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Abstract

Graph orientation is a well-studied area of combinatorial optimization, one that provides a link between directed and undirected graphs. An important class of questions that arise in this area concerns orientations with connectivity requirements. In this paper we focus on how similar questions can be asked about hypergraphs, and we show that often the answers are also similar: many known graph orientation theorems can be extended to hypergraphs, using the familiar uncrossing techniques. Our results also include a short proof and an extension of a theorem of S. Khanna, J. Naor and F. B. Shepherd [8], and a new orientation theorem that provides a characterization for $(2k+1)$-edge-connected graphs.

Keywords: Directed hypergraph; Connectivity; Orientation; Uncrossing

1 Introduction

One of the early examples of graph orientation results is the theorem of H. E. Robbins [11]: a graph has a strongly connected orientation if and only if it is 2-edge-connected. As an illustration, he rephrased this as a traffic control problem: decide whether the streets of a city can be turned into one-way streets such that any location in the city remains reachable from any other location. Robbins' problem of course has many different extensions; one possible direction of generalization could be illustrated by the following exercise. Consider m people and n computers, where each person

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has access to a given subset of the computers; the task is to decide whether the authorizations of each person can be restricted to read-only access on all but one of the computers, such that it remains possible to transfer data from any computer to any other. This can be seen as a hypergraph counterpart of Robbins' problem, the latter being equivalent to the case when everyone has access to exactly two computers. If the connectivity condition is required to hold even when $k-1$ people are absent, then we get a generalization of the k-edge-connected graph orientation problem solved by C. Nash-Williams [10]. However, one immediately sees that $2k$-edge-connectivity of the access hypergraph (i.e. no matter how the computers are divided into two groups, there are at least $2k$ people having access to machines in both groups), which was a sufficient condition for graphs to have a k-edge-connected orientation, is no longer sufficient.

The objective of the present paper is to study orientation problems where the graph case can be extended in the above manner to hypergraphs, and where good characterizations can be proved using more-or-less standard uncrossing techniques. After giving some preliminaries on directed hypergraphs in Section 2, we prove a hypergraph orientation theorem in Section 3 that provides hypergraph versions of some known graph theorems, including those of Robbins and Nash-Williams.

In [8], S. Khanna, J. Naor and F. B. Shepherd proposed a new framework, called network design with orientation constraints, that successfully integrated network design problems like minimum cost rooted k-edge-connected sub-digraphs, and orientation problems like rooted k-edge-connected orientation of a mixed graph. In Section 4, we show that their formulation is a TDI system, thus obtaining new min-max formulas, and we extend their result to hypergraphs, as well.

Finally, Section 5 includes a theorem on hypergraph orientations with a special local connectivity criterion, a result that is new even when specialized to graphs; in the latter case, it also gives a new characterization of $(2k+1)$-edge-connected graphs.

All of the results are based in some way on the uncrossing technique, so the notions related to it are presented here in some detail. On a ground set V, two subsets $X, Y \subseteq V$ are called *intersecting* if none of $X-Y$, $Y-X$ and $X \cap Y$ is empty; they are *crossing* if in addition $X \cup Y \neq V$. A *family of sets* is a collection of subsets of the ground set V, with possible repetition. The *union* of two families $\mathcal{F}_1$ and $\mathcal{F}_2$, denoted by $\mathcal{F}_1 + \mathcal{F}_2$, is the family where the multiplicity of every subset is the sum of its multiplicities in $\mathcal{F}_1$ and $\mathcal{F}_2$. A family is *cross-free* if it contains no crossing pairs of sets; it is *regular* if every node of the ground set is contained in the same number of members; this number is called the *covering number*. Given a family $\mathcal{F}$, $\mathrm{co}(\mathcal{F})$ denotes the family obtained by replacing every member of $\mathcal{F}$ by its complement. Clearly if $\mathcal{F}$ is cross-free or regular, then so is $\mathrm{co}(\mathcal{F})$. If $\mathcal{F}$ is a partition, then $\mathrm{co}(\mathcal{F})$ is called a *co-partition*. For a vector $x : V \to \mathbb{R}$ and a set $Y \subseteq V$, we use the notation $x(Y) := \sum_{v \in Y} x(v)$.

Let $y : 2^V \to \mathbb{Q}_+$ be a non-negative set function. By the *uncrossing operation* we mean the following modification of y: given two crossing sets X_1 and X_2 with $y(X_1), y(X_2) > 0$, decrease $y(X_1)$ and $y(X_2)$ by $\min\{y(X_1), y(X_2)\}$, and increase $y(X_1 \cap X_2)$ and $y(X_1 \cup X_2)$ by the same amount. If $y(X)$ is defined as the multiplicity of X in a family $\mathcal{F}$, then we speak of *uncrossing* $\mathcal{F}$.

Lemma 1.1. *After finitely many uncrossing operations y is positive only on a cross-free family of sets.*

Proof. This well-known result can be seen as a special case of the following claim (note that the claim does not hold for non-negative real numbers!):

Claim 1.2. *Let $x_1, \dots, x_n$ be non-negative rational numbers. Suppose that we apply repeatedly the following operation: for some indices $i < j < k < l$ where x_j and x_k are positive, decrease x_j and x_k by $\min\{x_j, x_k\}$, and increase x_i and x_l by $\min\{x_j, x_k\}$. Then this operation can be repeated only a finite number of times.*

Proof. By multiplying all x_i values by a suitable integer, we can assume that every x_i is integer. Now suppose that there is an infinite sequence of operations, and let m be the smallest index for which x_m decreases infinitely many times. Then one of $x_1, \dots, x_{m-1}$ increases infinitely many times by at least 1, but decreases finitely many times, which is impossible since $\sum x_i$ remains constant and $x_i \geq 0$ for every i. □

Let $X_1, \dots, X_t$ be an ordering of the subsets of V compatible with the standard partial order; let $x_i := y(X_i)$. Then it follows from the claim that after finitely many uncrossing steps uncrossing is impossible, thus y is positive on a cross-free family. □

