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Independence free graphs and vertex connectivity augmentation

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Abstract

Given an undirected graph G and a positive integer k, the k-vertex-connectivity augmentation problem is to find a smallest set F of new edges for which $G+F$ is k-vertex-connected. Polynomial algorithms for this problem have been found only for $k \leq 4$ and a major open question in graph connectivity is whether this problem is solvable in polynomial time in general.

In this paper we develop an algorithm which delivers an optimal solution in polynomial time for every fixed k. In the case when the size of an optimal solution is large compared to k, we also give a min-max formula for the size of a smallest augmenting set. A key step in our proofs is a complete solution of the augmentation problem for a new family of graphs which we call k-independence free graphs. We also prove new splitting off theorems for vertex connectivity.

1 Introduction

An undirected graph $G = (V, E)$ is *k-vertex-connected* if $|V| \geq k+1$ and the deletion of any $k-1$ or fewer vertices leaves a connected graph. Given a graph $G = (V, E)$ and a positive integer k, the k-vertex-connectivity augmentation problem is to find a smallest set F of new edges for which $G' = (V, E \cup F)$ is k-connected. This problem (and a number of versions with different connectivity requirements and/or edge weights) is an important and well-studied optimization problem in network design. The complexity of the vertex-connectivity augmentation problem is one of the most challenging open questions of this area. It is open even if the graph G to be augmented is $(k-1)$-vertex-connected. Polynomial algorithms have been developed only for $k = 2, 3, 4$ by Eswaran and Tarjan [4], Watanabe and Nakamura [21] and Hsu [10], respectively. Near optimal solutions can be found in polynomial time for every k, see [12], [11].

In this paper we give an algorithm which delivers an optimal solution in polynomial time for any fixed $k \geq 2$. Its running time is bounded by $O(n^5) + O(f(k)n^3)$, where

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n is the size of the input graph and $f(k)$ is an exponential function of k. We also obtain a min-max formula which determines the size of an optimal solution when it is large compared to k. In this case the running time of the algorithm is simply $O(n^5)$. A key step in our proofs is a complete solution of the augmentation problem for a new family of graphs which we call k-independence free graphs. We follow some of the ideas of the approach of [14], which used, among others, the splitting off method. We further develop this method for k-vertex-connectivity.

We remark that the other three basic augmentation problems (where one wants to make G k-edge-connected or wants to make a digraph k-edge- or k-vertex-connected) have been shown to be polynomially solvable. These results are due to Watanabe and Nakamura [20], Frank [5], and Frank and Jordán [7], respectively. For more results on connectivity augmentation and its algorithmic aspects, see the survey papers by Frank [6] and Nagamochi [19], respectively. In the rest of the introduction we introduce some definitions and our new lower bounds for the size of an augmenting set which makes G k-vertex-connected. We also state our main min-max results.

In what follows we deal with simple undirected graphs and k-connected refers to k-vertex-connected. For two disjoint sets of vertices X, Y in a graph $G = (V, E)$ we denote the number of edges from X to Y by $d_G(X, Y)$ (or simply $d(X, Y)$). We use $d(X) = d(X, V - X)$ to denote the *degree* of X. For a single vertex v we write $d(v)$. Let $G = (V, E)$ be a graph with $|V| \geq k + 1$. For $X \subseteq V$ let $N(X)$ denote the set of *neighbours* of X, that is, $N(X) = \{v \in V - X : uv \in E \text{ for some } u \in X\}$. Let $n(X)$ denote $|N(X)|$. We use X^* to denote $V - X - N(X)$. We call X a *fragment* if $X, X^* \neq \emptyset$. For two vertices x, y of G we shall use $\kappa(x, y, H)$ to denote the maximum number of openly disjoint paths from x to y in G. We use $\kappa(G)$ to denote the minimum of $\kappa(x, y, G)$ over all pairs of vertices of G. By Menger's theorem $\kappa(G)$ equals the size of a minimum vertex cut in G, unless G is complete. Equivalently, $\kappa(G)$ is the largest integer k for which G is k-connected.

Let $a_k(G)$ denote the size of a smallest augmenting set of G with respect to k. It is easy to see that every set of new edges F which makes G k-connected must contain at least $k - n(X)$ edges from X to X^* for every fragment X. By summing up these 'deficiencies' over pairwise disjoint fragments, we obtain a useful lower bound on $a_k(G)$, similar to the one used in the corresponding edge-connectivity augmentation problem. Let $t(G) = \max\{\sum_{i=1}^{r} k - n(X_i) : X_1, ..., X_r \text{ are pairwise disjoint fragments in } V\}$. Then

$$a_k(G) \geq \lceil t(G)/2 \rceil. \tag{1}$$

Another lower bound for $a_k(G)$ comes from 'shredders'. For $K \subset V$ let $b_G(K)$, or simply $b(K)$ when it is clear to which graph we are referring to, denote the number of components in $G - K$. Let $b(G) = \max\{b_G(K) : K \subset V, |K| = k - 1\}$. We call a set $K \subset V$ with $|K| = k - 1$ and $b_G(K) = q$ a *q-shredder*. Since $G - K$ has to be connected in the augmented graph, we have the second lower bound:

$$a_k(G) \geq b(G) - 1. \tag{2}$$

These lower bounds extend the two natural lower bounds used for example in [4, 10, 14]. Although these bounds suffice to characterize $a_k(G)$ for $k \leq 3$, there are examples showing that $a_k(G)$ can be strictly larger than the maximum of these lower bounds, consider for example the complete bipartite graph $K_{3,3}$ with target $k = 4$. We shall show in Section 3 that $a_k(G) = \max\{b(G) - 1, \lceil t(G)/2 \rceil\}$ when G is a 'k-independence free graph'. We use this result in Subsection 4.3 to show that if G is $(k-1)$-connected and $a_k(G)$ is large compared to k, then again we have $a_k(G) = \max\{b(G) - 1, \lceil t(G)/2 \rceil\}$. The same result is not valid if we remove the hypothesis that G is $(k-1)$-connected. To see this consider the graph G obtained from $K_{m,k-2}$ by adding a new vertex x and joining x to j vertices in the m set of the $K_{m,k-2}$, where $j < k < m$. Then $b(G) = m$, $t(G) = 2m + k - 2j$ and $a_k(G) = m - 1 + k - j$. We shall see in Subsection 4.4, however, that if we modify the definition of $b(G)$ slightly, then we may obtain an analogous min-max theorem for augmenting graphs of arbitrary connectivity. For a set $K \subset V$ with $|K| = k - 1$ we define $\delta(K) = \max\{0, \max\{k - d(x) : x \in K\}\}$ and $b^*(K) = b(K) + \delta(K)$. We let $b^*(G) = \max\{b^*(K) : K \subset V, |K| = k - 1\}$. It is easy to see that $a_k(G) \geq b^*(G) - 1$. We shall prove in Subsection 4.4 that if G is a graph of arbitrary connectivity and $a_k(G)$ is large compared to k, then $a_k(G) = \max\{b^*(G) - 1, \lceil t(G)/2 \rceil\}$.

2 Preliminaries

In this section we first introduce some submodular inequalities for the function n and then describe the 'splitting off' method. We also prove some preliminary results on edge splittings and shredders.

2.1 Submodular inequalities

Each of the following three inequalitites can be verified easily by counting the contribution of every vertex to the two sides. Inequality (3) is well-known, see for example [14]. Inequality (4) is similar.

Proposition 2.1. *In a graph $H = (V, E)$ every pair $X, Y \subseteq V$ satisfies*

$$\begin{aligned} n(X) + n(Y) &\geq n(X \cap Y) + n(X \cup Y) + |\,(N(X) \cap N(Y)) - N(X \cap Y)| \\ &+ |\,(N(X) \cap Y)) - N(X \cap Y)| + |\,(N(Y) \cap X)) - N(X \cap Y)|, \quad (3) \\ n(X) + n(Y) &\geq n(X \cap Y^*) + n(Y \cap X^*). \quad (4) \end{aligned}$$

The following new inequality is crucial in the proof of one of our main lemmas. It may be applicable in other vertex-connectivity problems as well.

Proposition 2.2. *In a graph $H = (V, E)$ every triple $X, Y, Z \subseteq V$ satisfies*

$$\begin{aligned} n(X) + n(Y) + n(Z) &\geq n(X \cap Y \cap Z) + n(X \cap Y^* \cap Z^*) + n(X^* \cap Y^* \cap Z) + \\ &\quad n(X^* \cap Y \cap Z^*) - |N(X) \cap N(Y) \cap N(Z)|. \quad (5) \end{aligned}$$

Proof. Readers may find it helpful to follow the proof given below if they imagine $V(G)$ represented by a $3 \times 3 \times 3$ cube, in which the three pairs of opposite faces represent (X, X^*), (Y, Y^*), and (Z, Z^*), respectively, and the 27 subcubes represent the corresponding partition of $V(G)$ into 27 subsets. We have

$$\begin{aligned} n(X) &= |N(X) \cap Y \cap Z| + |N(X) \cap N(Y) \cap Z| + |N(X) \cap Y^* \cap Z| + \\ &\quad + |N(X) \cap Y \cap N(Z)| + |N(X) \cap N(Y) \cap N(Z)| + |N(X) \cap Y^* \cap N(Z)| + \\ &\quad + |N(X) \cap Y \cap Z^*| + |N(X) \cap N(Y) \cap Z^*| + |N(X) \cap Y^* \cap Z^*|, \end{aligned}$$

and

$$\begin{aligned} n(X \cap Y \cap Z) &\leq |X \cap Y \cap N(Z)| + |X \cap N(Y) \cap Z| + |X \cap N(Y) \cap N(Z)| + \\ &\quad + |N(X) \cap Y \cap Z| + |N(X) \cap Y \cap N(Z)| + |N(X) \cap N(Y) \cap Z| + \\ &\quad + |N(X) \cap N(Y) \cap N(Z)|. \end{aligned}$$

The lemma follows from the above (in)-equalities and similar (in)-equalities for $n(Y)$, $n(Z)$, $n(X \cap Y^* \cap Z^*)$, $n(X^* \cap Y^* \cap Z)$ and $n(X^* \cap Y \cap Z^*)$. □

2.2 Extensions and Splittings

In the so-called 'splitting off method' one extends the input graph G by a new vertex s and a set of appropriately chosen edges incident to s and then obtains an optimal augmenting set by splitting off pairs of edges incident to s. This approach was initiated by Cai and Sun [1] for the k-edge-connectivity augmentation problem and further developed and generalized by Frank [5]. Here we adapt the method to vertex-connectivity and prove several basic properties of the extended graph as well as the splittable pairs.

Given the input graph $G = (V, E)$, an *extension* $G + s = (V + s, E + F)$ of G is obtained by adding a new vertex s and a set F of new edges from s to V. In $G + s$ we define $\bar{d}(X) = n_G(X) + d(s, X)$ for every $X \subseteq V$. We say that $G + s$ is (k, s)*-connected* if

$$\bar{d}(X) \geq k \text{ for every fragment } X \subset V, \tag{6}$$

and that it is a *k-critical extension* if F is an inclusionwise minimal set with respect to (6). The minimality of F implies that every edge su in a k-critical extension is *k-critical*, that is, deleting su from $G + s$ destroys (6). (Thus an edge su is k-critical if and only if there exists a fragment X in V with $u \in X$ and $\bar{d}(X) = k$.) A fragment X with $d(s, X) \geq 1$ and $\bar{d}(X) = k$ is called *tight*. A fragment X with $d(s, X) \geq 2$ and $\bar{d}(X) \leq k + 1$ is called *dangerous*. Observe that if G is l-connected then for every $v \in V$ we have $d(s, v) \leq k - l$ in any k-critical extension of G.

Since the function $d(s, X)$ is modular on the subsets of V in $G + s$, Propositions 2.1 and 2.2 yield the following inequalities.

Proposition 2.3. *In a graph $G + s$ every pair $X, Y \subseteq V$ satisfies*

$$\begin{aligned}\bar{d}(X)+\bar{d}(Y) &\geq \bar{d}(X\cap Y)+\bar{d}(X\cup Y)+|\,(N(X)\cap N(Y))-N(X\cap Y)| \\ &+ |\,(N(X)\cap Y)-N(X\cap Y)|+|\,(N(Y)\cap X)-N(X\cap Y)|, \quad (7)\end{aligned}$$

$$\bar{d}(X)+\bar{d}(Y) \geq \bar{d}(X\cap Y^*)+\bar{d}(Y\cap X^*)+d(s,X-Y^*)+d(s,Y-X^*). \quad (8)$$

Proposition 2.4. *In a graph $G+s$ every triple $X,Y,Z\subseteq V$ satisfies*

$$\begin{aligned}\bar{d}(X)+\bar{d}(Y)+\bar{d}(Z) &\geq \bar{d}(X\cap Y\cap Z)+\bar{d}(X\cap Y^*\cap Z^*)+\bar{d}(X^*\cap Y^*\cap Z)+ \\ &+\bar{d}(X^*\cap Y\cap Z^*)-|N(X)\cap N(Y)\cap N(Z)|+ \\ &+2d(s,X\cap Y\cap Z). \quad (9)\end{aligned}$$

Lemma 2.5. *Let $G+s$ be a (k,s)-connected extension of G. Then there exists an augmenting set F of G with respect to k with $V(F)\subseteq N(s)$.*

Proof. This follows from the fact that $N(s)$ covers all k-deficient fragments of G. □

We can use Lemma 2.5 to obtain good bounds on $a_k(G)$. The following result is an easy consequence of a theorem of Mader [17]. It was used in [14] in the special case when G is $(k-1)$-connected.

Theorem 2.6. *[14][17] Let F be a minimal augmenting set of $G=(V,E)$ with respect to k and let B be the set of those vertices of G which have degree at least $k+1$ in $G+F$. Then F induces a forest on B.*

Lemma 2.7. *Let $G+s$ be a (k,s)-connected extension of G and let A be a minimal augmenting set of G for which every edge in A connects two vertices of $N(s)$ in $G+s$. Then $|A|\leq d(s)-1$.*

Proof. Let $B=\{v\in N(s): d_{G+A}(v)\geq k+1\}$ and let $C=N(s)-B$. We have $d_A(x)\leq d(s,x)$ for each $x\in C$ and, by Theorem 2.6, B induces a forest in A. Let $e_A(B)$ and $e_A(C)$ denote the number of those edges of A which connect two vertices of B and of C, respectively. The previous observations imply the following inequality.

$$\begin{aligned}|A| &= e_A(C)+d_A(B,C)+e_A(B)\leq \sum_{x\in C} d_A(x)+|B|-1\leq \\ &\leq (d(s)-|B|)+|B|-1=d(s)-1.\end{aligned}$$

This proves the lemma. □

Lemma 2.8. *Let $G+s$ be a k-critical extension of a graph G. Then $\lceil d(s)/2\rceil\leq a_k(G)\leq d(s)-1$.*

Proof. The last inequality follows immediately from Lemma 2.7. To verify the first inequality we introduce a new lower bound on $a_k(G)$. Let us say that a fragment X *separates* a pair of vertices $u, v \in V$ if $\{u, v\} \cap X \neq \emptyset \neq \{u, v\} \cap X^*$. A family F of fragments of G is *half-disjoint* if every pair of vertices of G is separated by at most two fragments in F. Let $t'(G) = \max\{\sum_{X \in F} k - n(X_i)\}$ where the maximum is taken over all half-disjoint families F of fragments in G. Note that every family of pairwise disjoint fragments is half-disjoint and hence $t'(G) \geq t(G)$. Since every augmenting set with respect to k must contain at least $k - n(X)$ edges from X to X^* for any fragment X of G, we obtain the following lower bound:

$$a_k(G) \geq \lceil t'(G)/2 \rceil. \tag{10}$$

We shall prove that $d(s) \leq t'(G)$. This will imply the lemma by (10).

Let $\mathcal{X} = \{X_1, ..., X_m\}$ be a family of tight sets such that $N(s) \subseteq \cup_{i=1}^m X_i$ and such that m is minimum and $\sum_{i=1}^m |X_i|$ is minimum. Such a family exists since the edges incident to s in $G + s$ are k-critical. We claim that for every $1 \leq i < j \leq m$ either $X_i \cap X_j = \emptyset$ or at least one of $X_i^* \subseteq N(X_j)$ or $X_j^* \subseteq N(X_i)$ holds. Note that in the latter case no pair of vertices can simultaneously be separated by X_i and X_j.

To verify the claim, suppose that $X_i \cap X_j \neq \emptyset$. Then by the minimality of m the set $X_i \cup X_j$ cannot be tight. Thus (7) implies that $X_i^* \cap X_j^* = \emptyset$. Hence either one of $X_i^* \subseteq N(X_j)$ or $X_j^* \subseteq N(X_i)$ holds or $X_i \cap X_j^*$ and $X_j \cap X_i^*$ are both non-empty. In the former case we are done. In the latter case we apply (8) to X_i and X_j and conclude that $X_i \cap X_j^*$ and $X_j \cap X_i^*$ are both tight and all the edges from s to $X_i \cup X_j$ enter $(X_i \cap X_j^*) \cup (X_j \cap X_i^*)$. Thus we could replace X_i and X_j in $\mathcal{X}$ by two strictly smaller sets $X_i \cap X_j^*$ and $X_j \cap X_i^*$, contradicting the choice of $\mathcal{X}$. This proves the claim.

To finish the proof of the lemma observe that $\sum_{i=1}^m k - n(X_i) = \sum_{i=1}^m d(s, X_i) \geq d(s)$. In other words, the sum of 'deficiencies' of $\mathcal{X}$ is at least $d(s)$. Furthermore, our claim implies that $\mathcal{X}$ is a half-disjoint family of fragments. (Otherwise we must have, without loss of generality, $X_1, X_2, X_3 \in \mathcal{X}$ such that some pair $u, v \in V$ is separated by X_i for all $1 \leq i \leq 3$. This implies, by the claim above, that X_1, X_2, X_3 are pairwise disjoint, contradicting the fact that they each separate u, v.) Hence $d(s) \leq t'(G)$, as required. □

Let $G + s$ be a (k, s)-connected extension of G. *Splitting off* two edges su, sv in $G + s$ means deleting su, sv and adding a new edge uv. Such a split is *admissible* if the graph obtained by the splitting also satisfies (6). Notice that if $G + s$ has no edges incident to s then (6) is equivalent to the k-connectivity of G. Hence it would be desirable to know, when $G + s$ is a k-critical extension and $d(s)$ is even, that there is a sequence of admissible splittings which isolates s. In this case, using the fact that $a_k(G) \geq d(s)/2$ by Lemma 2.8, the resulting graph on V would be an *optimal* augmentation of G with respect to k. This approach works for the k-edge-connectivity augmentation problem [5] but does not always work in the vertex connectivity case. The reason is that such 'complete splittings' do not necessarily exist. On the other hand, we shall prove results which are 'close enough' to yield an optimal algorithm for k-connectivity augmentation, using the splitting off method, which is polynomial for k fixed.

Non-admissible pairs sx, sy can be characterized by tight and dangerous 'certificates' as follows. The proof of the following simple lemma is omitted.

Lemma 2.9. *Let $G+s$ be a (k, s)-connected extension of G. Then the pair sx, sy is not admissible for splitting in $G+s$ with respect to k if and only if one of the following holds:*
(i) there exists a tight set T with $x \in T$, $y \in N(T)$,
(ii) there exists a tight set U with $y \in U$, $x \in N(U)$,
(iii) there exists a dangerous set W with $x, y \in W$.

2.3 Local separators and shredders

For two vertices $u, v \in V$ a *uv-cut* is a set $K \subseteq V - \{u, v\}$ for which there is no uv-path in $G - K$. A set $S \subset V$ is a *local separator* if there exist $u, v \in V - S$ such that S is an inclusionwise minimal uv-cut. We also say S is a *local uv-separator* and we call the components of $G - S$ containing u and v *essential components* of S (with respect to the pair u, v). Note that S may be a local separator with respect to several pairs of vertices and hence it may have more than two essential components. Clearly, $N(C) = S$ for every essential component C of S. If S is a local uv-separator and T is a local xy-separator then we say T *meshes* S if T intersects the two essential components of S containing u and v, respectively.

