a mapping such that $\{\varphi^{-1}(W_e) \mid e \in B\} = \mathcal{F}$, where W_e is the component of $T - e$ entered by e. A *tree-composition* of $\emptyset \neq X \subset V$ is a cross-free composition of X which has a tree-representation $(T = (W, B), \varphi)$ such that $\varphi^{-1}(w) \neq \emptyset$ for every $w \in W$. Equivalently, a tree-composition of X is a cross-free composition of X that contains no partitions and co-partitions of V. For technical reasons, a partition or a co-partition of V will be regarded as a tree-composition of the empty set.

In this section we give a characterization for the degree-specified augmentation problem, by mainly the same methods as in Section 3, but instead of relying on the properties of base polyhedra, we use the following extension of the classical result on the TDI-ness of the intersection of base polyhedra:

Lemma 6.1. *Let $q_1 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be fully supermodular, and let $q_2 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a set function that is supermodular on the crossing pairs (X, Y) for which $q_1(X) < q_2(X)$ and $q_1(Y) < q_2(Y)$. Then the system*

$$\left\{x \in \mathbb{R}^V \mid x(V) = q_1(V);\ x(Y) \geq q_1(Y),\ x(Y) \geq q_2(Y)\ \forall Y \subseteq V\right\} \tag{22}$$

is TDI; it has a feasible solution if and only if

$$q_1(\overline{X}) + \sum_{Z \in \mathcal{F}} q_2(Z) \leq (\alpha + 1) q_1(V) \tag{23}$$

for every $X \subset V$ (including the empty set) and every tree-composition $\mathcal{F}$ of X with covering number α.

Proof. To prove TDI-ness, we have to show that the dual system

$$\max\{y_1q_1 + y_2q_2 - \beta q_1(V) : \ (y_1+y_2)A - \beta\mathbf{1} = c, \ y_1, y_2, \beta \geq 0\}$$

has an integral optimal solution for every integral c, where $y_1, y_2 : 2^V \to \mathbb{Q}_+$ are dual variables on the subsets of V, y_1 corresponding to the inequalities featuring q_1, y_2 corresponding to those featuring q_2, $\beta \in \mathbb{Q}_+$ is the dual variable for the inequality $x(V) \leq q_1(V)$, and A is the incidence matrix of all subsets of V. The main observation is that we can assume that y_1 is positive on a chain and y_2 is positive on a cross-free family in an optimal dual solution: this can be achieved by a slight modification of the usual uncrossing technique. Consider the following operations:

- If $y_1(X), y_1(Y) > 0$ and neither $X \subseteq Y$, nor $Y \subseteq X$, decrease y_1 on X and on Y by $\min\{y_1(X), y_1(Y)\}$, and increase y_1 by the same amount on $X \cap Y$ and on $X \cup Y$.
- If $y_2(X), y_2(Y) > 0$, $q_1(X) < q_2(X)$, $q_1(Y) < q_2(Y)$ and X, Y are crossing, then decrease y_2 on X and on Y by $\min\{y_2(X), y_2(Y)\}$, and increase y_2 by the same amount on $X \cap Y$ and on $X \cup Y$.
- If $y_2(X) > 0$ and $q_1(X) \geq q_2(X)$, then decrease y_2 on X to 0 and increase y_1 on X by the same amount.

Because of the properties of q_1 and q_2, these operations do not decrease $y_1q_1 + y_2q_2 - \beta q_1(V)$, and they maintain $(y_1+y_2)A - \beta\mathbf{1} = c$. We show that by repeatedly applying these operations (in any order), in a finite number of steps we get an optimal dual solution (y_1', y_2', β) such that y_1' is positive on a chain and y_2' is positive on a cross-free family.

Since $y_1, y_2 \in \mathbb{Q}_+$, there is a positive integer ν such that νy_1 and νy_2 are integral. The sum $\nu(2\sum_{X\subseteq V} y_1(X)|X|^2 + \sum_{X\subseteq V} y_2(X)|X|^2)$ increases by at least 1 during any of the above operations, and it is bounded from above by $2\nu|V|^2\,(\beta + \max_{v\in V} c(v))$. Thus the procedure terminates after a finite number of steps.

We proved that there is an optimal dual solution (y_1', y_2', β) where y_1' is positive on a chain and y_2' is positive on a cross-free family; but this means that this is also an optimal solution of the dual of the system we get if we restrict q_1 to the sets where y_1' is positive, and restrict q_2 to the sets where y_2' is positive (changing their value to $-\infty$ on all other sets). This system is the intersection of two base polyhedra, so it has an integral optimal dual solution, which is in turn optimal for the dual of the system (22); therefore the system (22) is TDI.

The proof of the non-emptiness condition (23) is similar: the infeasibility of the system is equivalent to the feasibility of its dual according to the Farkas Lemma. A feasible dual solution (y_1, y_2) can be uncrossed in the same way as above, yielding (y_1', y_2') where y_1' is positive on a chain and y_2' is positive on a cross-free family. This means that dual feasibility implies the emptiness of the intersection of the two base polyhedra given by p and q restricted to the sets where y_1' and y_2' are positive. Thus the non-emptiness condition for the intersection of base polyhedra (which is of the form (23)) is sufficient for the feasibility of the original system. □

Theorem 6.2. *Let $G = (V, E)$ be a graph, $h : 2^V \to \mathbb{Z}_+$ a positively crossing G–supermodular set function on V, and $m : V \to \mathbb{Z}_+$ a degree specification with $m(V)$ even; let*

$$h_m(X) := h(X) + \left(m(X) - \frac{m(V)}{2}\right)^+ .$$

There exists an undirected graph $H = (V, F)$ such that $G + H$ has an orientation covering h and $d_H(v) = m(v)$ for all $v \in V$ if and only if

$$\sum_{Z\in\mathcal{F}} h_m(Z) + \left(m(\overline{X}) - \frac{m(V)}{2}\right)^+ \leq e_G(\mathcal{F}) + (\alpha+1)\frac{m(V)}{2} \qquad (24)$$

for every $X \subset V$ and every tree-composition $\mathcal{F}$ of X with covering number α.