The usefulness of the uncrossing technique in combinatorial optimization follows from the nice properties of cross-free families, that are often linked to dual integrality properties. It is well-known that every cross-free family $\mathcal{F}$ has a *tree-representation* (T, φ), where $T = (W, A)$ is a directed tree, and $\varphi : V \to W$ is a mapping such that $\{\varphi^{-1}(W_e) \mid e \in A\} = \mathcal{F}$, where W_e is the component of $T - e$ entered by e. Here we only mention the following simple consequence:

Lemma 1.3. *A regular cross-free family decomposes into partitions and co-partitions.* □

2 Preliminaries on directed hypergraphs

The concept of directed hypergraphs was introduced in many different contexts, in areas like propositional logic, assembly, and relational databases, to efficiently model many-to-one relations; surveys of these applications can be found in [6] and [7]. In our terminology, a *directed hypergraph* is a pair $\vec{H} = (V, \vec{\mathcal{E}})$, where V is a finite ground set, and $\vec{\mathcal{E}}$ is a finite collection of so-called hyperarcs (possibly with repetition): a *hyperarc* is a subset $Z \subseteq V$ with a designated *head node* $v \in Z$, and it is denoted by Z^v. The nodes of $Z - v$ are called the *tail nodes* of Z^v. Clearly, a digraph is a directed hypergraph where every hyperarc has two nodes.

A natural way of looking at a directed hypergraph is that it is an *orientation* of a hypergraph $H = (V, \mathcal{E})$, i.e., a head node is assigned to every hyperedge in $\mathcal{E}$. To formulate orientation problems, some notions should be introduced on the connectivity properties of directed hypergraphs.

A *path* in a directed hypergraph is an alternating sequence, without repetition, of nodes and hyperarcs $v_1, e_1, v_2, e_2, \dots, e_k, v_{k+1}$, where v_i is one of the tail nodes of e_i, and v_{i+1} is the head node of e_i. The node t is said to be *weakly reachable* from the node s if there is a path from s to t. The reason for using the adjective "weakly" is that in some applications of directed hypergraphs such as assembly or databases, a different notion of reachability is appropriate (see [7]). However, that framework does not allow for an analogue of Menger's theorem, which restricts the possible discussion of connectivity. On the other hand, Menger's theorem extends naturally to directed hypergraphs with respect to weak reachability.

A hyperarc Z^v is said to *enter* a set X if $v \in X$ and $Z - X \neq \emptyset$. The set of hyperarcs of the directed hypergraph $\vec{H}$ entering a set $X \subseteq V$ is denoted by $\Delta^-_{\vec{H}}(X)$, and $\varrho_{\vec{H}}(X)$ denotes the number of hyperarcs in $\Delta^-_{\vec{H}}(X)$ (if it causes no ambiguity, then the indication of the hypergraph in the subscript is sometimes omitted, or the hyperarc set is indicated instead). For two nodes s and t, a set X is an $\overline{s}t$*-set* if $s \notin X$ and $t \in X$.

Proposition 2.1. *In a directed hypergraph $\vec{H} = (V, \vec{\mathcal{E}})$, there exist k edge-disjoint paths from node s to node t if and only if $\varrho_{\vec{H}}(X) \geq k$ for every $\overline{s}t$-set X.*

Proof. To reduce the problem to the digraph case, a new node v_e is added to V for every hyperarc $e \in \vec{\mathcal{E}}$, and the hyperarc $e = Z^v$ is replaced by arcs uv_e for every $u \in Z - v$, and an arc $v_e v$. There is a one-to-one correspondence between the paths from s to t in this digraph, and the paths from s to t in the original directed hypergraph. By applying Menger's theorem to the digraph, we get the conditions of the proposition. □

Let $d_{\vec{H}}(X, Y)$ be the number of hyperarcs $Z^v \in \vec{\mathcal{E}}$ with $Z \subseteq X \cup Y$, $Z - X \neq \emptyset$ and $Z - Y \neq \emptyset$. Analogously to the case of digraphs, the set function $\varrho_{\vec{H}}$ has the following property:

Claim 2.2. *Let $\vec{H}$ be a directed hypergraph, and $X, Y \subseteq V$. Then $\varrho_{\vec{H}}(X) + \varrho_{\vec{H}}(Y) = \varrho_{\vec{H}}(X \cap Y) + \varrho_{\vec{H}}(X \cup Y) + d_{\vec{H}}(X, Y)$.* □

Like Menger's theorem, J. Edmonds' disjoint branching theorem [1] can be easily adapted to directed hypergraphs. Given a set $S \subseteq V$, a directed hypergraph $\vec{H} = (V, \vec{\mathcal{E}})$ is *connected from* S if every node $v \in V$ is weakly reachable from some $s \in S$.

Proposition 2.3. *Let $\vec{H} = (V, \vec{\mathcal{E}})$ be a directed hypergraph, and $S_1, \dots, S_k$ subsets of V; for $X \subseteq V$, let $f(X)$ denote the number of sets S_i not disjoint from X. Then $\vec{H}$ can be decomposed into directed sub-hypergraphs $\vec{H}_1, \dots, \vec{H}_k$ such that $\vec{H}_i$ is connected from S_i, if and only if*

$$\varrho_{\vec{H}}(X) \geq k - f(X) \quad \textit{for every } \emptyset \neq X \subseteq V.$$

Proof. We prove the theorem by induction on the number of non-graph hyperarcs. If every hyperarc is a graph arc, then we can use Edmonds' theorem. Suppose that there is a hyperarc $e = Z^v \in \vec{\mathcal{E}}$ with $|Z| > 2$. Call a set $X \subseteq V - s$ *tight* if $\varrho_{\vec{H}}(X) = k - f(X)$. Let $\mathcal{F}$ be the family of tight sets entered by e. If $\mathcal{F} = \emptyset$ or $\mathcal{F}$ has a unique maximal element X, then we can replace the hyperarc Z^v by an arc

uv where u is an arbitrary node in $Z - X$, and use induction. If $\mathcal{F}$ has at least two maximal elements, say X and Y, then e cannot enter $X \cup Y$, since by Claim 2.2 the union would also be tight, which would contradict the maximality. But then $\varrho_{\vec{H}}(X \cap Y) + \varrho_{\vec{H}}(X \cup Y) = \varrho_{\vec{H}}(X) + \varrho_{\vec{H}}(Y) - d_{\vec{H}}(X, Y) < 2k - f(X \cap Y) - f(X \cup Y)$, so $X \cap Y$ or $X \cup Y$ would violate the condition. □