Lemma 2.10. *If T meshes S then S intersects every essential component of T (and hence S meshes T).*

Proof. Suppose S is a uv-separator and let C_u, C_v be the two essential components of S containing u and v respectively. Let C be an essential component of T. We need to show S meets C. Choose $w \in V(C)$. Without loss of generality $w \notin V(C_v)$. Choose $t \in T \cap C_v$. Let P be a path in the subgraph of $C \cup T$ from w to t. Then P contains a vertex of S since S separates w from t. Hence $C \cap S \neq \emptyset$. $\square$

Lemma 2.10 extends [3, Lemma 4.3(1)]. The next lemma extends one of the key observations of Cheriyan and Thurimella [3]. The proof is similar to that of [3, Proposition 3.1]. We shall use Lemma 2.11 in Section 5.

Lemma 2.11. *Let K be a local uv-separator of size $k-1$ and suppose that there exist $k-1$ vertex-disjoint paths $P_1, ..., P_{k-1}$ from u to v in G. Let $Q = \cup_{i=1}^{k-1} V(P_i)$. Then:*
(a) for each component C of $G - K$ either $C \cap \{u, v\} \neq \emptyset$ or C is a component of $G - Q$;
(b) if K has at least three essential components then $K = N(C)$ for some component C of $G - Q$.

Proof. Since K is a local uv-separator of size $k-1$, K contains exactly one vertex from each path $P_1, \ldots, P_{k-1}$. Let C_u, C_v, C be distinct components of K with $u \in C_u$ and $v \in C_v$. Then $Q - K \subseteq C_u \cup C_v$. Thus $C \cap Q = \emptyset$. Hence C is a component of $G - Q$. If C is an essential component of K then $K = N(C)$ holds. $\square$

Let K be a shredder of G and $G + s$ be a (k, s)-connected extension of G. A component C of $G - K$ is called a *leaf component* of K (in $G+s$) if $d(s, C) = 1$ holds. Note that $d(s, C') \geq 1$ for each component C' of $G - K$ by (6). The next lemma is easy to verify by (6).

Lemma 2.12. *Let K be a shredder in G and let C_1, C_2 be leaf components of K in $G + s$. Then there exist $k - 1$ vertex-disjoint paths in the subgraph of G induced by $C_1 \cup C_2 \cup K$ from every vertex of C_1 to every vertex of C_2.*

If $d(s) \leq 2b(G) - 2$ then every shredder K satisfying $b(K) = b(G)$ has at least two leaf components. Hence K is a local separator and every leaf component of K is an essential component of K in G.

3 Independence Free Graphs

Let $G = (V, E)$ be a graph and k be an integer. Let X_1, X_2 be disjoint subsets of V. We say (X_1, X_2) is a *k-deficient pair* if $d(X_1, X_2) = 0$ and $|V - (X_1 \cup X_2)| \leq k - 1$. We say two deficient pairs (X_1, X_2) and (Y_1, Y_2) are *independent* if for some $i \in \{1, 2\}$ we have either $X_i \subseteq V - (Y_1 \cup Y_2)$ or $Y_i \subseteq V - (X_1 \cup X_2)$, since in this case no edge can simultaneously connect X_1 to X_2 and Y_1 to Y_2. We say G is *k-independence free* if G does not have two independent k-deficient pairs. (Note that if G is $(k - 1)$-connected and (X_1, X_2) is a k-deficient pair then $X_2 = X_1^*$ and $V - (X_1 \cup X_2) = N(X_1) = N(X_2)$.) Thus:

- $(k-1)$-connected chordal graphs and graphs with minimum degree at least $2k-2$ are k-independence free;
- all graphs are 1-independence free and all connected graphs are 2-independence free;
- a graph with no edges and at least $k + 1$ vertices is not k-independence free for any $k \geq 2$;
- if G is k-independence free and H is obtained by adding edges to G then H is also k-independence free;
- a k-independence free graph is l-independence free for all $l \leq k$.

In general, a main difficulty in vertex-connectivity problems is that vertex cuts (and hence tight and dangerous sets) can cross each other in many different ways. In the case of an independence free graph G we can overcome these difficulties and prove the following results, including a complete characterisation of the case when there is no admissible split containing a specified edge in an extension of G.

Lemma 3.1. *Let $G+s$ be a (k, s)-connected extension of a k-independence free graph G and X, Y be fragments of G.*
(a) If X and Y are tight then either: $X \cup Y$ is tight, $X \cap Y \neq \emptyset$ and $\bar{d}(X \cap Y) = k$;

or $X \cap Y^*$ and $Y \cap X^*$ are both tight and $d(s, X - Y^*) = 0 = d(s, Y - X^*)$.
(b) If X is a minimal tight set and Y is tight then either: $X \cup Y$ is tight, $d(s, X \cap Y) = 0$ and $n_G(X \cap Y) = k$; or $X \subseteq Y$; or $X \subseteq Y^*$.
(c) If X is a tight set and Y is a maximal dangerous set then either $X \subseteq Y$ or $d(s, X \cap Y) = 0$.
(d) If X is a tight set, Y is a dangerous set and $d(s, Y - X^*) + d(s, X - Y^*) \geq 2$ then $X \cap Y \neq \emptyset$ and $\bar{d}(X \cap Y) \leq k + 1$.

Proof. (a) Suppose $X \cap Y^*, Y \cap X^* \neq \emptyset$. Then (8) implies that $\bar{d}(X \cap Y^*) = k = \bar{d}(Y \cap X^*)$ and $d(s, X - Y^*) = 0 = d(s, Y - X^*)$. Thus $X \cap Y^*$ and $Y \cap X^*$ are both tight. Hence we may assume that either $X \cap Y^*$ or $Y \cap X^*$ is empty. Since G is k-independence free, it follows that $X^* \cap Y^* \neq \emptyset \neq X \cap Y$ (for example if $X \cap Y^* = \emptyset = X^* \cap Y^*$ then $Y^* \subseteq V - (X \cup X^*)$, and (X, X^*) and (Y, Y^*) are independent k-deficient pairs). Thus $X \cup Y$ is a fragment in G. Using (7) we deduce that $X \cup Y$ is tight and $\bar{d}(X \cap Y) = k$.

(b) This follows from (a) using the minimality of X.

(c) Suppose $X \not\subseteq Y$ and $d(s, X \cap Y) \geq 1$. If $X \cap Y^* \neq \emptyset \neq Y \cap X^*$ then we can use (8) to obtain the contradiction

$$2k + 1 \geq \bar{d}(X) + \bar{d}(Y) \geq \bar{d}(X \cap Y^*) + \bar{d}(Y \cap X^*) + 2 \geq 2k + 2.$$

Thus either $X \cap Y^*$ or $Y \cap X^*$ is empty and, since G is k-independence free, $X^* \cap Y^* \neq \emptyset$. Thus $X \cup Y$ is a fragment in G. Using (7) we deduce that $X \cup Y$ is dangerous contradicting the maximality of Y.

(d) Using (8), we deduce that either $X \cap Y^*$ or $Y \cap X^*$ is empty and, since G is k-independence free, $X \cap Y \neq \emptyset \neq X^* \cap Y^*$. We can now use (7) to deduce that $\bar{d}(X \cap Y) \leq k + 1$. □

Using Lemma 3.1 we deduce

Corollary 3.2. *If $G + s$ is a k-critical extension of a k-independence free graph G then $d(s) = t(G)$. Furthermore there exists a unique minimal tight set in $G + s$ containing x for each $x \in N(s)$.*

Proof. Let $\mathcal{F}$ be a family of tight sets which cover $N(s)$ such that $\sum_{X \in \mathcal{F}} |X|$ is as small as possible. Since every edge incident to s is k-critical, such a family exists. We show that the members of $\mathcal{F}$ are pairwise disjoint. Choose $X, Y \in \mathcal{F}$ and suppose that $X \cap Y \neq \emptyset$. By Lemma 3.1(a) we may replace X and Y in $\mathcal{F}$ either by $X \cup Y$, or by $X \cap Y^*$ and $Y \cap X^*$. Both alternatives contradict the minimality of $\sum_{X \in \mathcal{F}} |X|$. Since the members of $\mathcal{F}$ are pairwise disjoint, tight, and cover $N(s)$, we have $d(s) = \sum_{X \in \mathcal{F}} (k - n_G(X)) \leq t(G)$. The inequality $d(s) \geq t(G)$ follows easily from (6). Thus $d(s) = t(G)$, as required.

The second assertion of the corollary follows immediately from criticality and Lemma 3.1(b). □

Lemma 3.3. *Let $G + s$ be a k-critical extension of a k-independence free graph G and $x_1, x_2 \in N(s)$. Then the pair sx_1, sx_2 is not admissible for splitting in $G+s$ with respect to k if and only if there exists a dangerous set W in $G + s$ with $x_1, x_2 \in W$.*

Proof. Suppose the lemma is false. Using Lemma 2.9 we may assume without loss of generality that there exists a tight set X_1 in $G+s$ such that $x_1 \in X_1$ and $x_2 \in N_G(X_1)$. Let X_2 be the minimal tight set in $G+s$ containing x_2. Since $x_2 \in N(s) \cap (X_2 - X_1^*)$, it follows from Lemma 3.1(a) that $X_1 \cup X_2$ is a tight, and hence dangerous, set in $G + s$ containing x_1, x_2. □

Theorem 3.4. *Let $G + s$ be a k-critical extension of a k-independence free graph G and $x_0 \in N(s)$.*
(a) There is no admissible split in $G + s$ containing sx_0 if and only if either: $d(s) = b(G)$; or $d(s)$ is odd and there exist maximal dangerous sets W_1, W_2 in $G+s$ such that $N(s) \subseteq W_1 \cup W_2$, $x_0 \in W_1 \cap W_2$, $d(s, W_1 \cap W_2) = 1$, $d(s, W_1 \cap W_2^*) = (d(s) - 1)/2 = d(s, W_1^* \cap W_2)$, and $W_1 \cap W_2^*$ and $W_2 \cap W_1^*$ are tight.
(b) Furthermore if there is no admissible split containing sx_0 and $3 \neq d(s) \neq b(G)$ then there is an admissible split containing sx_1 for all $x_1 \in N(s) - x_0$.

Proof. Note that since $G + s$ is a k-critical extension, $d(s) \geq 2$.
(a) Using Lemma 3.3, we may choose a family of dangerous sets $\mathcal{W} = \{W_1, W_2, \ldots, W_r\}$ in $G + s$ such that $x_0 \in \cap_{i=1}^r W_i$, $N(s) \subseteq \cup_{i=1}^r W_i$ and r is as small as possible. We may assume that each set in $\mathcal{W}$ is a maximal dangerous set in $G + s$. If $r = 1$ then $N(s) \subseteq W_1$ and

$$\bar{d}(W_1^*) = n_G(W_1^*) \leq n_G(W_1) \leq k + 1 - d(s, W_1) \leq k - 1,$$

since W_1 is dangerous. This contradicts the fact that $G+s$ is (k, s)-connected. Hence $r \geq 2$.

Claim 3.5. *Let $W_i, W_j \in \mathcal{W}$. Then $W_i \cap W_j^* \neq \emptyset \neq W_j \cap W_i^*$ and $d(s, W_i - W_j^*) = 1 = d(s, W_j - W_i^*)$.*

Proof. Suppose $W_i \cap W_j^* = \emptyset$. Since G is k-independence free, it follows that $W_i^* \cap W_j^* \neq \emptyset$ and hence $W_i \cup W_j$ is a fragment of G. The minimality of r now implies that $W_i \cup W_j$ is not dangerous, and hence $\bar{d}(W_i \cup W_j) \geq k + 2$. Applying (7) we obtain

$$2k + 2 \geq \bar{d}(W_i) + \bar{d}(W_j) \geq \bar{d}(W_i \cap W_j) + \bar{d}(W_i \cup W_j) \geq 2k + 2.$$

Hence equality holds throughout. Thus $\bar{d}(W_i \cap W_j) = k$ and, since $x_0 \in W_i \cap W_j$, $W_i \cap W_j$ is tight.

Choose $x_i \in N(s) \cap (W_i - W_j)$ and let X_i be the minimal tight set in $G+s$ containing x_i. Since $x_i \in N(s) \cap X_i \cap W_i$, it follows from Lemma 3.1(c) that $X_i \subseteq W_i$. Since G is k-independence free, $X_i \not\subseteq N(W_j)$. Thus $X_i \cap W_i \cap W_j \neq \emptyset$. Applying Lemma 3.1(b), we deduce that $X_i \cup (W_i \cap W_j)$ is tight. Now $X_i \cup (W_i \cap W_j)$ and W_j contradict Lemma 3.1(c) since $x_0 \in W_i \cap W_j$ and W_j is a maximal dangerous set. Hence we must have $W_i \cap W_j^* \neq \emptyset \neq W_j \cap W_i^*$. The second part of the claim follows from (8) and the fact that $x_0 \in W_i \cap W_j$. □

Suppose $r = 2$. Using Claim 3.5, we have $d(s) = 1 + d(s, W_1 \cap W_2^*) + d(s, W_2 \cap W_1^*)$. Without loss of generality we may suppose that $d(s, W_1 \cap W_2^*) \leq d(s, W_2 \cap W_1^*)$. Then

$$\bar{d}(W_2^*) = d(s, W_1 \cap W_2^*) + n_G(W_2^*) \leq d(s, W_2 \cap W_1^*) + n_G(W_2) = \bar{d}(W_2) - 1 \leq k.$$

Thus equality must hold throughout. Hence $d(s, W_1 \cap W_2^*) = d(s, W_2 \cap W_1^*) = (d(s) - 1)/2$, $d(s)$ is odd, $W_1 \cap W_2^*$ and $W_2 \cap W_1^*$ are tight and the second alternative in (a) holds.

Finally we suppose that $r \geq 3$. Choose $W_i, W_j, W_h \in \mathcal{W}$, $x_i \in (N(s) \cap W_i) - (W_j \cup W_h)$, $x_j \in (N(s) \cap W_j) - (W_i \cup W_h)$, and $x_h \in (N(s) \cap W_h) - (W_i \cup W_j)$. Then Claim 3.5 implies that $x_i \in W_i \cap W_j^* \cap W_h^*$. Applying (9), and using $d(s, W_i \cap W_j \cap W_h) \geq 1$, we get

$$\begin{aligned} 3k+3 \geq\ & \bar{d}(W_i) + \bar{d}(W_j) + \bar{d}(W_h) \geq \bar{d}(W_i \cap W_j \cap W_h) + \bar{d}(W_i \cap W_j^* \cap W_h^*) + \\ & + \bar{d}(W_j \cap W_i^* \cap W_h^*) + \bar{d}(W_h \cap W_i^* \cap W_j^*) - |N(W_i) \cap N(W_j) \cap N(W_h)| + \\ & + 2d(s, W_i \cap W_j \cap W_h) \geq 4k - |N(W_i) \cap N(W_j) \cap N(W_h)| + 2 \geq 3k + 3 \qquad (11) \end{aligned}$$

Thus equality must hold throughout. Hence $d(s, W_i \cap W_j \cap W_h) = 1$, and $W_i \cap W_j^* \cap W_h^*$ is tight. Furthermore, putting $S = N(W_i) \cap N(W_j) \cap N(W_h)$, we have $|S| = k - 1$ by (11). Since $n(W_i) \leq k - 1$ we must have $N(W_i) = S$ and hence $N(W_i) \cap W_j = \emptyset$. Thus $N(W_i \cap W_j \cap W_k) \subseteq S$. Since $d(s, W_i \cap W_j \cap W_h) = 1$, we must have $N(W_i \cap W_j \cap W_k) = S$. Since W_i is dangerous and $N(W_i) = S$, $d(s, W_i) = 2$ follows. Thus $d(s, W_i \cap W_j^* \cap W_h^*) = 1$ and $G - S$ has $r + 1 = d(s)$ components $C_0, C_1, ..., C_r$ where $C_0 = W_i \cap W_j \cap W_h$ and $C_i = W_i - C_0$ for $1 \leq i \leq r$. Thus S is a shredder in G with $b_G(S) = d(s)$. Since the (k, s)-connectivity of $G + s$ implies that $b(G) \leq d(s)$, we have $b(G) = d(s)$.

(b) Using (a) we have $d(s)$ is odd and there exist maximal dangerous sets W_1, W_2 in $G + s$ such that $N(s) \subseteq W_1 \cup W_2$, $x_0 \in W_1 \cap W_2$, $d(s, W_1 \cap W_2) = 1$, $d(s, W_1 \cap W_2^*) = d(s, W_1^* \cap W_2) = (d(s) - 1)/2 \geq 2$, and $W_1 \cap W_2^*$ and $W_1^* \cap W_2$ are tight. Suppose $x_1 \in N(s) \cap W_1 \cap W_2^*$ and there is no admissible split containing sx_1. Then applying (a) to x_1 we find maximal dangerous sets W_3, W_4 with $x_1 \in W_3 \cap W_4$ and $d(s, W_3 \cap W_4) = 1$. Using Lemma 3.1(c) we have $W_1 \cap W_2^* \subseteq W_3$ and $W_1 \cap W_2^* \subseteq W_4$. Thus $W_1 \cap W_2^* \subseteq W_3 \cap W_4$ and $d(s, W_3 \cap W_4) \geq 2$. This contradicts the fact that $d(s, W_3 \cap W_4) = 1$. □

We can use this result to determine $a_k(G)$ when G is k-independence free. We first solve the case when $b(G)$ is large compared to $d(s)$.

Lemma 3.6. *Let $G + s$ be a k-critical extension of a k-independence free graph G and K be a shredder in G. If $d(s) \leq 2b(K) - 2$ then $d(s, K) = 0$.*

Proof. Let $b(K) = b$. Suppose $x \in N(s) \cap K$ and let X be a minimal tight set in $G + s$ containing x. Let $\mathcal{L} = \{X_1, X_2, \ldots, X_r\}$ be the leaf components of K. Since $d(s) \leq 2b - 2$ we have $r \geq 2$. Choose $X_i \in \mathcal{L}$ and $x_i \in N(s) \cap X_i$. Then X_i is tight. Since $x \in K = N_G(X_i)$ we have $X \not\subseteq X_i^*$. Using Lemma 3.1(b), we deduce that $X \cup X_i$

is tight, $n_G(X \cap X_i) = k$ and $d(s, X \cap X_i) = 0$. Hence $x_i \notin X$ and $N(X) \cap X_i \neq \emptyset$. Since this holds for all $X_i \in \mathcal{L}$ and $x \in X \cap K$, we have

$$|N(X) \cap (X_1 \cup X_2 \dots X_r)| \geq r. \tag{12}$$

Furthermore, since $X \cap X_2 \neq \emptyset$ and $X \cap X_2 \subseteq X \cap X_1^*$ we have $X \cap X_1^* \neq \emptyset$. Using (8) and the fact that $d(s, X - X_1^*) \geq 1$ since $x \in X \cap N_G(X_1)$, it follows that $X^* \cap X_1 = \emptyset$. Using symmetry we deduce that $X^* \cap X_i = \emptyset$ for all $X_i \in \mathcal{L}$.