Proof. The necessity follows from the fact that if $\mathcal{F}'$ is a regular family with covering number $\alpha + 1$, then $\sum_{Z\in\mathcal{F}'} \varrho_{\vec{G}}(Z) \leq e_G(\mathcal{F}')$ for any orientation $\vec{G}$ of G, and $\sum_{Z\in\mathcal{F}'} \varrho_{\vec{H}}(Z) \leq (\alpha+1)m(V)/2 - \sum_{Z\in\mathcal{F}'} (m(Z) - m(V)/2)^+$ for any orientation $\vec{H}$ of a graph H satisfying the degree specification. Note that if we consider (24) only for partitions and co-partitions (that is, $X = \emptyset$), then it corresponds to (6)–(9).

The sufficiency can be proved in essentially the same way as in the proof of Theorem 3.1: define G' and h' similarly, and for $X \subseteq V$, let

$$q_1(X) := i_G(X) + \left(m(X) - \frac{m(V)}{2}\right)^+ ,$$

$$q_2(X) := h(X) + i_G(X) + \left(m(X) - \frac{m(V)}{2}\right)^+ .$$

In this case Lemma 3.2 implies that an orientation of G' covering h' exists if and only if the polyhedron

$$\{x : V \to \mathbb{R} \mid x(V) = q_1(V);\ x(Y) \geq q_2(Y),\ x(Y) \geq q_1(Y)\ \forall Y \subseteq V\}$$

has an integral point.

Claim 6.3. *The set function q_1 is fully supermodular, and the set function q_2 is supermodular on the crossing pairs (X, Y) for which $q_1(X) < q_2(X)$ and $q_1(Y) < q_2(Y)$.*

Proof. The set function q_1 is the sum of two fully supermodular functions (see the proof of Claim 3.3), so it is fully supermodular. Since h is positively crossing G–supermodular, q_2 is supermodular on the crossing pairs (X, Y) for which $h(X), h(Y) > 0$, and these are exactly the crossing pairs for which $q_1(X) < q_2(X)$ and $q_1(Y) < q_2(Y)$. □

Lemma 6.1 implies that an orientation of G' covering h' exists if and only if

$$q_1(\overline{X}) + \sum_{Z\in\mathcal{F}} q_2(Z) \leq (\alpha+1)q_1(V)$$

for every $X \subset V$ and every tree-composition $\mathcal{F}$ of X with covering number α. Using (5) and the fact that $e_G(\mathcal{F}) = e_G(\mathcal{F} + \{\overline{X}\})$, this is equivalent to the condition of the theorem.

From here we can follow the line of the proof of Theorem 3.1. Let $\vec{G}'$ be the orientation of G' covering h', and let $\vec{G}$ denote the digraph obtained from $\vec{G}'$ by deleting the node z. Let $m_i(v)$ be the multiplicity of the edge zv in $\vec{G}'$, and $m_o(v)$ the multiplicity of the edge vz in $\vec{G}'$. Define the set function $p(X) = (h(X) - \varrho_{\vec{G}}(X))^+$; p is positively crossing supermodular. As in the proof of Theorem 3.1, we can apply Theorem 2.3 (with the m_i, m_o and p defined above) to obtain a digraph D whose underlying undirected graph H is a good augmentation of G. This concludes the proof of Theorem 6.2. □

As it was shown in Section 4, the minimum cardinality augmentation problem is tractable for the non-negative crossing supermodular case, thanks to the polymatroidal structure of good degree specifications. However, we were not able to devise similar methods for the positively crossing supermodular case; indeed, in this more general setting, it remains an open question if a good min-max formula can be found for the minimum cardinality augmentation problem.

Remark. The appearance of tree-compositions in condition (24) may seem unfriendly, but it is unavoidable, even in the special case when the problem is to find an orientation of the undirected edges of a mixed graph such that the resulting digraph is k-edge-connected. This orientation problem was already considered in [5], where crossing G–supermodular set functions with possible negative values were studied. The following example shows that the positively G–supermodular case is more general, i. e. not every positively crossing G–supermodular set function h can be made crossing G–supermodular by decreasing the value of h on some of the sets where it is 0.

Let X_1, X_2, X_3 be three subsets of a ground set V, in general situation. Let $h(X_i) = 1$, $h(X_i \cup X_j) = 2$ $(i \neq j)$, $h(X_1 \cup X_2 \cup X_3) = 4$, and $h(X) = 0$ on the remaining sets; this is a positively crossing supermodular function. The value of $h(X_1 \cap X_2)$ cannot be decreased since

$$h(X_1 \cap X_2) \geq h(X_1) + h(X_2) - h(X_1 \cup X_2) = 0 \ .$$

Therefore it is impossible to correctly modify h so as to satisfy

$$h(X_1 \cap X_2) \leq h(X_1 \cap X_2 \cap X_3) + h(X_1 \cap X_2 \cup X_3) - h(X_3) \leq -1 \ .$$

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