These simple results show that weak connectivity of directed hypergraphs can be treated in essentially the same way as edge-connectivity of digraphs. Thus orientation problems can be formulated in the same general framework that is commonly used for graph connectivity orientation. Let $H = (V, \mathcal{E})$ be a hypergraph, and $h : 2^V \to \mathbb{Z}$ a set function, called the *requirement function*; we always assume that $h(\emptyset) = h(V) = 0$. An orientation $\vec{H} = (V, \vec{\mathcal{E}})$ of H is said to *cover* h if $\varrho_{\vec{H}}(X) \geq h(X)$ for every $X \subseteq V$. The *hypergraph orientation problem* is to find an orientation of a hypergraph (or of a sub-hypergraph with specified properties) that covers a given requirement function h. For example, for a positive integer k, if h equals k on all non-empty proper subsets of V, then the task is to find an orientation where every node is connected to every other node by k edge-disjoint paths (a *k-edge-connected orientation*).

In this paper we study the hypergraph orientation problem for supermodular-type requirement functions. A set function h on a ground set V is *intersecting* (respectively *crossing*) *supermodular*, if

$$h(X) + h(Y) \leq h(X \cap Y) + h(X \cup Y) \tag{1}$$

for any intersecting (respectively crossing) pair $X, Y \subseteq V$; it is *positively intersecting supermodular* if (1) holds whenever $h(X) > 0$, $h(Y) > 0$ and X, Y are intersecting.

3 Orientations covering non-negative crossing supermodular set functions

In [4], the graph orientation problem for non-negative crossing supermodular functions was solved, which includes as a special case k-edge-connected orientations with upper and lower bounds on the in-degrees of the nodes. In this section we extend this result to hypergraphs.

For a hypergraph $H = (V, \mathcal{E})$ and a set $X \subseteq V$, let $i_H(X)$ denote the number of hyperedges $Z \in \mathcal{E}$ with $Z \subseteq X$. For a family $\mathcal{F}$ let

$$e_H(\mathcal{F}) := \max\left\{ \sum_{X \in \mathcal{F}} \varrho_{\vec{H}}(X) \mid \vec{H} \text{ is an orientation of } H \right\}. \tag{2}$$

Note that if $\mathcal{F}$ is regular with covering number α, then clearly

$$e_H(\mathcal{F}) = \alpha|\mathcal{E}| - \sum_{X \in \mathcal{F}} i_H(X). \tag{3}$$

Claim 3.1. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are regular families, then $e_H(\mathcal{F}_1+\mathcal{F}_2)=e_H(\mathcal{F}_1)+e_H(\mathcal{F}_2)$.* □

If $\mathcal{F}$ is a partition, then $e_H(\mathcal{F})$ is the number of hyperedges that are not subsets of any member of the partition (these are called *cross-hyperedges*). It should be noted that for co-partitions it does not count the number of cross-hyperedges of the corresponding partition, except when H is a graph.

The main theorem of the section is the following:

Theorem 3.2. *Let $H=(V,\mathcal{E})$ be a hypergraph, and h a non-negative crossing supermodular set function. There is an orientation of H covering h if and only if*

$$\sum_{X\in\mathcal{F}} h(X) \le e_H(\mathcal{F}) \tag{4}$$

for every partition and co-partition $\mathcal{F}$.

Proof. The following hypergraph orientation lemma is a straightforward generalization of a graph orientation lemma that reduces the problem of giving a feasible orientation to the problem of finding suitable in-degrees.

Lemma 3.3. *Given a hypergraph H and a vector $x: V\to\mathbb{Z}_+$, there is an orientation $\vec{H}$ of H such that $\varrho_{\vec{H}}(v)=x(v)$ for every $v\in V$, if and only if $x(V)=|\mathcal{E}|$ and $x(Y)\ge i_H(Y)$ for every $Y\subseteq V$.*

Proof. The necessity is straightforward. We prove the sufficiency by induction on the number of hyperedges. Call a set Y tight if $x(Y)=i_H(Y)$. Let $Z\in\mathcal{E}$ be an arbitrary hyperedge; then $x(Z-X)\ge 1$ for any tight set $X\not\supseteq Z$ (including $X=\emptyset$), otherwise $Z\cup X$ would violate the condition. If there is a node $v\in Z$ with $x(v)>0$ such that $Z\subseteq X$ for every tight set X containing v, then we can remove the hyperedge Z, decrease $x(v)$ by one, find a feasible orientation of the resulting hypergraph by induction, and add the directed hyperedge Z^v. Otherwise, since a single tight set $X\not\supseteq Z$ cannot contain every node $v\in Z$ with $x(v)>0$, we can choose tight sets X_1, X_2, that are both maximal among the tight sets satisfying $X\cap Z\neq\emptyset$ and $Z-X\neq\emptyset$. Then $X_1\cup X_2$ is tight and $d_H(X_1,X_2)=0$, since $i_H(X_1)+i_H(X_2)=i_H(X_1\cap X_2)+i_H(X_1\cup X_2)-d_H(X_1,X_2)$; thus $Z-(X_1\cup X_2)\neq\emptyset$, which contradicts the maximality of X_1 and X_2. □

Call a partition or a co-partition $\mathcal{F}$ *tight* if $\sum_{X\in\mathcal{F}} h(X)=e_H(\mathcal{F})$. Observe that the crossing supermodularity remains valid if we increase the value of h on some singletons; we can thus assume that every singleton $\{v\}$ is in a tight partition $\mathcal{F}_v$. Let $\mathcal{F}:=\sum_{v\in V}\mathcal{F}_v$ be the union (with multiplicity) of these tight partitions; then $\sum_{X\in\mathcal{F}} h(X)=e_H(\mathcal{F})$. Our aim is to show that this implies $\sum_{v\in V} h(\{v\})=|\mathcal{E}|$. We can uncross $\mathcal{F}$ using the standard uncrossing operation, to obtain a cross-free regular family $\mathcal{F}'$ including all the singletons, for which

$$\sum_{X\in\mathcal{F}'} h(X) \ge e_H(\mathcal{F}'), \tag{5}$$

since an uncrossing step does not decrease $\sum_{X\in\mathcal{F}} h(X)$ thanks to the crossing supermodularity, and does not increase $e_H(\mathcal{F})$.