Since $X_1 \cup X_2$ is dangerous and $x_1, x_2 \notin X^*$, we can use Lemma 3.1(d) to deduce that $\bar{d}(X \cap (X_1 \cup X_2) \leq k+1$. Since $n_G(X \cap X_1) = k = n_G(X \cap X_2)$, we have $K = N_G(X \cap X_1) \cap N_G(X \cap X_2)$. Thus $x \in N_G(X \cap X_1)$, $K \subseteq X \cup N_G(X)$ and $X^* \cap K = \emptyset$. Since $X^* \cap X_i = \emptyset$ for all $X_i \in \mathcal{L}$, $X^* \cap Y \neq \emptyset$ for some non-leaf component Y of $G - K$. Using (12) and the facts that $N_G(X^* \cap Y) \subseteq (N_G(X) \cap Y) \cup (N_G(X) \cap K)$ and $n_G(X) \leq k-1$, we deduce that $n_G(X^* \cap Y) \leq k - 1 - r$. Since $G + s$ is (k, s)-connected we have $d(s, Y) \geq d(s, X^* \cap Y) \geq r + 1$. Thus

$$\begin{aligned} d(s) &= d(s,Y) + d(s, X_1 \cup X_2 \dots X_r) + d(s, (Y_1 \cup Y_2 \dots Y_{b-r}) - Y) + d(s, K) \\ &\geq (r+1) + r + 2(b - r - 1) + 1 \geq 2b. \end{aligned}$$

This contradicts the hypothesis that $d(s) \leq 2b - 2$. □

Lemma 3.7. *Let $G + s$ be a k-critical extension of a k-independence free graph G such that $b(G) + 1 \leq d(s) \leq 2b(G) - 2$. Then there exists an admissible split at s such that, for the resulting graph $G' + s$, we have $b(G') = b(G) - 1$.*

Proof. Let $b(G) = b$ and let K be a shredder in G with $b_G(K) = b$ and, subject to this condition, with the maximum number r of leaves in $G + s$. Let $C_1, C_2, \dots, C_r$ be the leaf components of K and let $N(s) \cap C_i = \{x_i\}$ for $1 \leq i \leq r$. Since $d(s) \leq 2b(G) - 2$ we have $r \geq 2$. Since $d(s) \geq b(G) + 1$ and $r \geq 2$, we may use Theorem 3.4 to deduce without loss of generality that there is an admissible split in $G + s$ containing sx_1. Choose sw such that sx_1, sw is an admissible split in $G + s$. Splitting sx_1, sw we obtain $G' + s$ where $d_{G'+s}(s) = d_{G+s}(s) - 2$. Adding the edge $x_1 w$ to G we obtain G'.

Suppose $b(G') = b(G)$. Then G has a shredder K' with $b(K', G) = b(G)$ such that x_1, w belong to the same component C' of $G - K'$. (Note that $\{x_1, w\} \cap K' = \emptyset$ by Lemma 3.6.) We shall prove that such a K' cannot exist in G.

Suppose $x_1, x_2, \dots, x_r \in V(C')$. Since w is also contained in C' we have $d(s, C') \geq r + 1$. Since $d(s) \leq 2b - 2$ it follows that K' has at least $r + 1$ leaf components, contradicting the maximality of r. Hence we may assume without loss of generality that

$$x_2 \notin C'. \tag{13}$$

Thus K' separates x_1 and x_2. Since, by Lemma 2.12, the subgraph of G induced by $C_1 \cup C_2 \cup K$ contains $k - 1$ openly disjoint x_1x_2-paths, we have

$$K' \subseteq C_1 \cup C_2 \cup K. \tag{14}$$

Claim 3.8. *K and K' are meshing local separators.*

Proof. Let C_2' be the component of $G - K'$ containing x_2. Since every x_1w-path in G contains a vertex of K we have $C' \cap K \neq \emptyset$. Also since G has $(k-1)$ x_1x_2-paths by Lemma 2.12, both C' and C_2' are essential K'-components. Hence we may assume $C_2' \cap K = \emptyset$. Hence $C_2' \subseteq C_2$ and $K' \cap C_2 \neq \emptyset$ (since $K \neq K'$). If K' does not mesh K we have $C_1 \cap K' = \emptyset$, so $C_1 \subseteq C'$. Since $N(C_1) = K$ we have $K - C' \subseteq K'$. Let C_1' be a leaf component of K' distinct from C_2'. Since C_1' is an essential K'-component, we have $N(C_1') = K'$. Since $K \cap C' \neq \emptyset$ and $K \subseteq K' \cup C'$ we have $C_1' \subseteq C_2 - C_2'$. But x_2 is the only s-neighbour in C_2 so $d(s, C_1') = 0$, a contradiction. □

Claim 3.9. *$r = 2$.*

Proof. Suppose $r \geq 3$. By Lemma 3.6, $x_1, x_2 \notin K'$. By Lemma 2.12, the subgraph of G induced by $C_1 \cup C_2 \cup K$ contains $k-1$ openly disjoint x_1x_2-paths. Since K and K' mesh by Claim 3.8, $K' \cap C_3 \neq \emptyset$, so $|K' \cap (C_1 \cup C_2 \cup K)| \leq k-2$. Hence at least one of the above $k-1$ openly disjoint x_1x_2-paths avoid K'. This contradicts (13). □

We can now complete the proof of the lemma. Let C_w be the component of $G - K$ containing w. Since sx_1, sw is an admissible split and C_1 is a leaf component of K, it follows that C_w is not a leaf component of K. Using (14), we deduce that C_w is a connected subgraph of $G - K'$ and hence $C_w \subseteq C'$. Since $d(s, C_w) \geq 2$ and $x_1 \in N(s) \cap (C' - C_w)$ we have $d(s, C') \geq 3$. Since $d(s) \leq 2b - 2$, it follows that K' has at least 3 leaf components. This contradicts the maximality of r by Claim 3.9. Thus K' does not exist and we have $b(G') = b(G) - 1$. □

Lemma 3.10. *Let $G + s$ be a k-critical extension of a k-independence free graph G and p be an integer such that $0 \leq p \leq \frac{1}{2}d(s) - 1$. Then there exists a sequence of p admissible splits at s if and only if $p \leq d(s) - b(G)$.*

Proof. We first suppose that there exists a sequence of p admissible splits at s in G. Let the resulting graph be $G_1 + s$. Then $d_{G_1+s}(s) = d_G(s) - 2p$ and $b(G_1) \geq b(G) - p$. Since $G_1 + s$ is (k, s)-connected we must have $d_{G_1+s}(s) \geq b(G_1)$ and hence $p \leq d(s) - b(G)$.

We next suppose that $p \leq d(s) - b(G)$. We shall show by induction on p that $G + s$ has a sequence of p admissible splits at s. If $p = 0$ then there is nothing to prove. Hence we may assume $p \geq 1$. Since $p \leq \frac{1}{2}d(s) - 1$ we have $d(s) \geq 4$. By Theorem 3.4 there is an admissible split at s. Let the resulting graph be $G_2 + s$. If $p - 1 \leq d_{G_2+s}(s) - b(G_2)$ then we are done by induction. Hence we may assume that $p \geq d_{G_2+s}(s) - b(G_2) + 2 \geq d_G(s) - b(G)$. Hence $p = d_G(s) - b(G)$. Since $p \leq \frac{1}{2}d_G(s) - 1$, we have $d_G(s) \leq 2b(G) - 2$. By Lemma 3.7 there exists an admissible split at s such that the resulting graph $G_3 + s$ satisfies $b(G_3) = b(G) - 1$. It now follows by induction that $G_3 + s$ has a sequence of $p - 1$ admissible splits at s. □

Lemma 3.11. *Let $G + s$ be a k-critical extension of a k-independence free graph G. If $d(s) \leq 2b(G) - 2$ then $a_k(G) = b(G) - 1$.*

Proof. Suppose $d(s) = b(G)$. Then all components of $G - K$ are leaf components. Let F be the edge set of a tree T on the vertices of $N(s)$. We shall show that $G + F$ is k-connected. If not, then we can partition V into three sets $\{X, Y, Z\}$ such that $|Z| = k - 1$ and no edge of $G + F$ joins X to Y. Since each pair of vertices of $N(s)$ are joined by k vertex-disjoint paths in $G+F$, $(k-1)$ paths in G by Lemma 2.12 and one path in T, either X or Y is disjoint from $N(s)$. Assuming $X \cap N(s) = \emptyset$, we have $\bar{d}(X) = n(X) \leq k - 1$, contradicting the fact that $G + s$ satisfies (6). Hence $G + F$ is a k-connected augmentation of G with $b(G) - 1$ edges.

Henceforth we may assume that $d(s) > b(G)$. By Lemma 3.7, there exists an admissible split at s such that, for the resulting graph $G' + s$, we have $b(G') = b(G) - 1$. Since $G' + s$ is a k-critical extension of G', the lemma follows by induction on $d_{G+s}(s) - b(G)$. □

Theorem 3.12. *If G is k-independence free then $a_k(G) = \max\{\lceil t(G)/2\rceil, b(G) - 1\}$.*

Proof. Let $G + s$ be a k-critical extension of G. By Corollary 3.2, $d(s) = t(G)$. If $d(s) \leq 3$ then $a_k(G) = \lceil t(G)/2\rceil$ by Lemma 2.8. Hence we may suppose that $d(s) \geq 4$. If $d(s) \leq 2b(G) - 2$ then $a_k(G) = b(G) - 1$ by Lemma 3.11. Hence we may suppose that $d(s) \geq 2b(G) - 1$.

By Lemma 3.10, there exists a sequence of $\lfloor d(s)/2\rfloor - 1$ admissible splits at s. Let the resulting graph be $G' + s$. Then $G' + s$ is a k-critical extension of G', $d_{G'+s}(s) \leq 3$, and $a_k(G') = \lceil d_{G'+s}(s)/2\rceil$ by Lemma 2.8. This gives the required augmenting set F for G with $|F| = \lceil d_{G+s}(s)/2\rceil = \lceil t(G)/2\rceil$. □

4 Graphs with Large Augmentation Number

Throughout this section we will be concerned with augmenting an l-connected graph G on at least $k + 1$ vertices for which $a_k(G)$ is large compared to k.

4.1 Unsplittable Extensions

In this subsection we consider a k-critical extension $G + s$ of G for which $d(s)$ is large, and characterise when there is no admissible split containing a given edge at s.

Lemma 4.1. *Let $X, Y \subset V$ be two sets with $X \cap Y \neq \emptyset$. Suppose $d(s) \geq (k - l)(k - 1) + 4$.*
(a) If X and Y are tight then $X \cup Y$ is tight.
(b) If X is tight and Y is dangerous then $X \cup Y$ is dangerous.
(c) If $d(s) \geq (k - l + 1)(k - 1) + 4$ and X and Y are dangerous then $X^* \cap Y^* \neq \emptyset$.

Proof. We prove (a). Let X, Y be tight sets with $X \cap Y \neq \emptyset$. By (7) we have

$$2k = \bar{d}(X) + \bar{d}(Y) \geq \bar{d}(X \cap Y) + \bar{d}(X \cup Y). \tag{15}$$

Clearly, $X \cap Y$ is a fragment and hence $\bar{d}(X \cap Y) \geq k$ by (6). Using (15) we have $\bar{d}(X \cup Y) \leq k$. Thus if $X^* \cap Y^* \neq \emptyset$ then $X \cup Y$ is also a fragment and hence is tight.

Suppose $X^* \cap Y^* = \emptyset$. Since $\bar{d}(X \cup Y) \le k$, we have $n(X \cup Y) \le k - d(s, X \cup Y)$. Thus

$$\begin{aligned} d(s) &\le d(s, X \cup Y) + d(s, N(X \cup Y)) \le d(s, X \cup Y) + (k-l)n(X \cup Y) \le \\ &\le d(s, X \cup Y) + (k-l)(k - d(s, X \cup Y)) = (k-l)k - (k-l-1)d(s, X \cup Y). \end{aligned}$$

Since $k-l-1 \ge 0$ and $d(s, X \cup Y) \ge 1$, this gives $d(s) \le (k-l)(k-1)+1$, contradicting the hypothesis on $d(s)$.

The proofs of (b) and (c) are similar, using the fact that $d(s, X \cup Y) \ge 2$ in (b) and (c). $\square$

Lemma 4.2. *Let sx_0 be a designated edge of a k-critical extension $G+s$ of G and suppose that there are $q \ge (k-l+1)(k-1)+4$ edges sy ($y \ne x_0$) incident to s for which the pair sx_0, sy is not admissible. Then there exists a shredder K in G such that K has $q+1$ leaves in $G+s$, and one of the leaves is the maximal tight set containing x_0.*

Proof. Let X_0 be the maximal tight set in $G+s$ containing x_0. Note that the set X_0 is uniquely determined by Lemma 4.1(a). Let $\mathcal{T} = \{X_1, ..., X_m\}$ be the set of all maximal tight sets which intersect $N(X_0)$. Note that $X_i \cap X_j = \emptyset$ for $0 \le i < j \le m$ by Lemma 4.1(a). Thus we have $d(s, \cup_{i=0}^m X_i) = d(s, X_0) + d(s, \cup_{i=1}^m X_i)$.

Since each $X_i \in \mathcal{T}$ contains a neighbour of X_0 and X_0 is tight, we have $m \le n(X_0) = k - d(s, X_0)$. Since each $X_i \in \mathcal{T}$ is tight and G is l-connected, we have $d(s, X_i) \le k - l$. So

$$d(s, \cup_{i=0}^m X_i) \le d(s, X_0) + (k-l)(k - d(s, X_0)) = k(k-l) - d(s, X_0)(k-l-1). \quad (16)$$

Let $M := \{y \in N(s) : sx_0, sy \text{ is not admissible}\}$. Since there exist $q \ge (k-l+1)(k-1)+4$ edges incident to s which are not admissible with sx_0, we can use (16) to deduce that $R := M - \cup_{i=0}^m X_i \ne \emptyset$. By Lemma 2.9 and by the choice of $\mathcal{T}$ there exists a family of maximal dangerous sets $\mathcal{W} = \{W_1, ..., W_r\}$ such that $x_0 \in W_i$ for all $1 \le i \le r$ and $R \subseteq \cup_{j=1}^r W_i$. By Lemma 4.1(b), $X_0 \subseteq W_i$ for all $1 \le i \le r$. Since $d(s, W_i - X_0) \le k + 1 - l - d(s, X_0)$, we can use (16) and the fact that $q \ge (k-l+1)(k-1)+4-l$ to deduce that $r \ge 2$. For $W_i, W_j \in \mathcal{W}$ we have $W_i^* \cap W_j^* \ne \emptyset$ by Lemma 4.1(c). Since $W_i \cup W_j$ is not dangerous by the maximality of W_i, we may apply (7) to obtain

$$k+1+k+1 \ge \bar{d}(W_i) + \bar{d}(W_j) \ge \bar{d}(W_i \cap W_j) + \bar{d}(W_i \cup W_j) \ge k+k+2. \quad (17)$$

Thus equality holds throughout and $W_i \cap W_j$ is tight. Since X_0 is a maximal tight set and $X_0 \subseteq W_i \cap W_j$ we have $X_0 = W_i \cap W_j$. Furthermore, since we have equality in (17), we can use (7) to deduce that $W_j \cap N(W_i) \subseteq N(W_i \cap W_j)$. So $W_j \cap N(W_i) \subseteq N(X_0)$ and $W_i \cap N(W_j) \subseteq N(X_0)$. Hence $N(s) \cap W_i \cap N(W_j) \subseteq \cup_{i=0}^m X_i$. (Note that every $z \in N(s) \cap N(X_0)$ is contained in one of the X_i's by the criticality of $G+s$.) So by the choice of $\mathcal{W}$, $R \cap W_i \cap W_j^* \ne \emptyset$ and $R \cap W_j \cap W_i^* \ne \emptyset$ follows.

By (8),

$$\begin{aligned}2k+2 &= \bar{d}(W_i)+\bar{d}(W_j)\geq\\ &\geq \bar{d}(W_i\cap W_j^*)+\bar{d}(W_i\cap W_j^*)+d(s,W_i-W_j^*)+d(s,W_j-W_i^*)\geq 2k+2,\end{aligned}$$

and so we have equality throughout. Thus all edges from s to W_i, other than the single edge sx_0, end in $W_i\cap W_j^*$ and $d(s,X_0)=1$. Hence $R\cap W_j\cap W_i^*=(R\cap W_j)-x_0$. Since $d(s,(W_i\cup W_j)-X_0)\leq k+2-l-d(s,X_0)$, we can use (16) and the fact that $q\geq(k-l+1)(k-1)+4$ to deduce that $r\geq 3$. Thus $\emptyset\neq(R\cap W_j)-x_0\subseteq W_j\cap W_i^*\cap W_k^*$ holds for every triple. Applying (9), and using $d(s,W_i\cap W_j\cap W_k)\geq 1$, we get

$$\begin{aligned}3k+3 &\geq \bar{d}(W_i)+\bar{d}(W_j)+\bar{d}(W_k)\geq\bar{d}(W_i\cap W_j\cap W_k)+\bar{d}(W_i\cap W_j^*\cap W_k^*)+\\ &+\bar{d}(W_j\cap W_i^*\cap W_k^*)+\bar{d}(W_k\cap W_i^*\cap W_j^*)-\\ &-|N(W_i)\cap N(W_j)\cap N(W_k)|+2\geq\\ &\geq 4k-|N(W_i)\cap N(W_j)\cap N(W_k)|+2\geq 3k+3. \qquad (18)\end{aligned}$$

For $S=N(W_i)\cap N(W_j)\cap N(W_k)$ we have $|S|=k-1$ by (18). Since $n(W_i)\leq k-1$ we must have $N(W_i)=S$ and hence $N(W_i)\cap W_j=\emptyset$. Thus $N(W_i\cap W_j\cap W_k)\subseteq S$. Since $W_i\cap W_j\cap W_k=X_0$ and $n(X_0)=k-1$ we have $N(W_i\cap W_j\cap W_k)=N(X_0)=S$. We may also deduce from (18) that $W_i\cap W_j^*\cap W_k^*$ is tight and $d(s,W_i\cap W_j\cap W_k)=1$. Since $n(W_i)=k-1$ and W_i is dangerous, $d(s,W_i)=2$ follows. Thus $d(s,W_i\cap W_j^*\cap W_k^*)=1$ and $G-S$ has $r+1$ components $C_0,C_1,...,C_r$, where $C_0=W_i\cap W_j\cap W_k=X_0$ and $C_i=W_i-X_0$ for $1\leq i\leq r$.

If $\{x_0\}\cup R=M$ then we are done. Hence suppose $\{x_0\}\cup R\neq M$. Choose $X_i\in\mathcal{T}$. Since $X_i\cap N(X_0)\neq\emptyset$, we have $X_i\cap S\neq\emptyset$. Since $X_i\cap R=\emptyset$, $N(X_i)\cap C_i\neq\emptyset$ for $0\leq i\leq r$. Using $r=|R|\geq q-d(s,\cup_{i=0}^m X_i)\geq k+2$ by (16) and the facts that $d(s,X_0)=1$, and $q\geq(k-l+1)(k-1)+4$, we deduce that $\bar{d}(X_i)\geq r+1+d(s,X_i)\geq k+4$ (since $d(s,X_i)\geq 1$). This contradicts the fact that X_i is tight. □

4.2 Graphs with Large Shredders

We show in this subsection that if $b^*(G)$ is large compared to k and $b^*(G)-1\geq\lceil t(G)/2\rceil$ then $a_k(G)=b^*(G)-1$. We need several new observations on shredders. We assume throughout this subsection that $G+s$ is a k-critical extension of G, and that, for some shredder K of G, we have $d(s)\leq 2b^*(K)-2$.

Lemma 4.3. *Suppose $b^*(K)\geq 4k+3(k-l)-1$. Then*

(a) the number of components C of $G-K$ with $d(s,C)\geq 3$ is at most $b(K)-2k-1$,

(b) $|N(s)\cap K|\leq 1$, and

(c) if $d(s,x)=j\geq 1$ for some $x\in K$ then $k-d_G(x)=j$.

Proof. Let w be the number of components C of $G-K$ with $d(s,C)\geq 3$. Then $d(s)\geq 3w+(b(K)-w)$. Thus

$$\begin{aligned}2w &\leq d(s)-b(K)\leq 2b^*(K)-2-b(K) \qquad (19)\\ &= 2b(K)+2\delta(K)-2-b(K)=2b(K)+3\delta(K)-2-b^*(K). \qquad (20)\end{aligned}$$

Since $\delta(K) \leq k-l$ and $b^*(K) \geq 4k+3(k-l)-1$, we have $w \leq b(K)-2k-1$. This proves (a).

Since $G+s$ is a critical extension of G, each vertex in $N(s)$ is contained in a tight set of $G+s$. Thus (b) will follow from the next claim.

Claim 4.4. *There is at most one vertex $x \in K$ for which $x \in Y$ holds for some tight set Y.*

Proof. Suppose that there exist two distinct vertices $x_1, x_2 \in K$ and tight sets Y_1, Y_2 in $G+s$ such that $x_1 \in Y_1$, $x_2 \in Y_2$. Let $Y = Y_1 \cup Y_2$ and let $\mathcal{D} = \{C : C \text{ is a component of } G-K, C \cap (Y \cup N(Y)) \neq \emptyset\}$. We have $|\mathcal{D}| \leq 2k$, since $\bar{d}(Y) \leq \bar{d}(Y_1)+\bar{d}(Y_2) \leq 2k$ and for every $C \in \mathcal{D}$ either $C-Y \neq \emptyset$, in which case $N(Y) \cap C \neq \emptyset$ holds, or $C \subset Y$, in which case $d(s, C \cap Y) \geq 1$ holds by (6).

Since $x_1, x_2 \in Y$, and by the definition of $\mathcal{D}$, every component C' of $G-K$ with $C' \notin \mathcal{D}$ satisfies $n(C') \leq k-3$, and hence $d(s, C') \geq 3$ by (6). This contradicts (a). □

To see (c) focus on a tight set X with $X \cap K \neq \emptyset$. By Claim 4.4 we have $X \cap K = \{x\}$ for some $x \in V$. If $X-K = \emptyset$ then $\{x\}$ is tight and hence we have $d(s,x) = k - d_G(x)$, as required. Suppose now that $X-K \neq \emptyset$ and let $M := X \cap C$ for some component C of $G-K$ for which $X \cap C \neq \emptyset$. Clearly, $N(M) \subseteq C \cup K$. Thus by (6) we obtain

$$\begin{aligned} k = \bar{d}(X) &\geq \bar{d}(M) - 1 + d(s,x) + |N(x) - M - N(M)| \\ &\geq k - 1 + d(s,x) + |N(x) - M - N(M)|. \end{aligned}$$

This implies $d(s,x) = 1$ and $N(x) \subseteq M \cup N(M)$. Therefore $b^*(K) \leq b(K)+1$, $x \notin N(C')$ for every component $C' \neq C$ of $G-K$ and hence $d(s,C') \geq k - n(C') \geq 2$. For C we have $d(s,C) \geq 1$ by (6). This gives $d(s) \geq 2(b(K)-1)+1+d(s,x) = 2b(K) \geq 2b^*(K)-2$. Thus equality must hold throughout and $\delta(K) = 1$. Since $N(s) \cap K = \{x\}$ by (b), we have $k - d_G(x) = \delta(K) = 1 = d(s,x)$. □

We shall use the following construction to augment G with $b^*(G)-1$ edges in the case when $d(s,K) = 0$ and $b(K) = b^*(G) =: b$. Let $C_1, ..., C_b$ be the components of $G-K$ and let $w_i = d_{G+s}(s, C_i)$, $1 \leq i \leq b$. Note that $w_i \geq 1$ by (6). Since $d(s) \leq 2b-2$, there exists a tree F' on b vertices with degree sequence $d_1, ..., d_b$ such that $d_i \geq w_i$, for $1 \leq i \leq b$. Let F be a forest on $N_{G+s}(s)$ with $d_F(v) = d_{G+s}(s,v)$ for every $v \in V(G)$ and such that the graph obtained from $(V-K, E(F))$ by contracting $C_1, C_2, \ldots, C_b$ to single vertices is F'. Thus $|E(F)| = |E(F')| = b-1$. We shall say that $G+F$ is a *forest augmentation* of G with respect to K and $G+s$, and prove that $G+F$ is k-connected. A set $X \subseteq V$ is said to be *deficient in* G if $n_G(X) \leq k-1$ and $X^* \neq \emptyset$. Note that since $d_{G+s}(s,K) = 0$, there are no sets X contained in K which are deficient in G, by (6).

Lemma 4.5. *Suppose $d(s,K) = 0$ and let $G+F$ be a forest augmentation of G with respect to K and $G+s$. If X is deficient in $G+F$ then $X \cap K \neq \emptyset$.*

Proof. Let X be a deficient set in G with $X \cap K = \emptyset$. We shall show that $n_{G+F}(X) \geq k$.

Case 1. $X \cap L = \emptyset$ for every leaf L of K in G.

Let $\mathcal{D} = \{C_i : X \cap C_i \neq \emptyset\}$. Choose $C_i \in \mathcal{D}$ arbitrarily and let $r := d(s, X \cap C_i)$. (Observe that $d_{G+s}(s, X \cap C_j) \geq 1$ for every $C_j \in \mathcal{D}$, otherwise $n_G(X) \geq n_G(X \cap C_j) \geq k - d_{G+s}(s, X \cap C_j) = k$ would follow.)

The tree F' has a vertex c_i of degree at least $d_{G+s}(s, C_i)$ corresponding to C_i. Let $e_1, ..., e_r$ be r edges incident to c_i corresponding to r edges in F leaving $X \cap C_i$. Choose r longest paths $P_1, ..., P_r$ in F' starting at c_i and containing the corresponding edges $e_1, ..., e_r$, such that each vertex on each P_i, other than the end vertex of P_i, corresponds to a component in $\mathcal{D}$. Such paths exist since no leaf of F' corresponds to a component in $\mathcal{D}$.

Let α be the number of paths whose last edge corresponds to an edge uv in F with $u \in X$. For every such path we have $v \in N_{G+F}(X) - N_G(X)$. Let β be the number of paths whose last edge $c_p c_q$ corresponds to an edge uv in F with $u \notin X$. Since every inner vertex of each P_i corresponds to a component in $\mathcal{D}$, we have $X \cap C_p \neq \emptyset$ and $u \in C_p - X$. Note also that $C_p \neq C_i$ since the first edge of each P_i corresponds to an edge in F which is incident to $X \cap C_i$. Since $N_G(X \cap C_i) \subseteq C_i \cup K$, there exists a vertex $w \in (N(X) \cap C_p) - N(X \cap C_i)$. Clearly, $\alpha + \beta = r$. Therefore we have the following inequalities.

$$n_G(X) \geq n_G(X \cap C_i) + \beta, \tag{21}$$

$$n_{G+F}(X) \geq n_G(X) + \alpha \geq n_G(X \cap C_i) + \alpha + \beta = n_G(X \cap C_i) + r. \tag{22}$$

Since $G + s$ is (k, s)-connected, $r \geq k - n_G(X \cap C_i)$, and hence X is not deficient in $G + F$. This solves Case 1.

Case 2. $X \cap L \neq \emptyset$ for some leaf L of K.

We consider two subcases. In the first subcase there exists a leaf L with $L \subseteq X$. Then $K \subseteq N_G(X)$ by Lemma 2.12 and hence if X properly intersects some component $C_i \neq L$ of $G - K$ then $n_G(X) \geq k$ follows, contradicting the fact that X is deficient. Thus (since $X^* \neq \emptyset$) there exists a component C of $G - K$ for which $C \cap X = \emptyset$. Now take a path P from L to C in F'. Let C' be the first component for which the edge on P which enters C' corresponds to an edge in F' which connects X to $V - X$. For this component we have $|N_{G+F}(X) \cap C'| \geq 1$, so $N_{G+F}(X) \geq |K| + 1 = k$, as required.

In the second subcase X properly intersects every leaf that it intersects. Let $a \in X \cap L_1$ for some leaf L_1. By Lemma 2.12 there exist $k - 1$ disjoint paths from a to K and hence, since $X \cap K = \emptyset$, $|N_G(X) \cap (K \cup L_1)| \geq k - 1$. If $X \cap L_2 \neq \emptyset$ for another leaf L_2 then $N_G(X) \cap L_2 \neq \emptyset$ and hence $n_G(X) \geq k$ follows. Hence $X \cap L_2 = \emptyset$ for every leaf L_2 distinct from L_1. Let P be a path in F' from L_1 to L_2. Let C' be the first component for which the edge on P which enters C' connects X to $V - X$. (Note that the first edge of P corresponds to an edge uv of F' with $u \in N_{G+s}(s) \cap L_1$. Since L_1 is a leaf component and $d_{G+s}(s, X \cap L_1) \geq 1$ we have $u \in X$.) For the component C' we have $(|N_{G+F}(X) \cap C'| \geq 1$, so $N_{G+F}(X) \geq |K| + 1 = k$, as required.

This completes the proof of Lemma 4.5. □

Lemma 4.6. *Suppose $b^*(K) \geq 4k + 3(k-l) - 1$ and $d(s,K) = 0$. Let $G + F$ be a forest augmentation of G with respect to K and $G + s$. Then $G + F$ is k-connected.*

Proof. We proceed by contradiction. Let X be a deficient set in $G + F$. Then X^* is also deficient so by Lemma 4.5, $X \cap K \neq \emptyset \neq X^* \cap K$. Suppose that $|X \cap K| \geq 2$ and $|X^* \cap K| \geq 2$. Since $|(G-K) - X - X^*| \leq |V - X - X^*| \leq k-1$, there are at least $b(K) - (k-1)$ components C of $G - K$ which are contained in $X \cup X^*$. There is no edge from X to X^*, so for each such component either $C \subseteq X$ or $C \subseteq X^*$ holds. Thus we have $N_G(C) \subseteq K - X^*$ or $N_G(C) \subseteq K - X$, and so $n_G(C) \leq k-3$. Hence $d_G(s,C) \geq 3$ by (6). This contradicts Lemma 4.3(a).

Thus we may assume without loss of generality that $|X \cap K| = 1$. Let $X \cap K = \{x\}$. Since there is no edge from s to K in G, $X - K \neq \emptyset$. Let C be a component of $G - K$ such that $M := X \cap C \neq \emptyset$. We have $N_{G+F}(M) - \{x\} \subseteq N_{G+F}(X)$. By Lemma 4.5, M is not deficient in $G + F$. Hence, if X remains deficient in $G + F$, then we have $N_{G+F}(X) \subseteq N_{G+F}(M)$, $n_{G+F}(M) = k$, $N_{G+F}(x) - X \subseteq N_{G+F}(M)$ and $n_{G+F}(X) = k-1$.

Therefore, since x has a neighbour in each leaf, we can deduce that $X \cap L_1 \neq \emptyset \neq X \cap L_2$ for (at least) two leaves L_1, L_2. If $L_1 \subset X$ then, since $K = N(L_1)$, we have $X^* \cap K = \emptyset$, contradicting Lemma 4.5. Hence $L_1 - X$ and $L_2 - X$ are both non-empty and $|N_G(X) \cap L_2| \geq 1$. Furthermore, since L_1 and L_2 are leaf components $N_{G+F}(X \cap L_1) \cap L_2 = \emptyset$. Letting $C = L_1$, we have $n_{G+F}(X) \geq n_{G+F}(L_1 \cap X) - 1 + |N_G(X) \cap L_2| \geq k$, contradicting the fact that X is deficient in $G + F$. □

Our final step is to show how to augment G with $b^*(K) - 1$ edges when $d(s,K) \neq 0$. In this case, Lemma 4.3(b) implies that there is exactly one vertex $x \in K$ which is adjacent to s. We use the next lemma to split off all edges from s to x and hence reduce to the case when $d(s,K) = 0$.

Lemma 4.7. *Suppose $d(s,x) \geq 1$ for some $x \in K$ and $d(s) \geq k(k-l+1) + 2$. Then there exists a sequence of $d(s,x)$ admissible splits at s which split off all edges from s to x.*

Proof. We have at most $k - l$ copies of sx. Suppose we get stuck after splitting some copies of sx, i.e. we obtain a graph $H + s$ where some edge sx cannot be split off. Since $d_H(s) \geq d_G(s) - (k-l-1) \geq (k-l+1)(k-1) + 4$, we can use Lemma 4.2 to deduce that there is a shredder K' in H with $b(K', H) = d_H(s)$ and x in one of the components of $H - K'$. Let u, v be two neighbours of s in H distinct from x and let C_u and C_v be the components of $H - K'$ containing u and v respectively. By Lemma 2.12, there exist $k-1$ openly disjoint paths between u and v in H containing only verticies of C_u, C_v and K', and hence avoiding x. Since all edges of $E(H) - E(G)$ are incident with x, these paths exist in G as well.

Since $b(K) \geq b^*(K) - (k-l) \geq (d(s)+2)/2 - (k-l) \geq k+1$, and each component of $G - K$ has an s-neighbour in G, we can choose the two neighbours u, v of s in $H + s$ to belong to different components in $G - K$. But for such a choice of u, v there do not exist $k-1$ disjoint paths from u to v in $G - x$, contradicting the above claim. □

We need one more lemma saying that if $d(s)$ is large enough compared to l and k, then $d(s)$ is equal to $t(G)$. Part (b) of the lemma, which gives a slight improvement of part (a) when $l = k-1$, is from [14].

Lemma 4.8. *(a) If $d(s) \geq (k-l)(k-1)+4$ then $d(s) = t(G)$.*
(b) If $l = k-1$ and $d(s) \geq k+1$ then $d(s) = t(G)$, [14].

Proof. We prove (a). Let $\mathcal{F}$ be a family of tight sets which cover $N(s)$ such that $|\mathcal{F}|$ is as small as possible. Since every edge incident to s is critical, such a family exists. We show that the members of $\mathcal{F}$ are pairwise disjoint. Choose $X, Y \in \mathcal{F}$ and suppose that $X \cap Y \neq \emptyset$. By Lemma 4.1(a) $X \cup Y$ is also tight. So replacing X and Y in $\mathcal{F}$ by $X \cup Y$ we contradict the minimality of $|\mathcal{F}|$.

Since the members of $\mathcal{F}$ are pairwise disjoint, tight, and cover $N(s)$, we have $d(s) = \sum_{X \in \mathcal{F}} k - n(X) \leq t(G)$. The inequality $d(s) \geq t(G)$ follows easily from (6). Thus $d(s) = t(G)$, as required. □

We can now prove our augmentation result for graphs with large shredders.

Theorem 4.9. *Suppose that G is l-connected, $b^*(G) \geq 4k + 4(k-l) - 1$, $t(G) \geq k(k-l+1)+2$ and $b^*(G) - 1 \geq \lceil t(G)/2 \rceil$. Then $a_k(G) = b^*(G) - 1$.*

Proof. Let $G+s$ be a k-critical extension of G. Then $d(s) = t(G)$ by Lemma 4.8(a). Let K be a shredder in G with $b^*(K) = b^*(G)$. Then $2b^*(K) - 2 \geq t(G) = d(s)$. Suppose $d(s, K) = 0$. Then $b^*(G) = b(K)$. Let $G + F$ be a forest augmentation of G with respect to K and $G+s$. Then $|F| = b(G) - 1$ and by Lemma 4.6, $G + F$ is the required k-connected augmentation of G. Hence we may assume that $d(s, K) \geq 1$.

Applying Lemma 4.3(c), we deduce that $\delta_G(K) = d_{G+s}(s, K) = d_G(s, x)$ for some $x \in K$. By Lemma 4.7, we can construct a graph $H + s$ by performing a sequence of $d_G(s, x)$ admissible splits at s which split off all edges from s to x in $G+s$. Since we only split edges incident to $x \in K$ to form $H + s$, we have $G - K = H - K$ and so $b_G(K) = b_H(K)$. Hence

$$\begin{aligned} d_{H+s}(s) &= d_{G+s}(s) - 2d_{G+s}(s,x) = d_{G+s}(s) - 2\delta_G(K) \leq 2b^*_G(K) - 2 - 2\delta_G(K) = \\ &= 2b_G(K) + 2\delta_G(K) - 2 - 2\delta_G(K) = 2b_G(K) - 2 = 2b_H(K) - 2. \end{aligned}$$

Thus we have $d_{H+s}(s) \leq 2b_H(K) - 2$, and $d_{H+s}(s, K) = 0$. Also note that the splittings add a set F_0 of $\delta_G(K)$ new edges to G to form H, and that $b_H(K) = b(K) \geq b^*_G(K) - (k-l) \geq 4k + 3(k-l) - 1$. Let $H + F_1$ be a forest augmentation of H with respect to K and $H + s$. Then $|F_1| = b_H(K) - 1 = b_G(K) - 1$, and $H + F_1$ is k-connected by Lemma 4.6. Thus $G + F_0 + F_1 = H + F_1$ is the required k-connected augmentation of G with $\delta_G(K) + b_G(K) - 1 = b^*_G(K) - 1$ edges. □

We will apply Theorem 4.9 to graphs which do not satisfy $b^*(G) - 1 \geq \lceil t(G)/2 \rceil$ using the following concept. A set F of new edges is *saturating* for G if $t(G+F) = t(G) - 2|F|$. Thus an edge $e = uv$ is *saturating* if $t(G+e) = t(G) - 2$.

Lemma 4.10. *If F is a saturating set of edges for an l-connected graph G with $b^*(G+F) \geq 4k + 4(k-l) - 1$, $t(G+F) \geq k(k-l+1)+2$, and $b^*(G+F) - 1 = \lceil t(G+F)/2 \rceil$, then $a_k(G) = \lceil t(G)/2 \rceil$.*

Proof. By Theorem 4.9 the graph $G+F$ can be made k-connected by adding a set F' of $\lceil t(G+F)/2\rceil$ edges. Since F is saturating, we have $t(G) = t(G+F)+2|F|$. Therefore the set $F \cup F'$ is an augmenting set for G of size $\lceil t(G)/2\rceil$. Since $a_k(G) \geq \lceil t(G)/2\rceil$, the lemma follows. □

We close this section by noting that Theorem 4.9 can be strengthened when $l = k-1$.

Theorem 4.11. *Suppose G is a $(k-1)$-connected graph such that $b(G) \geq k$ and $b(G) - 1 \geq \lceil t(G)/2\rceil$. Then $a_k(G) = b(G) - 1$, [14].*

Using the proof technique of Lemma 4.10 we deduce

Lemma 4.12. *If F is a saturating set of edges for a $(k-1)$-connected graph G with $b(G+F) - 1 = \lceil t(G+F)/2\rceil \geq k-1$ then $a_k(G) = \lceil t(G)/2\rceil$.*

4.3 Augmenting Connectivity by One

Throughout this subsection we assume that $G = (V, E)$ is a $(k-1)$-connected graph on at least $k+1$ vertices. We shall show that if $a_k(G)$ is large compared to k, then $a_k(G) = \max\{b(G)-1, \lceil t(G)/2\rceil\}$. Our proof uses Theorems 3.12 and 4.11. We shall show that if $a_k(G)$ is large, then we can add a saturating set of new edges F so that we have $t(G+F) = t(G) - 2|F|$ and $G+F$ is k-independence free.

In order to do this we need to measure how close G is to being k-independence free. We use the following concepts. Recall that a set $X \subset V$ is deficient in G if $n(X) = k-1$ and $V - X - N(X) \neq \emptyset$. Following [14], we call the (inclusionwise) minimal deficient sets in G the *cores* of G. A core B is *active* in G if there exists a $(k-1)$-cut K with $B \subseteq K$. Otherwise B is called *passive*. Let $\alpha(G)$ and $\pi(G)$ denote the numbers of active, respectively passive, cores of G. Since G is $(k-1)$-connected, the definition of k-independence implies that G is k-independence free if and only if $\alpha(G) = 0$. The following characterisation of active cores also follows easily from the above definitions.