Let $\mathcal{F}''$ be the family obtained by decreasing the multiplicity of every singleton in $\mathcal{F}'$ by 1. By Lemma 1.3, $\mathcal{F}''$ decomposes into partitions and co-partitions, and by Claim 3.1, (4), and (5), these must be tight partitions and co-partitions, and the partition formed of singletons is tight as well. As a consequence, if we define $x(v) := h(\{v\})$ for every $v \in V$, then $x(V) = |\mathcal{E}|$.

To complete the proof, it suffices to show that $x(Y) \geq i_H(Y) + h(Y)$ for every set $Y \subseteq V$, since in this case by Lemma 3.3 and the non-negativity of h there is an orientation with in-degree vector x, and since every set Y is entered by $x(Y) - i_H(Y)$ hyperarcs, this orientation covers h. To prove the inequality, define the partition $\mathcal{F}_Y = \{Y,\ \{v\} :\ v \in V - Y\}$ for every set $Y \subset V$. Using (4) on the partition $\mathcal{F}_Y$, we get

$$\begin{aligned} x(Y) &= |\mathcal{E}| - x(V-Y) = |\mathcal{E}| - \sum_{X\in\mathcal{F}_Y} h(X) + h(Y) \\ &\geq |\mathcal{E}| - e_H(\mathcal{F}_Y) + h(Y) = i_H(Y) + h(Y). \end{aligned}$$

□

Remark. In [5], S. Fujishige proved the following:

Theorem 3.4 (Fujishige [5]). *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a crossing supermodular function. There exists a vector $x : V \to \mathbb{Z}$ satisfying $x(V) = p(V)$ and $x(Y) \geq p(Y)\ \forall Y \subseteq V$, if and only if*

$$\sum_{i=1}^{t} p(X_i) \leq p(V),$$

$$\sum_{i=1}^{t} p(\overline{X_i}) \leq (t-1)p(V)$$

both hold for every partition $\{X_1, \ldots, X_t\}$ of V. □

Using Fujishige's theorem, a short alternative proof of Theorem 3.2 can be given. Define the set function $p(X) := h(X) + i_H(X)$; then p is crossing supermodular. If $\mathcal{F} = \{X_1, \ldots, X_t\}$ is a partition of V, then, by (4) and (3), $\sum p(X_i) = \sum h(X_i) + \sum i_H(X_i) \leq e_H(\mathcal{F}) + \sum i_H(X_i) = |\mathcal{E}| = p(V)$, and $\sum p(\overline{X_i}) = \sum h(\overline{X_i}) + \sum i_H(\overline{X_i}) \leq e_H(\mathrm{co}(\mathcal{F})) + \sum i_H(\overline{X_i}) = (t-1)|\mathcal{E}| = (t-1)p(V)$. Thus Theorem 3.4 implies that if the conditions (4) hold, then there is an integral vector $x : V \to \mathbb{Z}$ satisfying $x(V) = p(V) = |\mathcal{E}|$ and $x(Y) \geq p(Y) = i_H(Y) + h(Y) \geq i_H(Y)\ \forall Y \subseteq V$. By Lemma 3.3, H has an orientation with in-degree vector x, and it is easy to check that this orientation covers h.

Remark. The proofs show that Theorem 3.2 is true under the weaker assumption that h is non-negative and $h+i_H$ is crossing supermodular. If h is monotone decreasing (that is, $h(X) \geq h(Y)$ if $X \subseteq Y$), or it is symmetric, then the co-partition type constraints are unnecessary, since $\sum_{X\in\mathcal{F}} h(X) \geq \sum_{X\in\mathrm{co}(\mathcal{F})} h(X)$ and $e_H(\mathcal{F}) \leq e_H(\mathrm{co}(\mathcal{F}))$ for every partition $\mathcal{F}$.

A directed hypergraph is called (k, l)*-edge-connected* for non-negative integers $k \geq l$ if there is a node $s \in V$ such that there are k edge-disjoint paths from s to any other node, and there are l edge-disjoint paths to s from any other node. A hypergraph H is called (k, l)*-partition-connected* for non-negative integers $k \geq l$ if $e_H(\mathcal{F}) \geq k(t-1)+l$ for every partition $\mathcal{F}$ with t members. Using Proposition 2.1, we have the following corollary:

Corollary 3.5. *A hypergraph has a* (k, l)*-edge-connected orientation if and only if it is* (k, l)*-partition-connected.* □

Combining this result with Proposition 2.3, we get the following:

Corollary 3.6. *A hypergraph* H *has a* (k, l)*-edge-connected orientation, if and only if any hypergraph obtained from* H *by removing* l *hyperedges decomposes into* k *disjoint* $(1, 0)$*-partition-connected sub-hypergraphs.*

4 Orientations covering positively intersecting supermodular set functions

In [8], Khanna, Naor and Shepherd introduced the directed network design problem with orientation constraints. By this framework they gave a common generalization of subgraph problems such as finding a minimum cost rooted k-edge-connected subgraph of a digraph (that was solved in [3]), and orientation problems like rooted k-edge-connected orientation of mixed graphs, discussed in [2]. The basic problem is to find a minimum cost subgraph of a digraph that satisfies a prescribed connectivity property; however, there are also *orientation constraints*: additional constraints on some designated oppositely directed pairs of arcs, which require that at most one member of the pair can be chosen in the subgraph (the term "orientation constraint" is appropriate since a constrained pair of arcs can be thought of as a single undirected edge that has to be oriented or deleted, and the two possible orientations can have different costs). One of the main results in [8] stated that for the problem of finding a minimum cost subgraph that satisfies the orientation constraints and covers a given positively intersecting supermodular requirement function, the natural LP relaxation defines an integral polyhedron (note that for crossing supermodular requirement functions, this would include NP-complete problems).

In this section we extend this result to hypergraphs, and in addition show that the LP relaxation they used is in fact a TDI system; this latter result also enables us to formulate a min-max theorem. First, we show that in the more restricted case when the requirement function is intersecting supermodular, the orientation constraints can be incorporated into a construction of A. Schrijver [12] that transforms the problem without orientation constraints into a submodular flow problem; moreover, this construction can be easily extended to the hypergraph problem described below. A *mixed hypergraph* is a triple $M = (V; \mathcal{E}, \vec{\mathcal{A}})$, where $\mathcal{E}$ is a set of hyperedges and $\vec{\mathcal{A}}$ is a set of hyperarcs. An *oriented sub-hypergraph* of M is a sub-hypergraph of a directed hypergraph obtained from M by orienting the hyperedges in $\mathcal{E}$.

Theorem 4.1. *Let $M = (V; \mathcal{E}, \vec{\mathcal{A}})$ be a mixed hypergraph, and $h : 2^V \to \mathbb{Z}$ an intersecting supermodular requirement function. Suppose that a cost is assigned to each hyperarc in $\vec{\mathcal{A}}$, and to each possible orientation of every hyperedge in $\mathcal{E}$. Then the problem of finding a minimum cost oriented sub-hypergraph of M covering the requirement function h can be formulated as a submodular flow problem, solvable in polynomial time.*

Outline of Proof. Since the proof is a straightforward adaptation of a construction of Schrijver [12], only an outline is given here. We define a directed bipartite graph $G = (V, W; F)$ with arc costs, where the nodes of W correspond to the hyperedges and hyperarcs in $\mathcal{E} \cup \vec{\mathcal{A}}$; we denote a node corresponding to a hyperedge or hyperarc e by w_e. The arc set F contains an arc from w_e to the head of e (with arc cost equal to the cost of e) if e is a hyperarc; if e is a hyperedge, then F contains arcs from w_e to every node of e (each with cost equal to the cost of the corresponding orientation of e). A set function p is defined on the ground set $V \cup W$ as follows:

$$p(X) := \begin{cases} h(X \cap V) & \text{if } w_e \in X \text{ implies that the nodes of } e \text{ are in } X, \\ -1 & \text{if } X = \{w_e\} \text{ for some } e \in \mathcal{E}, \\ -\infty & \text{otherwise.} \end{cases}$$

The intersecting supermodularity of h implies that p is crossing supermodular. Consider the submodular flow problem of finding a minimum cost directed subgraph $G' = (V, W; F')$ of G that satisfies

$$\rho_{G'}(X) - \rho_{G'}(V \cup W - X) \geq p(X) \text{ for every } X \subseteq V \cup W. \tag{6}$$

Since $p(\{w_e\}) = -1$ if $e \in \mathcal{E}$, w_e is the tail of at most one arc of G'. Thus the subgraph G' corresponds to an oriented sub-hypergraph M' of M. It is easy to check that M' covers the requirement function h if and only if G' satisfies (6). □

If the requirement function is only positively intersecting supermodular, then the above construction does not lead to a submodular flow problem, and we do not know whether the problem defines a submodular flow polyhedron. The aim of the next paragraphs is to prove that a natural LP relaxation is nevertheless a TDI system. To formulate the appropriate linear program, the hypergraph analogue of the orientation constraints must be defined. A set $\vec{\mathcal{E}}$ of hyperarcs is called *parallel* if every hyperarc in $\vec{\mathcal{E}}$ is an orientation of the same hyperedge. In Theorem 4.1, we would obtain an equivalent problem if we replaced every hyperedge of the mixed graph by a set of parallel hyperarcs (consisting of all possible orientations of that hyperedge), and imposed the additional constraint that at most one of these parallel hyperarcs can be in the chosen sub-hypergraph. This concept of orientation constraints can be further generalized: we allow arbitrary disjoint sets of parallel hyperarcs, and arbitrary lower and upper bounds on the number of hyperarcs selectable from such a set.

Theorem 4.2. *Let $\vec{H} = (V, \vec{\mathcal{E}})$ be a directed hypergraph, with $f : \vec{\mathcal{E}} \to \mathbb{Z}_+$ and $g : \vec{\mathcal{E}} \to \mathbb{Z}_+$ lower and upper integral capacities on the hyperarcs. Let $\vec{\mathcal{E}}_1, \dots, \vec{\mathcal{E}}_t \subseteq \vec{\mathcal{E}}$ be disjoint parallel sets of hyperarcs, with corresponding lower and upper bounds l_i, u_i*

$(i = 1, \dots, t)$. Let h be a positively intersecting supermodular set function on V, and $c : \vec{\mathcal{E}} \to \mathbb{Z}$ a cost function. Then the system

$$
\begin{aligned}
(\mathbf{S}) \qquad & \min \textstyle\sum_{e\in\vec{\mathcal{E}}} c(e)z(e) && && (7)\\
& \textstyle\sum_{e\in\Delta^-_{\vec{H}}(X)} z(e) \geq h(X) && \textit{for every } X \subseteq V && (8)\\
& f(e) \leq z(e) \leq g(e) && \textit{for every } e \in \vec{\mathcal{E}} && (9)\\
& l_i \leq \textstyle\sum_{e\in\vec{\mathcal{E}}_i} z(e) \leq u_i && (i = 1, \dots, t). && (10)
\end{aligned}
$$

is TDI. Moreover, the values of an optimal dual solution corresponding to the inequalities (8) *can be assumed to be positive on a laminar family of sets.*

Proof. Let c be an integral objective vector. Let y_1 denote the dual variables associated with the inequalities in (8), and let y_2 denote the dual variables associated with the other inequalities. For a hyperedge $e \in \vec{\mathcal{E}}$, the dual constraints are of the form

$$\left(\sum_{X:\ e\in\Delta^-_{\vec{H}}(X)} y_1(X) \right) + y_2 b_e \leq c(e) \tag{11}$$

for appropriate vectors b_e. For an appropriate vector b, the dual objective function is

$$\max \left(\sum_{X\subseteq V} y_1(X)h(X) + y_2 b \right). \tag{12}$$

Let $(y_1^*, y_2^*) \geq 0$ be an optimal dual solution. The main observation is that we can assume that y_1^* is positive only on a laminar family $\mathcal{F}$. If y_1^* is positive on a set X with $h(X) = 0$, then we can decrease $y_1^*(X)$ to 0. Suppose that y_1^* is positive on two intersecting sets X and Y where $h(X), h(Y) > 0$; let $\alpha = \min\{h(X), h(Y)\}$. Decrease $y_1^*(X)$ and $y_1^*(Y)$ by α, and increase $y_1^*(X \cap Y)$ and $y_1^*(X \cup Y)$ by α. Since $\varrho_e(X) + \varrho_e(Y) \geq \varrho_e(X \cap Y) + \varrho_e(X \cup Y)$ for each edge e, the inequality (11) is preserved. The positively intersecting supermodularity of h implies that the dual objective function (12) does not decrease. By Claim 1.2, after a finite number of uncrossing steps, we obtain an optimal dual solution where y_1^* is positive on a laminar family $\mathcal{F}$.