Lemma 4.13. *Let B be a core in G. Then B is active if and only if $\kappa(G - B) = k - |B| - 1$.*

A set $S \subseteq V$ is a *deficient set cover* (or *$\mathcal{D}$-cover* for short) if $S \cap T \neq \emptyset$ for every deficient set T. Clearly, S covers every deficient set if and only if S covers every core. Note that S is a minimal $\mathcal{D}$-cover for G if and only if the extension $G+s$ obtained by joining s to each vertex of S is k-critical.

Lemma 4.14. *[14, p 16, Lemma 3.2] (a) Every minimal augmenting set for G induces a forest.*
(b) For every $\mathcal{D}$-cover S for G, there exists a minimal augmenting set F for G with $V(F) \subseteq S$.
(c) If F is a minimal augmenting set for G, $e = xy \in F$, and $H = G+F-e$, then H has precisely two cores X, Y. Furthermore $X \cap Y = \emptyset$; $x \in X$, $y \in Y$; for any edge $e' = x'y'$ with $x' \in X, y' \in Y$, the graph $H + e'$ is k-connected; and, for every deficient set Z on H, we have $X \subseteq Z$ or $Y \subseteq Z$.

Proof. (a) is given in [14, p 16].
(b) Since S covers all deficient sets, G becomes k connected when we add all edges between the vertices of S.
(c) follows from [14, Lemma 3.2]. $\square$

Based on these facts we can prove the following lemma.

Lemma 4.15. *Let S be a minimal $\mathcal{D}$-cover in G and let F be a minimal augmenting set with $V(F) \subseteq S$. Let $d_F(v) = 1$ and let $e = uv$ be the leaf of F incident with v. Let X and Y be the cores of $G + F - e$ and suppose that for a set F' of edges we have $\kappa(x, y, G + F') \geq k$ for some vertices $x \in X, y \in Y$. Then $S - \{v\}$ is a $\mathcal{D}$-cover of $G + F'$.*

Proof. Without loss of generality we may assume that $u \in X$ and $v \in Y$. By the minimality of S, there exists a core Z of G such that $Z \cap S = \{v\}$. Since Z is also deficient in $G + F - e$, it must contain a core of $G + F - e$, so $Y \subseteq Z$ by Lemma 4.14(c). Now, since Y is also deficient in G and Z is a core in G, we must have $Z = Y$ and $Y \cap S = \{v\}$. For a contradiction suppose that there is a deficient set P in $G + F'$ which is not covered by $S - \{v\}$. Then $P \cap S = \{v\}$ and so P is also deficient in $G + F' + F - e$ and in $G + F - e$. Thus, by Lemma 4.14(c), $Y \subseteq P$ and $y \in P$ hold. Furthermore, since $G + F' + F - e + xy$ is k-connected by Lemma 4.14(c), we must have $x \notin P \cup N(P)$ in $G + F' + F - e$. Thus $x \notin P \cup N(P)$ holds in $G + F'$ as well. This contradicts the fact that $\kappa(x, y, G + F') \geq k$. $\square$

We need some further concepts and results from [14].

Lemma 4.16. *[14, Lemma 2.1, Claim I(a)] Suppose $t(G) \geq k$. Then the cores of G are pairwise disjoint and the number of cores of G is equal to $t(G)$. Furthermore, if $t(G) \geq k + 1$, then for each core X, there is a unique maximal deficient set $S_X \subseteq V$ with $X \subseteq S_X$ and $S_X \cap Y = \emptyset$ for every core Y of G with $X \neq Y$. In addition, for two different cores X, Y we have $S_X \cap S_Y = \emptyset$.*

Lemma 4.17. *[14, Lemma 2.2] Let K and L be distinct $(k - 1)$-cuts in G with $b(K) \geq k$. Then L intersects precisely one component D of $G - K$.*

(The proof of Lemma 4.17 in [14] is similar to our proof of Lemma 4.1.)

Lemma 4.18. *Let K be a shredder in G with $b(K) \geq k$. Then*
(a) if $C = S_X$ for some component C of $G - K$ and for some core X then X is passive,
(b) if some component D of $G - K$ contains precisely two cores X, Y and no edge of G joins S_X to S_Y then both X and Y are passive.

Proof. (a) Suppose that X is active and let L be a $(k - 1)$-cut with $X \subseteq L$. Since $b(K) \geq k$, we have $L \subset K \cup C$ by Lemma 4.17. Since G is $(k - 1)$-connected and $L \neq K$, $G - L - C$ is connected. Hence $G - L$ has a component C'' with $C'' \subset C$. Therefore C contains a (minimal) deficient set X' with $X \cap X' = \emptyset$, contradicting

$C = S_X$.
(b) Notice that $b(K) \geq k$ and the existence of X, Y implies $t(G) \geq k+1$. Suppose X is active and let L be a $(k-1)$-cut with $X \subseteq L$. As in the proof of (a), this implies that $G - L$ has a component C with $C \subseteq D - L$. Since D contains precisely two cores, $Y \subset C$ and hence, since S_Y is the unique maximal deficient set containing Y which is disjoint from every core, $C \subseteq S_Y$ must hold. On the other hand, since C is a component of $G - L$, we have $X \subseteq N(C)$ and so $X \cap N(S_Y) \neq \emptyset$. This contradicts our assumption that no edge of G joins S_X to S_Y. □

Recall that an edge $e = uv$ is *saturating* if $t(G+e) = t(G)-2$. We say that two cores X, Y form a *saturating pair* if there is a saturating edge $e = xy$ with $x \in X, y \in Y$. For a core X let $\nu(X)$ be the number of cores Y $(Y \neq X)$ for which X, Y is not a saturating pair. The following lemma implies that an active core cannot belong to many non-saturating pairs.

Lemma 4.19. *Suppose $t(G) \geq k$ and let X be an active core in G. Then $\nu(X) \leq 2k+1$.*

Proof. Suppose that $\nu(X) \geq 2k+2$. Let $G+s$ be a k-critical extension of G. By Lemma 4.16, the cores of G are pairwise disjoint. Hence $d(s) = t(G) \geq \nu(X)+1 \geq 2k+3$, and there is exactly one edge from s to each core of G. Let Y be a core of G distinct from X and let x and y be neighbours of s in X and Y, respectively. Let $G'+s$ be the graph obtained by splitting sx, sy from s in G. Suppose that this split is admissible. Then $G'+s$ is a k-critical extension of G'. Applying Lemma 4.8(b) to $G'+s$ we deduce that $t(G') = d_{G'+s}(s) = t(G)-2$. Thus X, Y is a saturating pair. Since $\nu(X) \geq 2k+2$, it follows that there are at least $2k+2$ edges sy in G for which the pair sx, sy is not admissible. Then Lemma 4.2 implies that there exists a shredder K in G with at least $2k$ components and such that $S_X = C$ for some component C of $G - K$. By Lemma 4.18(a) this contradicts the fact that X is active. □

We shall also need the following characterisation of saturating pairs.

Lemma 4.20. *[14, p.13-14] Let $t(G) \geq k+2$ and suppose that two cores X, Y do not form a saturating pair. Then one of the following holds: (a) $X \subseteq N(S_Y)$, (b) $Y \subseteq N(S_X)$, (c) there exists a deficient set M with $S_X, S_Y \subset M$, which is disjoint from every core other than X, Y.*

For every passive core B_i $(1 \leq i \leq \pi(G))$ let $\mathcal{F}_i = \{X \subset V$: X is deficient in G, $B_i \subseteq X$, the subgraph $G[X]$ is connected, and X contains at most $4k-8$ active cores$\}$. Let $M_i = \cup_{X \in \mathcal{F}_i} X$ and let $T(G) = \cup_{i=1}^{\pi(G)} M_i \cup N(M_i)$.

Lemma 4.21. *Let B_i be a passive core for some $1 \leq i \leq \pi(G)$ and let $\mathcal{X} = \{X_1, ..., X_t\}$ be a minimal family of members of $\mathcal{F}_i$ for which $\cup_{j=1}^{t} X_j = M_i$. Then $t \leq k$ and $n(M_i) \leq k(k-1)$. Moreover, if $\alpha(G) \geq 5k-8$, then M_i intersects at most $k(4k-8)$ active cores.*

Proof. First we prove that $t \leq k$. For a contradiction suppose that $t \geq k+1$. By the minimality of the family $\mathcal{X}$ we have that $\hat{X}_j := X_j - \cup_{r \neq j} X_r$ is non-empty for all $1 \leq j \leq t$. Note that the sets $\hat{X}_j$ are pairwise disjoint. By applying (3) to a pair $X_r, X_j \in \mathcal{X}$, and using the facts that $X_r \cap X_j \neq \emptyset$, $t \geq k+1$, and that G is $(k-1)$-connected, we deduce that $X_r \cap X_j$ is deficient in G. Since $B_i \subseteq X_r$ for each $X_r \in \mathcal{X}$, a similar argument shows that $P := \cup_{1 \leq i < j \leq t} X_r \cap X_j$ is also deficient. Note that $M_i - P = \cup_{j=1}^{t} \hat{X}_j$. Since $X_r = \hat{X}_r \cup (P \cap X_r)$ and $G[X_r]$ is connected, there exists a neighbour of P in $\hat{X}_r$, and since the sets X_r are pairwise disjoint, these neighbours are distinct. Hence $n(P) \geq t \geq k+1$, contradicting the fact that P is deficient. Thus $t \leq k$. Hence, since each neighbour of M_i is a neighbour of some set in $\mathcal{X}$, and $\mathcal{X}$ consists of deficient sets, we have $n(M_i) \leq k(k-1)$.

To see the second part of the statement suppose that for some $X_r \in \mathcal{X}$ and for some active core A we have $X_r \cap A \neq \emptyset$ and $X_r - A \neq \emptyset \neq A - X_r$. Since $\alpha(G) \geq 5k-8$, X_r contains at most $4k-8$ active cores, and the (active) cores are pairwise disjoint, we have $|V - (X_r \cup A)| \geq k-1$. Now (3) implies that $X_r \cap A$ is deficient, a contradiction. Thus every active core A for which $A \cap M_i \neq \emptyset$ satisfies $A \subset X_r$ for some $X_r \in \mathcal{X}$. Hence the definition of $\mathcal{F}_i$ implies that M_i intersects at most $k(4k-8)$ active cores. □

We shall use the following lemmas to find a saturating set F for G such that $G+F$ has many passive cores. Informally, the idea is to pick a properly chosen active core B and, by adding a set F of at most $2k-2$ saturating edges between the active cores of G other than B, make $\kappa(G+F-B) \geq k-|B| =: r$. By Lemma 4.13, this will make B passive, and will not eliminate any of the passive cores of G. We shall increase the connectivity of $G-B$ by choosing a minimal r-deficient set cover S for $G-B$ of size at most $k-1$ and then iteratively add one or two edges so that the new graph has an r-deficient set cover properly contained in S. Thus after at most $k-1$ such steps (and adding at most $2k-2$ edges) we shall make B passive. The first lemma tells us how to choose the active core B.

Lemma 4.22. *Suppose $\pi(G) \leq 4(k-1)$ and $\alpha(G) \geq 20k(k-1)^2$. Then there exists an active core B with $B \cap T(G) = \emptyset$.*

Proof. Since $\alpha(G) \geq 20k(k-1)^2 \geq 5k-8$, Lemma 4.21 implies that for any passive core B_i we have M_i intersects at most $k(4k-8)$ active cores, and $N(M_i)$ intersects at most $k(k-1)$ active cores. Thus $T(G)$ intersects at most $\pi(G)(k(5k-9)) < 4(k-1)k(5k-5) = 20k(k-1)^2$ active cores. Since $\alpha(G) \geq 20k(k-1)^2$, the lemma follows. □

Lemma 4.23. *Suppose $\pi(G) \leq 4(k-1)$ and $\alpha(G) \geq 8k^3+6k^2-23k-16$. Let B be an active core in G, $H = G-B$, and $r = k-|B|$. Suppose that every r-deficient set Z of H contains an active core of G. Let S be a minimal r-deficient set cover of H with $S \subseteq N_G(B)$. Then there exists a saturating set of edges F for G such that $|F| \leq 2$, F is not incident with B, and either $\pi(G+F) > \pi(G)$; or $\pi(G+F) = \pi(G)$, B is an active core in $G+F$, and $H+F$ has an r-deficient set cover S' which is properly contained in S.*

Proof. Since B is active, $\kappa(H) = k - 1 - |B| = r - 1$. Since B is deficient in G, we have $|S| \leq n_G(B) = k - 1$. By Lemma 4.14 there exists a minimal r-augmenting set F^* for H such that F^* is a forest and $V(F^*) \subseteq S$. Let $d_{F^*}(v) = 1$ and let $e = uv$ be a leaf of F^*. By Lemma 4.14(c), there exist precisely two r-cores Z, W in $H + F^* - e$ with $u \in Z, v \in W$. Then Z, W are r-deficient in H. By an hypothesis of the lemma, there exist active k-cores X, Y of G with $X \subseteq Z$ and $Y \subseteq W$.

Suppose X and Y form a saturating pair in G. We may choose a saturating edge xy with $x \in X$ and $y \in Y$. Then $xy \notin E$ and hence $\kappa(x, y, G + xy) \geq k$ and $\kappa(x, y, H + xy) \geq r$ holds. Hence either $\pi(G + xy) > \pi(G)$; or every active k-core of G other than X, Y remains active in $G + xy$. If the second alternative holds then B remains active in $G + F$ and, by Lemma 4.15, $S' = S - v$ is an r-deficient set cover in $H + xy$.

Hence we may assume that X, Y is not a saturating pair in G. By Lemma 4.20 either

(i) there exists a k-deficient set M in G with $S_X \cup S_Y \subseteq M$ which is disjoint from every k-core other than X, Y, or

(ii) $Y \subseteq N(S_X)$ or $X \subseteq N(S_Y)$.

Choose $x \in X$ and $y \in Y$ arbitrarily and let $P_1, P_2, ..., P_{k-1}$ be $k-1$ openly disjoint xy-paths in G. Let $Q = \cup_{i=1}^{k-1} V(P_i)$. It is easy to see that if some edge of G joins S_X to S_Y, then one of the paths, say P_1, satisfies $V(P_1) \subseteq S_X \cup S_Y$. On the other hand, if no edge of G joins S_X to S_Y, then (ii) cannot hold. Hence (i) holds and, either one of the paths, say P_1, satisfies $V(P_1) \subseteq M$, or each of the $k-1$ paths intersects $N(M)$. In the latter case, since $n(M) = k-1$, we have $N(M) \subset Q$ and $Q \subset M \cup N(M)$ hold. We shall handle these two cases separately.

Case 1. No edge of G joins S_X to S_Y, (i) holds, and we have $N(M) \subset Q \subset M \cup N(M)$.

Let $C_1, C_2, ..., C_p$ be the components of $G - N(M)$. Using the properties of M (M intersects exactly two cores, M is the union of one or more components of $G - N(M)$, and $n(M) = k-1$) we can see that either, one component C_i contains S_X and S_Y and is disjoint from every core of G other than X, Y, or each of S_X and S_Y corresponds to a component of $G - N(M)$.

Since X and Y are active cores, Lemma 4.18, with $K = N(U)$, implies that $p \leq k - 1$. Since $\alpha(G) \geq (k-2)(2k+2) + k + 3$, G has at least $(k-2)(2k+2) + 1$ active cores disjoint from B, X, Y, and $N(M)$. Thus some component C_j of $G - N(M)$ is disjoint from M and contains at least $2k+3$ active cores distinct from B. By Lemmma 4.19, there exists a saturating edge xa_1 with $a_1 \in A_1$ for some active core $A_1 \subset C_j$, $A_1 \neq B$. If $\pi(G + xa_1) \geq \pi(G) + 1$ then we are done. Otherwise all the active cores in G other than X, A_1 remain active in $G + xa_1$. Applying Lemma 4.19 again, we may pick a saturating edge ya_2 with $a_2 \in A_2$ for some active core A_2 of $G + xa_1$, with $A_2 \subset C_j$, $A_2 \neq B$.

We have $\kappa(x, y, G + xa_1 + ya_2) \geq k$, since there is a path from x to y, using the edges xa_1, ya_2, and vertices of C_j only, and thus this path is openly disjoint from Q

(since $Q \subseteq M \cup N(M)$). Hence $\kappa(x, y, H + xa_1 + ya_2) \geq r$. Thus by Lemma 4.15, $S' = S - v$ is an r-deficient set cover in $H + xa_1 + ya_2$.

Case 2. Either $V(P_1) \subseteq S_X \cup S_Y$ or (i) holds and $V(P_1) \subseteq M$.

Let us call a component D of $G - Q$ *essential* if D intersects an active core other than X, Y or B. Let $D_1, D_2, ..., D_p$ be the essential components of $G - Q$. We say that a component D_i is *attached to* the path P_j if $N(D_i) \cap V(P_j) \neq \emptyset$ holds. Let $R = S_X \cup S_Y$ if $V(P_1) \subseteq S_X \cup S_Y$ holds and let $R = M$ if $V(P_1) \subseteq M$. Then, R is disjoint from every active core other than X, Y.

Claim 4.24. *At most $2k - 2$ essential components are attached to P_1.*

Proof. Focus on an essential component D which is attached to P_1 and let $w \in W \cap D$ for some active core $W \neq X, Y, B$ which has a vertex in D. There exists a path P_D from w to a vertex of P_1 whose inner vertices are in D. Since $w \notin R$ and $V(P_1) \subseteq R$, we have $D \cap N(R) \neq \emptyset$. The claim follows since the essential components are pairwise disjoint and $n(R) \leq 2k - 2$. □

Suppose that one of the paths P_i intersects at least $4k + 4$ active cores in G other than X , Y or B. For every such active core A intersecting P_i choose a representative vertex $a \in A \cap P_i$. Since the cores are pairwise disjoint, the representatives are pairwise distinct. Order the active cores intersecting P_i following the ordering of their representatives along the path P_i from x to y. By Lemma 4.19, we may choose a saturating edge xa_1 in G, where a_1 is among the $2k + 2$ rightmost representatives and a_1 belongs to an active core A_1. If $\pi(G + xa_1) \geq \pi(G) + 1$ then we are done. Otherwise all the active cores of G other than X, A_1 remain active in $G + xa_1$. Again using Lemma 4.19, we may choose a saturating edge ya_2 in $G + xa_1$, where a_2 is among the $2k + 2$ leftmost representatives. By the choice of a_1 and a_2 there exist two openly disjoint paths from x to y in $G + xa_1 + ya_2$ using vertices of $V(P_i)$ only. Thus $\kappa(x, y, G + xa_1 + ya_2) \geq k$. Hence, by Lemma 4.15, $S' = S - v$ is an r-deficient set cover in $H + xa_1 + ya_2$.

Thus we may assume that each path P_i intersects at most $4k + 3$ active cores in G other than X , Y or B. Hence there are at least

$$\alpha(G) - 3 - (k-1)(4k+3) \geq (8k^3 + 6k^2 - 23k - 19) - (k-1)(4k+3) = (2k+2)(4k^2 - 3k - 8)$$

active cores other than B contained in $G - Q$. Note that since cores are minimal deficient sets, they induce connected subgraphs in G. Hence each core contained in $G - Q$ is contained in a component of $G - Q$. If some component of $G - Q$ contains at least $2k + 3$ active cores of G other than B then the lemma follows as in Case 1. Hence we may assume that there are at least $4k^2 - 3k - 8$ essential components in $G - Q$ and each such component contains an active core distinct from X, Y, and B.

Using Claim 4.24 we deduce that there are at least $4k^2 - 3k - 8 - (2k - 2) = (4k + 3)(k - 2) + 1$ essential components D_i with all their attachments on $P_2, P_3, \ldots, P_{k-1}$. Since G is $(k-1)$-connected, $n(D_i) \geq k-1$ and hence D_i has at least two attachments on at least one of the paths $P_2, P_3, \ldots, P_{k-1}$. Relabelling the components $D_1, \ldots, D_p$

and the paths $P_2, \ldots, P_{k-1}$ if necessary, we may assume that D_i has at least two attachments on P_{k-1} for $1 \leq i \leq 4k+4$.