Modify the system $(\mathbf{S})$ by replacing (8) with

$$\sum_{e\in\Delta^-_{\vec{H}}(X)} z(e) \geq h(X) \text{ for every } X \in \mathcal{F}; \tag{13}$$

let us denote this system by $(\mathbf{S}')$. Then (y_1^*, y_2^*) remains a feasible dual solution, and of course it is optimal. Thus if the modified system has an integral optimal dual solution, it is optimal for the dual of $(\mathbf{S})$ as well. The rest of the proof consists of showing that the system $(\mathbf{S}')$ can be described by a network matrix, hence it has an integral dual optimal solution, since network matrices are totally unimodular.

The rows of the network matrix will correspond to the arcs of a directed tree $T' = (W', A')$, and the corresponding lower and upper bounds will be denoted by l' and u'. The laminar family $\mathcal{F}$ has a tree-representation (T, φ) where $T = (W, A)$ is an arborescence; T' will include T as a subtree; for an arc a of T let $l'(a) = -\infty$ and $u'(a) = -h(\varphi^{-1}(W_a))$, where W_a is the component of $T - a$ entered by a. The node set W' is obtained by adding new nodes w_i ($i = 1, \dots, t$) to W (that is, one new node w_i for each orientation constraint $\mathcal{E}_i$). For every $Z \subseteq V$ let $w_Z \in W$ denote the root node of the minimal subtree of T containing all nodes of $\varphi(Z)$. To finish the construction of T', add an arc $a_i = w_{Z_i} w_i$ to A' for $i = 1, \dots, t$, where Z_i is the hyperedge whose orientations are in $\vec{\mathcal{E}}_i$. Define the corresponding lower and upper bounds as $l'(a_i) = l_i$, $u'(a_i) = u_i$.

The columns of the matrix will represent a set B' of arcs, with a one-to-one correspondence between the hyperarcs in $\vec{\mathcal{E}}$ and the arcs in B'. To a hyperarc $Z^v \in \vec{\mathcal{E}}_i$, assign an arc $w_i v$. To hyperarcs $Z^v \in \vec{\mathcal{E}}$ not appearing in any $\vec{\mathcal{E}}_i$, assign an arc $w_Z v$.

Let N denote the network matrix given by the above network $(W'; A', B')$. Then the matrix of the system

$$\{z : B' \to \mathbb{R} \mid l' \le zN \le u',\ f \le z \le g\}$$

is totally unimodular. Moreover, by the one-to-one correspondence between the arcs in B' and the hyperarcs in $\vec{\mathcal{E}}$, this system is equivalent to the system $(\mathbf{S}')$. This implies that $(\mathbf{S}')$ has an integral dual optimal solution, which in turn is an optimal dual solution for $(\mathbf{S})$. □

The theorem implies that the polyhedron described by $(\mathbf{S})$ is integral, and for every integer cost function there exists an integral optimal dual solution where the family of the sets with positive dual variable is laminar. This allows us to formulate fairly friendly new min-max formulas for some graph problems. For example, what is the maximum number of undirected edges, or the maximum number of arcs, that can be removed from a mixed graph such that the obtained subgraph has an orientation covering a given set function h? The following corollary describes a min-max formula that involves both of these problems. The notation $e_E(\mathcal{F})$ is used for the value of (2) corresponding to the undirected graph H defined by a set E of undirected edges.

Corollary 4.3. *Let $G = (V; E, A)$ be a mixed graph (where E is the set of undirected edges and A is the set of arcs). Let $c : E \cup A \to \{0, 1\}$ be a cost function, and $h : 2^V \to \mathbb{Z}_+$ a positively intersecting supermodular set function. Then the minimum cost of a subgraph that has an orientation covering h equals*

$$\max_{\mathcal{F} \text{ laminar}} \sum_{X \in \mathcal{F}} h(X) - e_E(\mathcal{F}) - \sum_{X \in \mathcal{F}} \varrho_A(X) + q(\mathcal{F}), \tag{14}$$

where $q(\mathcal{F})$ is the sum of the costs of the edges and arcs that enter at least one member of $\mathcal{F}$.

Proof. To formulate the problem in the terms of Theorem 4.2, let the directed hypergraph $\vec{H} = (V, \vec{\mathcal{E}})$ be the digraph obtained by replacing the undirected edges of G by a pair of oppositely directed edges; assign an orientation constraint to every such pair with $l_i = 0$, $u_i = 1$. The cost of the arcs in a pair is the cost of the corresponding undirected edge in E. The capacities of the arcs are bounded by $f \equiv 0$, $g \equiv 1$.

For a $\{0,1\}$-valued cost function c, consider the system (**S**), and let the dual solutions be denoted by (y_1, y_2), where y_1 consists of the dual variables associated to the constraints in (8). Take an integral dual optimal solution $(y_1^*, y_2^*) \geq 0$, where y_1^* is positive on a laminar family, and $|y_2^*|$ is minimal. Let $\mathcal{F}$ be the laminar family where every set X has multiplicity $y_1^*(X)$. Then the value of this dual solution is

$$\sum_{X\in\mathcal{F}} h(X) - e_E(\mathcal{F}) - \sum_{X\in\mathcal{F}} \varrho_A(X) + q(\mathcal{F}).$$

Conversely, the value of (14) corresponds to the value of the following dual solution (y_1^*, y_2^*). Let $\mathcal{F}$ be a laminar family where the maximum is attained in (14). For $X \subseteq V$, let $y_1^*(X)$ be the multiplicity of X in $\mathcal{F}$. Define the values of the dual variables in y_2^* as required by the dual constraints, always setting a variable corresponding to an arc to 0 if the arc also belongs to an orientation constraint. In this case the dual objective value is equal to the expression (14). □

5 Special k-edge-connected orientations

A natural generalization of the orientation problems discussed so far would be the study of orientations satisfying local edge-connectivity requirements. A classical result in this area is the Strong Orientation Theorem of Nash-Williams [10]; however, its known proofs require more sophisticated methods than the uncrossing techniques discussed here. Furthermore, given an undirected graph $G = (V, E)$ and $r : V \times V \to \mathbb{Z}_+$, the problem of deciding whether there is an orientation of G with at least $r(x, y)$ edge-disjoint paths from x to y for each $x, y \in V$ is NP-complete; the following is a sketch of the reduction of 3-SAT.