Let z_i be the leftmost attachment of D_i on P_{k-1}. Without loss of generality we may assume that $z_1, z_2, \ldots, z_{4k-2}$ occur in this order on P_{k-1} as we pass from x to y. By Lemma 4.19, there exists a saturating edge ya_i where $a_i \in A_i$ for some active core $A_i \subseteq D_i$, where $A_i \neq B$ and $1 \leq i \leq 2k+2$. If $\pi(G + ya_i) \geq \pi(G) + 1$ then we are done. Otherwise every active core in G other than Y, A_i remains active in $G + ya_i$. Using Lemma 4.19 again, there exists a saturating edge xa_j where $a_j \in A_j$ for some active core $A_j \subseteq D_j$, where $A_j \neq B$ and $2k+3 \leq j \leq 4k+4$. Note that z_i is either to the left of z_j or $z_i = z_j$. Hence, using the fact that D_j has at least two attachments on P_{k-1} and by the choice of z_i, z_j, there exist two openly disjoint paths in $G + xa_j + ya_i$, using vertices from $V(P_{k-1}) \cup D_i \cup D_j$ only. Therefore $\kappa(x, y, G + xa_j + ya_i) \geq k$, and we are done as above. This completes the proof of the lemma. □

Lemma 4.25. *Suppose $\pi(G) \leq 4(k-1)$ and $\alpha(G) \geq 20k(k-1)^2$. Then there exists a saturating set of edges F for G such that $|F| \leq 2k-2$ and $\pi(G+F) \geq \pi(G)+1$.*

Proof. Let B be an active core in G with $B \cap T(G) = \emptyset$. Such a set exists by Lemma 4.22. Let $H = G - B$, and $r = k - |B|$. Since B is active, $\kappa(H) = r - 1$. Every r-deficient set X in H is k-deficient in G and $N_G(B) \cap X \neq \emptyset$. Hence the set of vertices in H corresponding to $N_G(B)$ is an r-deficient set cover of H. Let $S \subseteq N_G(B)$ be a minimal r-deficient set cover of H. Since B is k-deficient in G, we have $|S| \leq n(B) = k-1$.

We shall prove by induction on i that, for $0 \leq i \leq k-1$, there exists a saturating set of edges F_i for G such that $|F_i| \leq 2i$, F_i is not incident with B, and either $\pi(G+F_i) \geq \pi(G)+1$; or $\pi(G+F_i) = \pi(G)$, B is an active core of $G+F_i$, and $H+F_i$ has an r-deficient set cover $S_i \subseteq S$ with $|S_i| \leq |S| - i$. The lemma will follow since the second alternative cannot hold with $|S_i| = 0$ (since this would imply that $H + F_i$ is r-connected and hence that B is passive in $G + F_i$).

The statement is trivially true for $i = 0$ taking $F_i = \emptyset$. Hence suppose that there exists a set F_i satisfying the above statement for some $0 \leq i \leq k-2$. If $\pi(G + F_i) \geq \pi(G) + 1$ then we can put $F_{i+1} = F_i$. Hence we may suppose that $\pi(G+F_i) = \pi(G)$, B is an active core of $G+F_i$, and $H+F_i$ has an r-deficient set cover $S_i \subseteq S$ with $|S_i| \leq |S| - i$. We would like to apply Lemma 4.23 to B and $G + F_i$. To do this we must show that $G+F_i$, B and S_i satisfy the hypotheses of this lemma. We have $\pi(G+F_i) = \pi(G) \leq 4(k-1)$ and $\alpha(G+F_i) = \alpha(G) - 2|F_i| \geq 8k^3 + 6k^2 - 23k - 16$. Clearly, $S_i \subseteq S \subseteq N_{G+F_i}(B)$ holds. Thus the last property we need to verify is that every r-deficient set Z in $G + F_i - B$ contains at least one active core of $G + F_i$. Note that, since F_i is a saturating set for G, each core of $G + F_i$ is a core of G. Furthermore, since $\pi(G + F_i) = \pi(G)$, if A is an active core of G and A is a core of $G + F_i$ then A is an active core of $G + F_i$. Since Z is r-deficient in $G + F_i - B$, it is k-deficient in $G + F_i$. Thus Z contains at least one core in $G + F_i$. If Z contains an active core in $G + F_i$, then we are done, so suppose that every core of $G + F_i$ in Z is passive. Let B_j be such a core. Then B_j is passive in G. Let Z' be a maximal subset of Z for which $G[Z']$ is connected and $B_j \subseteq Z'$. Since Z is deficient in G, it can be seen that such a Z' exists, Z' is deficient in G, and $B \subseteq N_G(Z')$. Since $B \cap T(G) = \emptyset$, it follows that $Z' \notin \bar{\mathcal{F}}_j$ and hence Z' contains at least $4k-7$ active cores in G. Since $|F_i| \leq 2(k-2) = 2k-4$ and each edge of F_i is incident to at most two cores of G, it follows that there exists an active core A in G with $A \subset Z'$ which is still an (active) core in $G+F_i$, contradicting the assumption that every core of $G+F_i$ in Z is passive. Hence $G+F_i$, B and S_i satisfy the hypotheses of Lemma 4.23. Thus there exists a saturating set of edges F for $G+F_i$ such that $|F| \leq 2$, F is not incident with B, and either $\pi(G+F_i+F) > \pi(G+F_i) = \pi(G)$; or $\pi(G+F_i+F) = \pi(G+F_i) = \pi(G)$ and $G+F_i+F-B$ has an r-deficient set cover S_{i+1} which is properly contained in S_i. Hence the inductive statement holds with $F_{i+1} = F_i \cup F$. □

Lemma 4.26. *Suppose $t(G) \geq 20k(k-1)^2 + (4k-3)(4k-4)$. Then there exists a saturating set of edges F for G such that $G+F$ is k-independence free and $t(G+F) \geq 2k-4$.*

Proof. Since every graph is 1-independence free and every connected graph is 2-independence free, we may suppose that $k \geq 3$. If $\pi(G) \leq 4(k-1)$ then we may apply Lemma 4.25 recursively $4k-3-\pi(G)$ times to G to find a saturating set of edges F_1 for G such that $\pi(G+F_1) \geq 4k-3$. If $\pi(G) \geq 4k-3$ we set $F_1 = \emptyset$. Applying Lemma 4.19 to $G+F_1$, we can add saturating edges joining pairs of active cores until the number of active cores is at most $2k+1$. Thus there exists a saturating set of edges F_2 for $G+F_1$ such that $\alpha(G+F_1+F_2) \leq 2k+1$ and $\pi(G+F_1+F_2) \geq 4k-3$. Applying Lemma 4.19 to $G+F_1+F_2$, we can add saturating edges joining pairs consisting of one active and one passive core until the number of active cores decreases to zero. Thus there exists a saturating set of edges F_3 for $G+F_1+F_2$ such that $\alpha(G+F_1+F_2+F_3) = 0$ and $\pi(G+F_1+F_2+F_3) \geq 2k-4$. □

The main theorem of this subsection is the following.

Theorem 4.27. *If $a_k(G) \geq 20k^3$ then*

$$a_k(G) = \max\{\lceil t(G)/2 \rceil, b(G)-1\}.$$

Proof. Since every graph is 1-independence free and every connected graph is 2-independence free, the result follows from Theorem 3.12 if $k \leq 2$. Hence we may suppose that $k \geq 3$. We have $t(G) \geq a_k(G)+1 \geq 20k^3$ by Lemmas 2.8 and 4.8. If $b(G)-1 \geq \lceil t(G)/2 \rceil$ then $a_k(G) = b(G)-1$ by Theorem 4.11 and we are done. Thus we may assume that $\lceil t(G)/2 \rceil > b(G)-1$ holds. We shall show that $a_k(G) = \lceil t(G)/2 \rceil$. By Lemma 4.26, there exists a saturating set of edges F for G such that $G+F$ is k-independence free and $t(G+F) \geq 2k-4$. Note that adding a saturating edge to a graph H reduces $\lceil t(H)/2 \rceil$ by exactly one and $b(H)$ by at most one. Thus, if $\lceil t(G+F)/2 \rceil \leq b(G+F)-1$, then there exists $F' \subseteq F$ such that $\lceil t(G+F')/2 \rceil = b(G+F')-1$ and the theorem follows by applying Lemma 4.12. Hence we may assume that $\lceil t(G+F)/2 \rceil > b(G+F)-1$. Since $G+F$ is k-independence free, we can apply Theorem 3.12 to deduce that $a_k(G+F) = \lceil t(G+F)/2 \rceil$. Using (1) and the fact that $t(G) = t(G+F)+2|F|$ we have $a_k(G) = \lceil t(G)/2 \rceil$, as required. □

Theorem 4.27 gives an affirmative answer to a conjecture from [15, p.300].

4.4 Augmenting Connectivity by at least Two

Throughout this subsection we assume that $G = (V, E)$ is an l-connected graph on at least $k+1$ vertices and that $l \leq k-2$. We shall show that if $a_k(G)$ is large compared to k, then $a_k(G) = \max\{b^*(G)-1, \lceil t(G)/2 \rceil\}$. Our proof uses Theorems 4.9 and 4.27. We shall show that if $a_k(G)$ is large then either we can add a saturating set of edges F so that $G+F$ is $(k-1)$-connected, or else G has a large shredder K with $|K| = k-2$. If the latter occurs then we show directly that we can make G k-connected by adding $\lceil t(G)/2 \rceil$ edges.

Let $G+s$ be a k-critical extension of G. Construct a $(k-1)$-critical extension $H+s$ of G from $G+s$ by deleting edges incident to s. Let $f = (k-l+1)(k-1)+4$ be the upper bound on the number of non-admissible pairs containing a fixed edge given by Lemma 4.2.

Lemma 4.28. *If $d_{G+s}(s) \geq f(k-l+1)/(k-l)$ then $d_{G+s}(s) - d_{H+s}(s) \geq d_{G+s}(s)/(k-l+1)$.*

Proof. If $d_{H+s}(s) \leq f$ then the lemma is trivial. Otherwise by Lemma 4.8(a) there exists a family $\mathcal{F}$ of pairwise disjoint $(k-1)$-tight sets in H such that $d_{H+s}(s) = \sum_{\mathcal{F}}(k-1-n(X))$. Since $G+s$ is (k,s)-connected we have $d_{G+s}(s) \geq \sum_{\mathcal{F}}(k-n(X))$. Hence $d_{G+s}(s) \geq d_{H+s}(s) + |\mathcal{F}|$. Since $d_{H+s}(s, X) \leq k-l$ for each $X \in \mathcal{F}$, we have $|\mathcal{F}| \geq d_{H+s}(s)/(k-l)$. Thus $d_{G+s}(s) \geq d_{H+s}(s) + d_{H+s}(s)/(k-l) = (k-l+1)d_{H+s}(s)/(k-l)$. Hence $d_{G+s}(s) - d_{H+s}(s) \geq d_{G+s}(s)/(k-l+1)$. □

We next perform a sequence of $(k-1)$-admissible splits in $H+s$ and obtain H^*+s. We do this according to the following rules. If $d_{H+s}(s) \leq f+2$ then we put $H^*+s = H+s$. If $d_{H+s}(s) \geq f+3$ then we perform $(k-1)$-admissible splits until either $d_{H^*+s}(s) \leq f+2$, or $d_{H^*+s}(s) \geq f+3$ and there is no $(k-1)$-admissible split at s in H^*+s. We then add all the edges of $(G+s)-(H+s)$ to H^*+s and obtain G^*+s. We shall refer to the edges of $(G+s)-(H+s)$ as *new edges* of G^*+s.

Lemma 4.29. *If $d_{G+s}(s) \geq f(k+l-1)$ then G^*+s is a critical (k,s)-connected extension of G^*.*

Proof. Suppose G^*+s is not (k,s)-connected. If $d_{H^*+s}(s) \leq f$ then $H^*+s = H+s$ and $G^*+s = G+s$, contradicting the assumption that $G+s$ is (k,s)-connected. Hence $d_{H^*+s}(s) \geq f+1$. Choose a minimal k-deficient set X in G^*+s. Since $\bar{d}_{H^*+s}(X) \geq k-1$ we have $\bar{d}_{G^*+s}(X) = k-1 = \bar{d}_{H^*+s}(X)$ and no new edge of G^*+s is incident with X. Since X is not k-deficient in $G+s$, there exists an edge sx in $G+s$ with $x \in X$. Then $sx \in E(H+s)$, since no new edge is incident with X. Hence sx is $(k-1)$-critical in $H+s$ so there exists a minimal tight set Y with $x \in Y$ and $\bar{d}_{H+s}(Y) = k-1$. Hence $\bar{d}_{H^*+s}(Y) = k-1$. Working in H^*+s we may use the facts that $d_{H^*+s}(s) \geq f+1$ and H^*+s is $(k-1)$-critical to deduce that $X \cup Y$ is a fragment. Submodularity of $\bar{d}$ now implies that $\bar{d}_{H^*+s}(X \cap Y) = k-1$. Since there are no new edges incident to X, this gives $\bar{d}_{G^*+s}(X \cap Y) = k-1$. Now the minimality of X implies that $X \subseteq Y$. Since $\bar{d}_{H+s}(Y) = \bar{d}_{H^*+s}(Y)$, we now deduce that $\bar{d}_{H+s}(X) = \bar{d}_{H^*+s}(X)$. Thus $\bar{d}_{H+s}(X) = k-1$ and the minimality of Y gives

$X = Y$. Since no new edge is incident with X this gives $\bar{d}_{G+s}(Y) = \bar{d}_{H+s}(Y) = k-1$. Thus Y is k-deficient in $G+s$, contradicting the fact that $G+s$ is (k,s)-connected.

Criticality of $G^* + s$ follows from the criticality of $G+s$, since splitting cannot increase $\bar{d}$. □

Using Lemma 4.2, we can either construct $H^* + s$ such that $d_{H^*+s}(s)$ is small or else there exists $K \subseteq V$ such that $|K| = k-2$ and $H^* - K$ has $d_{H^*+s}(s)$ components. In the first case, we perform a sequence of admissible splits in $G^* + s$ such that, in the resulting graph $G' + s$, G' is $(k-1)$-connected and then apply Theorems 4.9 and 4.27. We accomplish this by ensuring that $\kappa(x, y, G') \geq k-1$ for every $x, y \in N_{H^*+s}(s)$. Since there are many new edges, this is feasible. In detail, we proceed incrementally using the lemmas below. In the second case, we show directly that we can make G k-connected by adding $\lceil t(G)/2 \rceil$ edges.

Henceforth we shall assume that $G' + s$ is obtained from $G^* + s$ by performing a sequence of k-admissible splits and that $T \subseteq V(G)$ is a cover of all fragments X in G' with $n_{G'}(X) \leq k-2$. (In proving the theorem we will take $T = N_{H^*+s}(s)$.) Let $|T| = \tau$.

Lemma 4.30. *If $\kappa(u, v, G') \geq k-1$ for all $u, v \in T$ then G' is $(k-1)$-connected.*

Proof. Suppose G' has a fragment X with $n(X) \leq k-2$. Then we may choose $u \in X \cap T$ and $v \in X^* \cap T$, contradicting the fact that $\kappa(u, v) \geq k-1$. □

Lemma 4.31. *Let $sz, sw \in E(G' + s)$ and suppose that the pair sz, sw is not k-admissible. If $\kappa(z, w, G') \leq k-2$ then sz belongs to at most f non-admissible pairs in $G' + s$.*

Proof. Suppose that sz has $r > f$ non-admissible partners. Then by Lemma 4.2, there is a shredder K in G' with $r+1$ leaf components in $G' + s$ such that z as well as each non-admissible partner x of z is in one of these components. By Lemma 2.12, $\kappa(z, x) \geq k-1$ for every such x. Taking $x = w$ gives a contradiction. □

Lemma 4.32. *Suppose that $d_{G'+s}(s) \geq (f+1)(2(k-2)(f+2)+\tau)$. Choose $u, v \in T$ and suppose that $\kappa(u, v, G') = m \leq k-2$. Then there exists a sequence of at most two k-admissible splits such that, for the resulting graph $G'' + s$, we have $\kappa(u, v, G'') = m+1$.*

Proof. Let T_u (T_v) be the smallest set which contains u (v, respectively), separates u and v, and has precisely m neighbours. It is well-known that these unique smallest separators exist. Since $n_{G'}(T_u) = n_{G'}(T_v) = m \leq k-2$, there exist vertices $x \in T_u \cap N_{G'}(s)$ and $y \in T_v \cap N_{G'}(s)$. It is also known that there exist m paths $P_1, ..., P_m$ from u to v, and two paths P_0 and P_{m+1}, one from u to x and the other from v to y such that all these $m+2$ paths are vertex-disjoint apart from u and v. (Note that $u = x$ or $v = y$ is possible.) We may assume, without loss of generality, that $N_{G'}(s) \cap (V(P_0) - x) = \emptyset$ and $N_{G'}(s) \cap (V(P_{m+1}) - y) = \emptyset$. Let $Q = \cup_1^m V(P_i) - \{u, v\}$. If the pair sx, sy is admissible, we have $\kappa(u, v, G' + xy) \geq m+1$, as required. If not, we need to choose splittable pairs in a more complicated way, as in the proof of Lemma 4.23.

Suppose there exists a path P_i $(1 \leq i \leq m)$ with $d_{G'}(s, V(P_i)) \geq 2f + 2$. By Lemma 4.31, there is an admissible pair sx, sa in $G' + s$, where a is one of the $f + 1$ neighbours of s on P_i closest to v. If $\kappa(u, v, G' + xa) \geq m + 1$ then we are done. Otherwise we may split sy, sb in $G' + s + xa$, where b is one of the $f + 1$ neighbours of s on P_i closest to u. The vertices x, b, a, y appear on P_i in this order. Hence there exist two vertex-disjoint uv-paths on vertex set $V(P_i) \cup V(P_0) \cup V(P_{m+1})$, showing $\kappa(u, v, G' + xa + yb) \geq m + 1$, as required. Thus we may assume that no such path exists and hence $d_{G'}(s, V - Q) > d(s) - m(2f + 2) \geq (f + 1)(2(k - 2)(f + 1) + \tau)$.

Let F be the graph obtained from $G' - Q$ by deleting any edges joining u and v. Let $C_0, C_1, ..., C_{p+1}$ be the components of F which each contain at least one neighbour of s, where $u, x \in V(C_0)$ and $v, y \in V(C_{p+1}$. Suppose $d(s, C_j) \geq f + 2$ for some $1 \leq j \leq p$. We may split sx, sa for some $a \in C_j$ which is admissible with sx and we split sy, sb for some $b \in C_j$ which is admissible with sy in $G' + s + sa$. These admissible pairs exist by Lemma 4.31. It is easy to see that $\kappa(u, v, G' + xa + yb) \geq m + 1$, as required. Thus we may assume that no such component exists. Similarly, if $d(s, C_0) \geq f + 1$, then we may split sy, sc for some $c \in C_0$ which is admissible with sy in $G' + s$, and we again have $\kappa(u, v, G' + yc) \geq m + 1$, as required. A similar construction holds if $d(s, C_{p+1}) \geq f + 1$. Hence we have at least $d(s, V - Q)/(f + 1) \geq 2(k - 2)(f + 1) + \tau$ components in F, each containing at least one neighbour of s.