Consider a collection $\mathcal{C}$ of clauses, and construct the following graph G. For every pair $\{x, \overline{x}\}$ of complementary literals, create two nodes v_x and $v_{\overline{x}}$, and an edge $v_x v_{\overline{x}}$. For each clause $c \in \mathcal{C}$, add nodes s_c, t_c, w_c, z_c; for each literal $y \in c$, add edges $v_{\overline{y}} s_c$, $v_y t_c$, $v_{\overline{y}} w_c$, and $v_y z_c$. Consider the problem of finding an orientation of G such that for every clause $c \in \mathcal{C}$ there are at least 3 edge-disjoint paths from s_c to w_c, 3 edge-disjoint paths from z_c to t_c, and 1 path from s_c to t_c. It is easy to see that the existence of such an orientation is equivalent to the satisfiability of $\mathcal{C}$.

In this light, a solution to the following rather restricted local edge-connectivity orientation problem may have some interest. We consider k-edge-connected orientations of graphs and hypergraphs, where the number of edge-disjoint paths required between two designated special nodes can be more than k. First we formulate a partition-type condition for the hypergraph case, and prove its sufficiency using a modified uncrossing method; then we show that for graphs a cut-type condition is sufficient; this latter result is also proved directly using Mader's splitting off theorem. As a special case we give an orientation-type characterization of $(2k + 1)$-edge-connected graphs.

Theorem 5.1. *Let $H = (V, \mathcal{E})$ be a hypergraph, $s, t \in V$ and $k_1, k_2 \geq k$ positive integers. For a non-empty subset $X \subset V$ let $h(X) := k_1$ if X is an $\overline{s}t$-set, $h(X) := k_2$ if X is a $\overline{t}s$-set, and $h(X) := k$ otherwise. Then H has a k-edge-connected orientation such that there are k_1 edge-disjoint paths from s to t and k_2 edge-disjoint paths from t to s, if and only if*

$$e_H(\mathcal{F}) \geq \sum_{X \in \mathcal{F}} h(X) \tag{15}$$

for every partition $\mathcal{F}$ (where $e_H(\mathcal{F})$ is defined in (2)).

Proof. The goal is to find an orientation that covers h. Observe that the set function h has none of the properties discussed in the previous sections. As in the proof of Theorem 3.2, we increase the value of h on the singletons so that every singleton $\{v\}$ is in a tight partition $\mathcal{F}_v$ (a partition that satisfies (15) by equality); let $\mathcal{F} := \sum_{v \in V} \mathcal{F}_v$ be the union of these partitions, and let h' denote the modified set function; then

$$\sum_{X \in \mathcal{F}} h'(X) = e_H(\mathcal{F}). \tag{16}$$

Apply one of the following three operations on $\mathcal{F}$ as long as one of them can be applied:

1. Uncross X and Y if they are crossing unless one of them is an $\overline{s}t$-set and the other is a $\overline{t}s$-set;
2. If $\mathcal{F}$ contains a co-partition, replace it by the partition obtained by taking the complement of every member;
3. If X is an $\overline{s}t$-set, Y is a $\overline{t}s$-set, and there is a sub-family $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathrm{co}(\mathcal{G})$ is a partition of $X \cap Y$ or $\mathrm{co}(\mathcal{G}) = \{X \cap Y\}$, replace X, Y and $\mathcal{G}$ in $\mathcal{F}$ by $X - Y$, $Y - X$ and $\mathrm{co}(\mathcal{G})$.

Claim 5.2. *These operations do not increase $e_H(\mathcal{F})$, and do not decrease the value of $\sum_{X \in \mathcal{F}} h'(X)$.*

Proof. A simple case analysis shows that the operations do not increase $e_H(\mathcal{F})$, as it suffices to check that the operations do not increase $\sum_{X \in \mathcal{F}} \varrho_e(X)$ for any hyperarc e. An even more simple case analysis shows that the operations do not decrease $\sum_{X \in \mathcal{F}} h(X)$, consequently they cannot decrease the value $\sum_{X \in \mathcal{F}} h'(X)$, since singletons are never removed from the family. □

Obviously $\mathcal{F}$ remains regular, and since the second and the third operations decrease the covering number, and by Claim 1.2 the first operation can be applied only finitely many times consecutively, after a finite number of steps none of the three operations can be applied. Let us denote the obtained regular family by $\mathcal{F}'$; Claim 5.2 and (16) imply that $\sum_{X \in \mathcal{F}'} h'(X) \geq e_H(\mathcal{F}')$. Let $\mathcal{F}''$ be the regular family obtained from $\mathcal{F}'$ by decreasing the multiplicity of every singleton by one.

Proposition 5.3. *$\mathcal{F}''$ decomposes into partitions.*

Proof. We can assume that there is an $\bar{s}t$-set and a $\bar{t}s$-set, otherwise by the unavailability of the first and the second operation $\mathcal{F}''$ is a cross-free family that decomposes into partitions. The $\bar{s}t$-sets in $\mathcal{F}''$ form a chain, the $\bar{t}s$ sets likewise. Let X be the minimal $\bar{s}t$-set, and Y the maximal $\bar{t}s$-set in $\mathcal{F}''$.