Since each component C_i with $n_{G'}(C) \leq k - 2$ must contain a vertex from T, and $u, v \in T$, there are at least $2(k - 2)(f + 1)$ components C_i, $1 \leq i \leq p$, with at least $k - 1$ attachments on Q. Since $m \leq k - 2$, we have at least $2f + 2$ components $D_1, ..., D_r$ which have two attachments on the same path, P_1 say. Now we proceed as in the final part of the proof of Lemma 4.23. Let a_j be the attachment of D_j on P_1 closest to u for $1 \leq j \leq r$. We first pick a D_i where a_i is among the $f + 1$ attachment vertices a_j closest to u on P_1 and we choose an admissible pair sy, sb with $b \in D_i$. This edge exists by Lemma 4.31. Then we pick a D_h where a_h is among the $f + 1$ attachment vertices a_j closest to v on P_i and we choose an admissible pair sx, sa with $a \in D_h$. The edge xa exists by Lemma 4.31. Note that a_i either occurs before a_h on P_1 or $a_i = a_h$. Hence, using the fact that the components D_j have at least two attachments on P_1 and by the choice of a_i, a_h, there exist two openly disjoint uv-paths in $G' + xa + yb$, using vertices from $V(P_1) \cup V(P_0) \cup V(P_{m+1}) \cup D_i \cup D_h$ only. Therefore $\kappa(u, v, G' + xa + yb) \geq m + 1$, as required. □

Applying this lemma iteratively to all pairs of vertices in T, starting with $G' + s = G^* + s$ and using the fact that f is a decreasing function of l, we obtain:

Corollary 4.33. *Suppose that $d_{G^*+s}(s) \geq (f+1)(2(k-2)(f+2)+\tau)+2\tau^2(k-l-1)$. Then there exists a sequence of at most $\tau^2(k - l - 1)$ k-admissible splits such that, for the resulting graph $G' + s$, we have $\kappa(G') \geq k - 1$.*

Theorem 4.34. *If G is l-connected and $a_k(G) \geq 10(k - l + 2)^3(k + 1)^3$ then $a_k(G) =$* $\max\{b^*(G) - 1, \lceil t(G)/2 \rceil\}$.

Proof. We have $d_{G+s}(s) = t(G) \geq a_k(G) + 1 \geq 10(k-l+2)^3(k+1)^3$ by Lemmas 2.8 and 4.8. If $b^*(G) - 1 \geq \lceil t(G)/2 \rceil$ then $a_k(G) = b^*(G) - 1$ by Theorem 4.9 and we are done. Thus we may assume that $\lceil t(G)/2 \rceil \geq b^*(G)$ holds. We shall show that $a_k(G) = \lceil t(G)/2 \rceil$. We construct $H + s$, $H^* + s$, and $G^* + s$ as above. By Lemma 4.29, G^* is obtained from G by adding a saturating set F of edges. Note that adding a saturating edge to a graph G_0 reduces $\lceil t(G_0)/2 \rceil$ by exactly one and $b^*(G_0)$ by at most one. Thus, if $\lceil t(G+F)/2 \rceil \leq b^*(G+F) - 1$, then there exists $F' \subseteq F$ such that $\lceil t(G+F')/2 \rceil = b^*(G+F') - 1$ and the theorem follows by applying Lemma 4.10. Hence we may assume that $\lceil t(G^*)/2 \rceil \geq b^*(G^*) - 1$. We have

$$t(G^*) = d_{G^*+s}(s) \geq d_{G+s}(s) - d_{H+s}(s) \geq 10(k-l+2)^2(k+1)^3$$

by Lemma 4.28. Using Lemma 4.2, we either have $d_{H^*+s}(s) \leq 2f$ or else there exists $K \subseteq V$ such that $|K| = k-2$ and $H^* - K$ has $d_{H^*+s}(s)$ components.

Case 1: $d_{H^*+s}(s) \leq 2f$.

Let $T = N_{H^*+s}(s)$. Then $|T| = \tau \leq 2f$. By Corollary 4.33, there exists a sequence of at most $4(k-l+2)^3(k+1)^2$ k-admissible splits in $G^* + s$ such that, for the resulting graph $G' + s$, we have $\kappa(G') \geq k-1$. Note that $d_{G'+s}(s) \geq 2(k-l+2)^2(k+1)^3$. Thus there exists a saturating set of edges F for G such that $G' = G + F$ is $(k-1)$-connected and $t(G+F) \geq 2(k-l+2)^2(k+1)^3$. As above, we may assume that $\lceil t(G+F)/2 \rceil \geq b^*(G+F) - 1 \geq b(G+F) - 1$ (otherwise we are done by Lemma 4.10). Since $G + F$ is $(k-1)$-connected, we can apply Theorem 4.27 to deduce that $a_k(G+F) = \lceil t(G+F)/2 \rceil$. Using (1) and the fact that $t(G) = t(G+F) + 2|F|$ we have $a_k(G) = \lceil t(G)/2 \rceil$, as required.

Case 2: $d_{H^*+s}(s) \geq 2f + 1$ and there is no $(k-1)$-admissible split at s in $H^* + s$.

By Lemma 4.2, there exists $K \subseteq V$ such that $|K| = k-2$ and $H^* - K$ has $d_{H^*+s}(s)$ components each of which is a leaf component. Using Lemma 2.12, and the fact that $N_{H^*+s}(s)$ covers all fragments X in H^* with $n_{H^*}(X) \leq k-2$ we deduce

Claim 4.35. *H^* is $(k-2)$-connected.*

Since G^*+s is k-critical, it follows from Caim 4.35 that $d_{G^*+s}(s,v) \leq 2$ for all $v \in V$. For each vertex $v \in V$ such that $d_{G^*+s}(s,v) = 2$, let $X_v \subseteq V$ be the minimal set such that $v \in X_v$ and $\bar{d}_{G^*+s}(X_v) = k$. (The existence of X_v follows from the k-criticality of $G^* + s$, uniqueness follows from submodularity and the fact that $d_{G^*+s}(s)$ is large.) If $|X_v| \geq 2$ then we modify $G^* + s$ by choosing a vertex $v' \in X_v$ and replacing one of the edges from s to v by a new edge sv'. It can be seen that this operation preserves (k,s)-connectivity and criticality. Repeating this procedure for each such vertex v, the resulting graph $G_0 + s$ satisfies:

Claim 4.36. *For all $v \in V$ we have $d_{G_0+s}(s,v) \leq 2$. Furthermore $d_{G_0+s}(s,v) = 2$ if and only if $d_{G_0}(v) = k-2$.*

Let $\hat{G}+s$ be the graph obtained from G_0+s by splitting off as many k-admissible pairs of edges sx, sy as possible in G_0+s such that x and y belong to the same component of G_0-K. Then $\hat{G}+s$ is a k-critical extension of $\hat{G}$. Let $C_1, C_2, \ldots, C_r$ be the components of $\hat{G}-K$. Note that these components have the same vertex sets as the components of H^*-K and hence $r = d_{H^*+s}(s) \geq 2f+1$. Let $d_{\hat{G}+s}(s, C_i) = d_i$. Relabelling if necessary, we have $d_1 \geq d_2 \geq \ldots \geq d_r$.

Claim 4.37. $d_{\hat{G}+s}(s, K) = 0$.

Proof. Suppose $\hat{G}+s$ has an edge sx with $x \in K$. By criticality there exists a fragment $X \subset V$ such that $\bar{d}_{\hat{G}+s}(X) = k$. Since, by Caim 4.35, $x \in N_{H^*}(C_i)$ for all $1 \leq i \leq r$, we have $x \in N_{\hat{G}}(C_i)$. Hence either $N_{\hat{G}}(X) \cap C_i \neq \emptyset$, or $C_i \subseteq X$ and $d_{\hat{G}+s}(s, X \cap C_i) \geq 1$, for all $1 \leq i \leq r$. Thus $\bar{d}_{\hat{G}+s}(X) \geq r > k$. □

Using Lemma 4.10 we may suppose that

$$b^*(\hat{G}) \leq \lceil t(\hat{G})/2 \rceil = \lceil d_{\hat{G}+s}(s)/2 \rceil. \tag{23}$$

Claim 4.38. $d_1 \leq (\sum_{i=2}^{r} d_i) - 1$.

Proof. Suppose $d_1 \geq (\sum_{i=2}^{r} d_i)$. Since $d_1 + (\sum_{i=2}^{r} d_i) = d_{H^*}(s) \geq 2f+1$, we have $d_1 \geq f+1$. Since there is no k-admissible pair of edges joining s to C_1 in $\hat{G}+s$, it follows from Lemma 4.2 that there is a shredder K in $\hat{G}$ with each of the d_1 neighbours of s in C_1 in distinct components of $\hat{G}-K$ and at least one other component containing the remaining neighbours of s in $\hat{G}$. Thus $b(\hat{G}) \geq d_1+1$, and $b^*(\hat{G}) \geq b(\hat{G}) \geq d_1+1 \geq (d_{\hat{G}+s}(s)/2)+1$. This contradicts (23). □

Claim 4.39. *Suppose X is a fragment in $\hat{G}$ with $|X \cap K| \leq |X^* \cap K|$.*
(a) If $n_{\hat{G}}(X) = k-2$, then either $X = C_{i_1} \cup C_{i_2} \cup \ldots \cup C_{i_p}$ for some $\{i_1, i_2, \ldots, i_p\} \subseteq \{1, 2, \ldots, r\}$; or $X = Z_i \subset C_i$ for some $1 \leq i \leq r$;
(b) If $n_{\hat{G}}(X) = k-1$, then either $X = Z_{i_1} \cup C_{i_2} \cup \ldots \cup C_{i_p}$ for some $\{i_1, i_2, \ldots, i_p\} \subseteq \{1, 2, \ldots, r\}$ and $Z_{i_1} \subseteq C_{i_1}$; or $X = Z_{i_1} \cup Z_{i_2}$ for some $1 \leq i_1 < i_2 \leq r$, $Z_{i_1} \subseteq C_{i_1}$, $Z_{i_2} \subseteq C_{i_2}$, and $n_{\hat{G}}(Z_{i_1}) = k-2 = n_{\hat{G}}(Z_{i_2})$.

Proof. Suppose $X \cap K \neq \emptyset$. Then $X^* \cap K \neq \emptyset$. Since $N_{\hat{G}}(C_i) = K$ by Claim 4.35, it follows that $C_i \not\subseteq X$ and $C_i \not\subseteq X^*$ for all i, and hence that $n_{\hat{G}}(X) = |V - (X \cup X^*)| \geq r > k$. Thus we may suppose that $X \cap K = \emptyset$. Let

$$S = \{i \ : \ X \cap C_i \text{ is a proper subset of } C_i,\ 1 \leq i \leq r\}.$$

Since the claim holds when $S = \emptyset$ we may suppose that $|S| \geq 1$. Let $Z_i = X \cap C_i$ for $i \in S$. By Claim 4.35, $n_{\hat{G}}(Z_i) \geq k-2$. Hence $|N_{\hat{G}}(X) \cap (K \cup C_i)| \geq k-2$ and $|N_{\hat{G}}(X) \cap C_i| \geq 1$ for all $i \in S$. The claim now follows using the hypothesis of (a) and (b) that $n_{\hat{G}}(X) = k-2$ and $n_{\hat{G}}(X) = k-1$, respectively. □

Claim 4.40. *For each i, $1 \leq i \leq r$, there exists a unique minimal subset $Y_i \subseteq C_i$ such that $n_{\hat{G}}(Y_i) = k-2$.*

Proof. The existence of such a set follows from the fact that $n_{\hat{G}}(C_i) = k-2$. To prove uniqueness we suppose to the contrary that X_1 and X_2 are two minimal subsets of C_i satisfying $n_{\hat{G}}(X_1) = k-2 = n_{\hat{G}}(X_2)$. Then $n_{H^*}(X_1) = k-2 = n_{H^*}(X_2)$, since H^* is $(k-2)$-connected by Claim 4.35, and the operations used in going from H^* to $\hat{G}$ (adding edges incident to s and splitting off pairs of edges from s) cannot decrease $n(X_i)$. Let sw be the unique edge of $H^* + s$ from s to C_i. Since $H^* + s$ is $(k-1, s)$-connected, we must have $w \in X_1 \cap X_2$. Since $X_1 \cup X_2 \subseteq C_i$, $X_1 \cup X_2$ is a fragment of $\hat{G}$, and hence we have $n_{\hat{G}}(X_1 \cup X_2) \geq k-2$, by Claim 4.35. Submodularity of $n_{\hat{G}}$, now implies that $n_{\hat{G}}(X_1 \cap X_2) \leq k-2$, contradicting the minimality of X_1 and X_2. □

For each i, $1 \leq i \leq r$, choose two distinct edges sy_i, sy'_i in $\hat{G}+s$ with $y_i, y'_i \in Y_i$. Note that these edges exist by the (k,s)-connectivity of $\hat{G}$. Furthermore, by Claim 4.36, $y_i = y'_i$, if and only if $Y_i = \{y_i\}$ and $d_{\hat{G}}(y_i) = k-2$.

We are now ready to construct the required augmentation of G. Let $G' + s$ be the graph obtained from $\hat{G} + s$ by adding an extra edge from s to C_2 if $d_{\hat{G}+s}(s)$ is odd. Thus $d_{G'+s}(s) = 2\lceil t(\hat{G})/2 \rceil$ is even. First we try to define a good augmenting set by the method we used when we defined a forest augmentation. Since we want to increase the connectivity of G' by two, we now look for a loopless 2-connected graph G_3 on r vertices whose degree sequence is $d_1, d'_2, ..., d_r$, where $d'_2 = d_{G'+s}(s, C_2)$ (so d'_2 is either d_2 or $d_2 + 1$, depending on whether $d_{\hat{G}+s}(s)$ is even or odd). If such a graph exists, it leads to a good augmenting set in a natural way, as we shall see in Subcase 2.1. However, such a graph may not exist, as the following example shows: let G be obtained from $K_{r,k-2}$ by replacing some vertex v in the r-set by a copy of $K_{k-1,4}$ and then connecting each vertex of the $(k-2)$-set to each vertex of the $(k-1)$-set. It can be seen that the degree sequence defined by the corresponding extension $G'+s$ of G is $4, 2, 2, ..., 2$. There is no loopless 2-connected graph with this degree sequence. When such a graph does not exist, we need a somewhat more involved method to define the augmenting set. This will be described in Subcase 2.2.

Subcase 2.1 There exists a loopless 2-connected graph G_3 on r vertices with degree sequence $d_1, d'_2, ..., d_r$.

Let F be a set of edges joining the components of $G' - K$ such that $d_F(v) = d_{G'+s}(s, v)$ for all $v \in V$ and such that the graph obtained from $(V - K, F)$ by contracting each component C_i to a single vertex c_i, is G_3. Since G_3 is 2-connected, each vertex $c_i \in V(G_3)$ in has at least two distinct neighbours in G_3. Let y_i, y'_i be the neighbours of s in C_i defined after Claim 4.40. Since we may interchange the end vertices of the edges of F within each component, we may choose F to have the additional property that, for each $1 \leq i \leq r$, the two edges of F incident to y_i and y'_i join C_i to different components of $\hat{G}-K$. We can now use Claim 4.39 to deduce that $\hat{G}+F$ is k-connected. Suppose to the contrary that $\hat{G} + F$ has a fragment X with $n_{\hat{G}+F}(X) \leq k-1$. Replacing X by X^* if necesssary we may assume that $|X \cap K| \leq |X^* \cap K|$. By Claim 4.35, $n_{\hat{G}}(X) \geq k-2$ and by Claim 4.39, we have one of the following four alternatives.

(a1) $n_{\hat{G}}(X) = k-2$ and $X = C_{i_1} \cup C_{i_2} \cup \ldots \cup C_{i_p}$ for some $p \leq r-1$. Suppose $p \leq r-2$. Then the 2-connectivity of G_3 implies that there are two edges of F from

X to distinct components C_{j_1}, C_{j_2} disjoint from X. Hence $n_{\hat{G}+F}(X) \geq k$. Suppose $p = r-1$. There are at least two edges from X to C_{i_r}. If C_{i_r} has only one vertex then $N_{\hat{G}+F}(X) = V$ and X is not a fragment. If all edges of F join X to the same vertex $v \in C_{i_r}$, then we have $n_{\hat{G}}(C_{i_r} - v) \leq k-1$ and $d_{\hat{G}+s}(s, C_{i_r} - v) = 0$, contradicting the (k,s)-connectivity of $\hat{G} + s$. Thus at least two edges of F join X to distinct vertices of C_{i_r} and we again have $n_{\hat{G}+F}(X) \geq k$.
(a2) $n_{\hat{G}}(X) = k-2$ and $X = Z_i \subset C_i$ for some $1 \leq i \leq r$. By Claim 4.40, $y_i, y'_i \in X$. Since y_i, y'_i are joined by F to distinct components C_{j_1}, C_{j_2} disjoint from C_i, we again have $n_{\hat{G}+F}(X) \geq k$.
(b1) $n_{\hat{G}}(X) = k-1$, and $X = Z_{i_1} \cup C_{i_2} \cup \ldots \cup C_{i_p}$ for some $p \leq r$ and $Z_{i_1} \subseteq C_{i_1}$. Suppose $2 \leq p \leq r-1$. Then the 2-connectivity of G_3 implies that there is at least one edge of F from $X - C_{i_1}$ to a component C_{j_1} disjoint from X. Hence $n_{\hat{G}+F}(X) \geq k$. Suppose $p = r$. Since $\hat{G} + s$ is (k,s)-connected, it has an edge from s to a vertex $v \in X^* \subseteq C_{i_1} - Z_{i_1}$. Since all edges of F are incident to distinct components v is joined by an edge of F to some vertex of $X - C_{i_1}$, and again we have $n_{\hat{G}+F}(X) \geq k$. Suppose $p = 1$. Since $\hat{G} + s$ is (k,s)-connected, it has an edge from s to at least one vertex $v \in Z_{i_1}$. Since all edges of F are incident to distinct components, v is joined by an edge of F to some component distinct from C_{i_1}, and again we have $n_{\hat{G}+F}(X) \geq k$.
(b2) $n_{\hat{G}}(X) = k-1$ and $X = Z_{i_1} \cup Z_{i_2}$ for some $Z_{i_1} \subseteq C_{i_1}$, $Z_{i_2} \subseteq C_{i_2}$, and $n_{\hat{G}}(Z_{i_1}) = k-2 = n_{\hat{G}}(Z_{i_2})$. By Claim 4.40, $y_{i_1}, y'_{i_1} \in Z_{i_1}$. Since y_{i_1}, y'_{i_1} are joined by F to two distinct components C_{j_1}, C_{j_2} disjoint from C_{i_1}, at least one of these components is also disjoint from C_{i_2} and we again have $n_{\hat{G}+F}(X) \geq k$.

Thus $\hat{G} + F$ is k-connected. Putting $F_0 = E(\hat{G}) - E(G)$, we deduce that $F_0 \cup F$ is the required augmenting set of edges for G of size $\lceil d_{G+s}(s)/2 \rceil = \lceil t(G)/2 \rceil$.

Subcase 2.2 There is no loopless 2-connected graph on r vertices with degree sequence $d_1, d'_2, ..., d_r$.

Hakimi [9] characterised the degree sequences of loopless 2-connected graphs, see also [13, Corollary 3.2].

Theorem 4.41. *There exists a 2-connected loopless graph with degree sequence* $d_1 \geq d_2 \geq \ldots d_r$ *if and only if* $d_1 + d_2 + \ldots d_r$ *is even and* $d_1 \leq d_2 + \ldots + d_r - 2r + 4$.

This characterisation implies that in Subcase 2.2 we have either: $d_1 \geq d'_2$ and $d_1 \geq d'_2 + d_3 + ... + d_r - 2r + 5$; or $d_1 = d'_2 - 1$ and $d'_2 \geq d_1 + d_3 + ... + d_r - 2r + 5$. Since $d_{G'+s}(s) = d_1 + d'_2 + d_3 + ... + d_r$ and $d_{G'+s}(s)$ is even, this implies that

$$d_{G'+s}(s) \leq 2d_1 + 2r - 4. \tag{24}$$

We shall use the following concept to find a good augmenting set in this subcase. Let $H + s = (V + s, E)$ be a graph and $m_1, m_2, \ldots, m_q$ be a partition of $d_{H+s}(s)$. Then a $(m_1, m_2, \ldots, m_q)$-*detachment* of $H + s$ at s is a graph obtained from $H + s$ by 'splitting' s into q vertices with degrees $m_1, m_2, \ldots, m_q$, respectively. In [13, Corollary 3.3] we characterise when a graph $H + s$ has a loopless 2-connected $(m_1, m_2, \ldots, m_q)$-detachment at a given vertex s. (This result generalises Hakimi's above mentioned result, which corresponds to the special case when the graph consists of a single vertex

s and some loops incident to s.) For $v_1, v_2, \ldots, v_m \in V(H+s)$ let $b(v_1, v_2, \ldots, v_m)$ be the number of components of $(H+s) - \{v_1, v_2, \ldots, v_m\}$.