If $X \cap Y \neq \emptyset$, then for every $v \in X \cap Y$ there is a $\bar{v}t$-set in $\mathcal{F}''$, since $\mathcal{F}''$ is regular; let $\mathcal{A}$ denote the family of these sets. By the minimality of X, the members of $\mathcal{A}$ are not $\bar{s}t$-sets. Furthermore, they are neither crossing each other, nor X, nor Y, since the first operation cannot be applied. This is only possible if the minimal sets in $\mathrm{co}(\mathcal{A})$ define a partition of $X \cap Y$ (or $X \cap Y \in \mathrm{co}(\mathcal{A})$). But then the third operation would have been applicable, contradicting the assumption.

Thus X and Y are disjoint. For every $v \in V - X - Y$ there is a $\bar{t}v$-set in $\mathcal{F}''$, since $\mathcal{F}''$ is regular. By the maximality of Y, these sets are not $\bar{t}s$-sets, so they are disjoint from X and Y, otherwise they would cross X or Y. The minimal such sets are also disjoint from each other, so they form a partition $\mathcal{P}$ of $V - X - Y$. Thus $\mathcal{F}''$ contains the partition $\mathcal{P} + \{X, Y\}$. By induction, $\mathcal{F}''$ decomposes into partitions. □

If the conditions of the theorem are met, then $\sum_{X \in \mathcal{F}''} h'(X) \leq e_H(\mathcal{F}'')$ must hold, so $\sum_{X \in \mathcal{F}'} h'(X) \geq e_H(\mathcal{F}')$ implies that the partition formed by the singletons must be tight. Let the vector $x : V \to \mathbb{Z}$ be defined by $x(v) := h'(\{v\})$; then $x(V) = |\mathcal{E}|$.

The end of the proof is the same as for Theorem 3.2. Define the partition $\mathcal{F}_Y := \{Y,\ \{v\} :\ v \in V - Y\}$ for every set $Y \subset V$. The conditions of the theorem imply that

$$\begin{aligned} x(Y) &= |\mathcal{E}| - x(V - Y) = |\mathcal{E}| - \sum_{X \in \mathcal{F}_Y} h'(X) + h'(Y) \\ &\geq |\mathcal{E}| - e_H(\mathcal{F}_Y) + h'(Y) = i_H(Y) + h'(Y). \end{aligned}$$

Thus $x(Y) \geq i_H(Y) + h'(Y)$ for every set $Y \subseteq V$, and by Lemma 3.3 there is an orientation with in-degree vector x that covers h', hence it covers h. □

A simple observation shows that for graphs the condition of Theorem 5.1 can be further simplified.

Theorem 5.4. *Let $G = (V, E)$ be an undirected graph, with $s, t \in V$ special nodes, and let k, k_1, k_2 be positive integers for which $k_1, k_2 \geq k$. Then G has a k-edge-connected orientation such that there are k_1 edge-disjoint paths from s to t and k_2 edge-disjoint paths from t to s if and only if $d_G(X) \geq 2k$ for every $\emptyset \neq X \subset V$, and $d_G(X) \geq k_1 + k_2$ for every $\bar{s}t$-set.*

Proof. Let the definition of h be the same as in Theorem 5.1. Suppose indirectly that the conditions of the theorem hold, yet there is a partition $\mathcal{F}$ such that

$$e_H(\mathcal{F}) = \sum_{X \in \mathcal{F}} \frac{d_G(X)}{2} < \sum_{X \in \mathcal{F}} h(X).$$

From this, there is a member X of $\mathcal{F}$ such that $h(X) > d_G(X)/2$; X must separate s and t, otherwise it would violate the conditions. Let Y be the other member of $\mathcal{F}$ separating s and t. Then either $k_1 + k_2 > (d_G(X) + d_G(Y))/2$, or $k > d_G(Z)/2$ for some other member Z of $\mathcal{F}$, contradicting the conditions. □

Theorem 5.4 can also be proved using a different approach that does not seem to extend to hypergraphs, namely a simple application of the undirected splitting off theorem of Mader.

Alternative proof of Theorem 5.4. We use induction on the number of edges of G. Call a set X tight if $d(X) = k_1 + k_2$ and X separates s and t, or $d(X) = 2k$ and X does not separate s and t. We can assume that every edge enters a tight set, otherwise it can be deleted. If every edge of G enters a tight set separating s and t, then the edge set of G can be partitioned into $k_1 + k_2$ simple paths between s and t, and every node v is reached by at least k such paths, since $d(v) \geq 2k$. Let $\vec{G}$ be the digraph obtained by orienting k_1 paths from s to t, and k_2 paths from t to s; then $\varrho_{\vec{G}}(X) \geq k$ and $\varrho_{\vec{G}}(V - X) \geq k$ for every set $\emptyset \neq X \subset V - \{s, t\}$, thus $\vec{G}$ is a good orientation.

We can now assume that there exists a minimal tight set W not containing s and t . Observe that if X and Y are crossing tight sets, then either one of them is an $\bar{s}t$-set and the other is a $\bar{t}s$-set, or $X \cap Y$ is tight; thus the minimality of W implies that an edge spanned by W could not enter a tight set, hence $i_G(W) = 0$. Thus W is a singleton $\{w\}$ with $d(w) = 2k$. A *complete splitting* at w consists of partitioning the neighbours of w into k pairs, for every pair $\{u, v\}$ adding an edge uv to the graph, and deleting w. The following splitting-off theorem by Mader [9] can be applied to this node:

Theorem 5.5 (Mader [9]). *Let $G = (V + w, E)$ be a connected graph, where $d(w)$ is even and there is no cut-edge incident to w. Then there is a complete splitting at w that does not decrease local edge-connectivity.* □

By Mader's theorem, there is a complete splitting at w that preserves the conditions of the Theorem. By induction, the resulting graph has a good orientation, which can be transformed into a good orientation of the original graph by the inverse operation of edge splitting: if the orientation of a split off edge is an arc uv, then replace it by arcs uw and wv. □

Finally, let us mention a corollary regarding $(2k + 1)$-connected graphs. Nash-Williams [10] proved that $2k$-edge-connected graphs are exactly those that have a k-edge-connected orientation. There is no similarly elegant characterization of $(2k+1)$-edge-connected graphs, but Theorem 5.4 implies the following:

Corollary 5.6. *A graph is $(2k+1)$-edge-connected if and only if for every pair $s, t \in V$ it has a k-edge-connected orientation with $k + 1$ edge-disjoint paths from s to t.* □

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