Lemma 4.42. *[13] Let $H+s = (V+s, E)$ be a graph and $m_1, m_2, \ldots, m_q$ be a partition of $d(s)$ into at least two positive integers, such that $m_1 \geq m_2 \geq \ldots \geq m_q$. Let $e(u)$ denote the number of loops incident to some vertex u in $H+s$. Then $H+s$ has a loopless 2-connected $(m_1, m_2, \ldots, m_q)$-detachment if and only if*
(a) $H+s$ is 2-edge-connected,
(b) $b(v)+e(v) = 1$ for all $v \in V$,
(c) $m_2 + m_3 + \ldots + m_q \geq b(s) + e(s) + q - 2$, and
(d) $d(s, V-v) + e(s) \geq q + b(s,v) - 1$ for all $v \in V$.

Let $G_4 + s$ be the graph obtained from $(G'+s) - K - \cup_{i=2}^{r} C_i$ by adding $p := (d_{G'+s}(s) - 2d_1)/2 - 1$ loops at s. Note that p is a nonnegative integer by Claim 4.38 and the fact that $d_{G'+s}(s)$ is even. Applying Lemma 4.42 to $G_4 + s$ we deduce:

Claim 4.43. *G_4+s has a loopless 2-connected $(d_2^*, d_3, \ldots, d_{r-1})$-detachment G_5, where $d_2^* = d_2' + d_r - 2$.*

Proof. Since $G'+s$ is (k,s)-connected and G_4 is connected and loopless, it follows that G_4+s satisfies Lemma 4.42(a) and Lemma 4.42(b). Using $d_i \geq 2$ for all $3 \leq i \leq r-1$ and (24), we have $d_2' + d_r \leq d_{G'+s}(s) - d_1 - 2(r-3) \leq d_{G'+s}(s) - d_{G'+s}(s)/2 + r - 2 - 2(r-3) = d_{G'+s}(s)/2 - r + 4$. Thus $d_3 + \ldots + d_{r-1} = d_{G'+s}(s) - d_1 - d_2' - d_r \geq d_{G'+s}(s) - d_1 - d_{G'+s}(s)/2 + r - 4 = 1 + e(s) + r - 4$, proving that Lemma 4.42(c) holds for G_4+s. To show that Lemma 4.42(d) holds focus on a vertex $v \in V(C_1)$. Considering the graph $G' - (K+v)$ and using Claim 4.36, we have $b^*(G') \geq b_{G_4}(v) + r - 1 + \beta$, where $\beta = 2$ if $d_{G'+s}(s,v) = 2$ and $\beta = 0$, otherwise, since if $d_{G'+s}(v) = 2$ then $d_{G'}(v) = k-2$. Thus, by (23), $b^*(G') = b^*(\hat{G}) \leq \lceil t(\hat{G})/2 \rceil = d_{G'+s}(s)/2$, we have $d_{G'+s}(s)/2 \geq b_{G_4}(v) + r - 1 + \beta$. Thus

$$\begin{aligned} d_{G_4+s}(s, V(C_1) - v) + e(s) &= d_1 - d_{G'+s}(s,v) + e(s) \\ &= d_{G'+s}(s)/2 - 1 - d_{G'+s}(s,v) \\ &\geq b_{G_4}(v) + r - 1 + \beta - 1 - d_{G'+s}(s,v) \\ &\geq (r-2) + b_{G_4+s}(s,v) - 1, \end{aligned}$$

as required. □

Label the detached vertices of G_5 as $c_2, c_3, c_4 \ldots, c_{r-1}$ where $d_{G_5}(c_i) = d_i$ for $3 \leq i \leq r-1$ and $d_{G_5}(c_2) = d_2^*$. The edge $e = c_j y_1$ is in $E(G_5)$ for some $2 \leq j \leq r-1$. We next subdivide the edge e with a new vertex c_r to form the graph G_5', and then 'flip' some edges from c_2 to c_r in G_5' preserving 2-connectivity and increasing the degree of c_r up to d_r while maintaining the property that y_1 and y_1' are joined to different detached vertices. We use the following result [13, Corollary 2.17].

Lemma 4.44. *[13] Let $t \geq 3$ be an integer. Let H be a loopless 2-connected graph, $x, y \in V(H)$ and $xz_i \in E(H-y)$ for $1 \leq i \leq t$. If $t \geq d(y) - d(y,x) + 1$, then $H - xz_i + yz_i$ is loopless and 2-connected for some i, $1 \leq i \leq t$.*

We construct a new graph G_6 from G_5' as follows. If $d_r = 2$ then we put $G_6 = G_5'$. If $d_r \geq 3$ then we use Lemma 4.44 to find a set of edges $S = \{c_2 z_i \in E(G_5') : 1 \leq i \leq d_r - 2\}$ such that $c_2 y_1' \notin S$ and $G_6 = G_5' - S + \{c_r z_i : 1 \leq i \leq d_r - 2\}$ is 2-connected and loopless. This is possible since $d_{G_5'}(c_r) = 2$, $d_{G_5'}(c_r, c_2) \leq 1$, and $d_{G_5'}(c_2) = d_2' + d_r - 2 \geq d_r + d_r - 2$. In G_6 we have $y_1 c_r \in E(G_6)$, $y_1' c_r \notin E(G_6)$, $d_{G_6}(c_i) = d_i$ for $3 \leq i \leq r$, and $d_{G_6}(c_2) = d_2'$. (Note that we could have used Lemma 4.42 directly to construct a 2-connected loopless detachment with the same degree sequence as G_6 from $G_4 + s$ plus one extra loop at s. The reason we go via G_5 is to ensure that $y_1 . y_1'$ are adjacent to distinct detached vertices.)

Let F be a set of edges joining the components of $G' - K$ such that $d_F(v) = d_{G'+s}(s, v)$ for all $v \in V - K$ and such that the graph obtained from $(V - K, F)$ by contracting $C_2, \ldots, C_r$ to $c_2, c_3, \ldots, c_r$, respectively, is G_6. Since G_6 is 2-connected, each vertex c_i in G_6 has at least two distinct neighbours. Let y_i, y_i' be the neighbours of s in C_i defined after Claim 4.40. Since we may interchange the end vertices of the edges of F within each component, C_i, for $2 \leq i \leq r$ we may choose F to have the additional property that, for $2 \leq i \leq r$, the two edges of F incident to y_i and y_i' join C_i to different vertices of $G - K - C_i$, which either belong to different components of $G - K - C_i$, or both belong to C_1. Furthermore, since y_1 and y_1' are joined to different detached vertices in G_6, the two edges of F incident to y_1 and y_1' join C_1 to different components of $G' - K - C_1$.

We can now use Claim 4.39 to deduce that $\hat{G} + F$ is k-connected as in Subcase 2.1. Putting $F_0 = E(\hat{G}) - E(G)$ we deduce that $F_0 \cup F$ is the required augmenting set of edges for G of size $\lceil d_{G+s}(s)/2 \rceil = \lceil t(G)/2 \rceil$. □

5 Algorithmic aspects and corollaries

In this section we discuss the algorithmic aspects of our results and also show that our main theorems imply (partial) solutions to a number of conjectures in this area.

The proofs of our min-max theorems (Theorems 4.27 and 4.34) are algorithmic and lead to a polynomial algorithm which finds an optimal augmenting set with respect to k for any l-connected input graph G and target $k \geq l + 1$, provided $a_k(G) \geq 10(k - l + 2)^3 (k + 1)^3$ (or $a_k(G) \geq 20k^3$, if $k = l + 1$). As we shall see, the running time in this case can be bounded by $O(n^5)$, even if k is part of the input. Our algorithm for the general case first decides whether $a_k(G)$ is large, compared to k, or not. Since, by Lemma 2.8, $a_k(G)$ is large if and only if $d(s)$ is large in a k-critical extension $G + s$ of G, the first step is to create such an extension. If $a_k(G)$ is small then our algorithm performs an exhaustive search on all possible augmenting sets F with $V(F) \subseteq N(s)$ and outputs the smallest augmenting set which makes G k-connected. The number of possibilities depends only on k, since $|N(s)|$ is also small. It follows from the next lemma that the output is indeed an optimal solution.

Lemma 5.1. *Let $G + s$ be a (k, s)-connected extension of G. Then there exists an optimal augmenting set F of G with respect to k with $V(F) \subseteq N(s)$.*

Proof. Let $S := N(s)$ and let F be an optimal augmenting set with respect to k for which $c(F) := \sum_{uv \in F} |\{u, v\} - S|$ is as small as possible. Suppose $c(F)$ is positive and let $uv \in F$ be an edge with $\{u, v\} - S \neq \emptyset$. Since F is optimal, we have $\kappa(G+F-uv) = k-1$ and, by Lemma 4.14(c), it follows that $G+F-uv$ has precisely two cores (i.e. minimal k-deficient sets) X, Y. Clearly, X and Y are k-deficient fragments in G. Thus, since $G+s$ is (k, s)-connected, we must have $S \cap X \neq \emptyset \neq S \cap Y$. Lemma 4.14(c) also implies that by taking $F' = F - uv + xy$ for a pair x, y of vertices with $x \in S \cap X$ and $y \in S \cap Y$ we have that $G + F'$ is k-connected. Now $|F'| = |F|$ and $c(F') < c(F)$, contradicting the choice of F. This proves that $c(F) = 0$ must hold, and hence the required augmentning set exists. □

Thus if $a_k(G)$ is small then we need to perform c_k k-connectivity tests, where c_k depends only on k, to find an optimal solution. If $a_k(G)$ is large then our algorithm has several steps, according to the different subcases in our proofs. In what follows we give a sketch of the algorithm to verify that it can be run in polynomial time. We do not attempt to work out the details of an efficient implementation. The input of the algorithm is a graph $G = (V, E)$ with $\kappa(G) = l$, and a positive integer $k \geq l + 1$, satisfying $|V| \geq k + 1$.

Algorithm

Step 1. (Extension) Create a k-critical extension $G + s$ of G. If $l = k - 1$ and $d_{G+s}(s) \geq 20k^3+1$ then go to Step 3. If $l \leq k-2$ and $d_{G+s}(G) \geq 10(k-l+2)^3(k+1)^3+1$ then go to Step 4. Else go to Step 2.

Step 2. (Exhaustive search) Check all possible augmenting sets F of size at most $d_{G+s}(s) - 1$ with $V(F) \subseteq N(s)$. Output the smallest set F for which $G + F$ is k-connected.

Step 3. (Augment by one) Check if G is k-independence free. If not, go to Step 3B.

Step 3.A (Independence free case) Make G k-connected by iteratively splitting off pairs of edges incident to s. (If $d(s) = 3$ or $d(s) = b(G)$ in the current graph then add a tree on $N(s)$ and terminate. If $d(s) \geq 2b(G) - 1$ then split off an arbitrary admissible pair. If $d(s) \leq 2b(G) - 2$ then split off as desribed in the proof of Lemma 3.7.)

Step 3.B (Make it independence free) Make G k-independence free by splitting off pairs of edges as described in Lemmas 4.25 and 4.26. (Note that $T(G)$ need not be computed when we want to increase the number of passive cores by making an active core passive. If we fail to make an active core B passive, which means $B \cap T(G) \neq \emptyset$, then we can try another one.) Before performing the next split always check whether $d(s) \leq 2b(G) - 2$ holds. If yes, go to Step 3.C. Otherwise, continue splitting until G becomes k-independence free, and then go to Step 3.A with the current graph and its extension.

Step 3.C (Forest augmentation) Make G k-connected by a forest augmentation, as in Theorem 4.11. (This is a special case of the general forest augmentation defined after Lemma 4.3.)

Step 4. (Large shredder) If there is a shredder K with $d(s) \leq 2b^*(G) - 2 = 2b^*(K) - 2$ then find one and go to Step 4.A. Else go to Step 5.

Step 4.A (Forest augmentation) Make G k-connected by splitting off all edges from s to K and then adding a forest augmentation, as described in Lemma 4.7 and after Lemma 4.3.

Step 5. (Augment by at least two) Construct $H+s$, H^*+s, and G^*+s as described before Lemma 4.29. Check whether any subsequence of the splittings, found in $H+s$, makes $d(s) = 2b^*(G) - 2$ if we perform it in $G+s$. If yes, go to Step 4.A with the graph in which the equality is first attained. If not, check if $d_{H^*+s}(s) \geq 2f+1$ holds. If yes, go to Step 5.B. Else go to Step 5.A.

Step 5.A (Make it $(k-1)$-connected) Make G $(k-1)$-connected by splitting off pairs of edges, as described in Lemma 4.32. If $d(s) = 2b^*(G) - 2$ holds after some iteration, go to Step 4.A. When G becomes $(k-1)$-connected, go to Step 3.

Step 5.B (Augment by detachments) Augment G following of the steps of the proof of Case 2 of Theorem 4.34. (In detail, find a big shredder K of size $k-2$, rearrange edges to make G_0+s, split off edges within components of H^*-K as long as possible to make $\hat{G}+s$, possibly add an edge to make $d(s)$ even, and then find a good augmenting set obtained from a loopless 2-connected graph G_3 or G_5, possibly after flipping some edges.) If $d(s) = 2b^*(G) - 2$ occurs after splitting off some edges, while making $\hat{G}+s$, go to Step 4.A.

This algorithm is well-defined and outputs an optimal augmenting set of G with respect to k by the results and proofs of this paper. Most of the steps of this algorithm are easy to implement in polynomial time by network flow techniques. There are two exceptions: (a) how to find a shredder K with $d(s) \leq 2b(G) - 2 = 2b(K) - 2$ (in Step 3) or with $d(s) \leq 2b^*(G) - 2 = 2b^*(K) - 2$ (in Steps 4-5), if it exists; (b) how to find the required loopless 2-connected graphs (detachments) in Section 6. We shall not discuss (b) in this paper but remark that there is a simple algorithm which finds G_3, if it exists, and we also have a similarly simple and efficient algorithm which finds G_5, if it exists.

Question (a) was answered in [14], and in more detail in [3], when G is $(k-1)$-connected and we search for a shredder K with $d(s) \leq 2b(G) - 2 = 2b(K) - 2$. The next lemma provides an answer in the general case. Note that we need to answer this question only if $d(s)$ is large compared to k.

Lemma 5.2. *If $d(s) \geq k(k-l+1)+2$ then we can decide in polynomial time whether $d(s) \leq 2b^*(G) - 2$ holds, and if yes, a shredder with $b^*(K) = b^*(G)$ can also be found in polynomial time.*

Proof. We show how to find a family $\mathcal{K}$ of shredders in such a way, that $|\mathcal{K}|$ is polynomial in $|V|$ and if there is a shredder K with $d(s) \leq 2b^*(G) - 2 = 2b^*(K) - 2$ then $K \in \mathcal{K}$. Once we have $\mathcal{K}$, we can answer question (a) by computing $b^*(K')$ for all $K' \in \mathcal{K}$.

To generate the family $\mathcal{K}$ we proceed as follows. We choose the neighbours of s one by one and for each vertex $x \in N(s)$ we try to split off all copies of the edge sx by admissible splittings. If this is not possible, we continue with the next vertex of $N(s)$. If it is possible, we find $k-1$ vertex-disjoint paths $P_1, P_2, ..., P_{k-1}$ between all pairs of vertices of $N(s) - x$ for which such paths exist. Let u, v be such a pair, let $Q = \cup_{i=1}^{k-1} V(P_i)$, and let $C_1, ..., C_l$ be the components of $G - Q$. For each component C_i, $1 \leq i \leq l$, if $n_G(C_i) = k - 1$ then we put $N_G(C_i) \in \mathcal{K}$ and if $n_G(C_i) = k - 2$ then we put $(N_G(C_i) \cup q) \in \mathcal{K}$ for all $q \in Q - \{u, v\}$. Family $\mathcal{K}$ is complete when this procedure has been executed for all neighours of s. Clearly, $|\mathcal{K}|$ is polynomial in $|V|$.

Suppose there is a shredder K with $d(s) \leq 2b^*(G) - 2 = 2b^*(K) - 2$. Then Lemma 4.3 and Lemma 4.7 imply that $|N(s) \cap K| \leq 1$ and if $x \in N(s) \cap K$ then $b^*(K) = b(K) + d(s, x)$ and we can split off all copies of sx (in any order) by admissible splittings. By splitting off these copies $d(s)$ is reduced by $2d(s, x)$ and $b^*(K)$ is reduced by $d(s, x)$. Hence $d(s) \leq 2b(K) - 2$ holds in the resulting graph. This implies that K has at least two leaves C_1, C_2 and, by Lemma 4.3(a), $G - K$ has at least one component $C \neq C_1, C_2$ with $n_G(C) \geq k - 2$. By Lemma 2.12 there exist $k - 1$ vertex-disjoint paths from $u \in N(s) \cap C_1$ to $v \in N(s) \cap C_2$, and for the union Q of these paths we have that each component $D \neq C_1, C_2$ of $G - K$ is a component of $G - Q$ by Lemma 2.11. Thus C is a component of $G - Q$ and, since $K \subseteq Q - \{u, v\}$, either $K = N(C)$ or $K = N(C) + q$ holds for some $q \in Q - \{u, v\}$. This proves $K \in \mathcal{K}$, as required. $\square$

Before stating our bound on the running time we note that by inserting a preprocessing step, which works in linear time, we can make the input graph sparse, and hence reduce the running time, as follows. Let $G = (V, E)$ and k be the input of our problem. Let $n = |V|$ and $m = |E|$. It was shown in [2] and [18] that $G = (V, E)$ has a spanning subgraph $G' = (V, E')$ with $|E'| \leq k(n - 1)$ satisfying $\kappa(u, v, G') \geq \min\{k, \kappa(u, v, G)\}$ for each pair $u, v \in V$. It can be seen that by replacing G by G' we do not change the k-deficient fragments (or their deficiencies) and that for any augmenting set F the graph $G + F$ is k-connected if and only if $G' + F$ is k-connected. Thus we can work with G' and assume that $m = O(kn)$. Note also that $d(s) = O(kn)$ in any extension $G + s$ of G we work with in the algorithm. By using these facts and efficient network flow algorithms for the basic operations (such as finding admissible splittings, checking whether an edge is k-critical, etc) we can conclude with the following theorem.

Theorem 5.3. *Given a graph $G = (V, E)$ and a positive integer k, our Algorithm finds an optimal augmenting set with respect to k. If $a_k(G) \geq 10(k - l + 2)^3(k + 1)^3$ then the running time is $O(n^5)$. Otherwise the running time is $O(c_k n^3)$.*

5.1 Corollaries

Our main results (Theorems 4.27 and 4.34) imply (partial) solutions to several related conjectures. The extremal version of the connectivity augmentation problem is to find, for given parameters n, k, t, the smallest integer m for which every k-connected graph on n vertices can be made $(k + t)$-connected by adding m new edges. Several special cases of this problem were solved in [16] and it was conjectured that (at least if n is large enough compared to k) the extremal value for $t \geq 2$, $k \geq 2$ is $\lceil nt/2 \rceil$ (or $\lfloor nt/2 \rfloor$, depending on the parities of n, k, t). Since $b^*(G) - 1 \leq n$, the min-max equality of Theorem 4.34 shows that if n is large enough and $t \geq 2$ then the $a_k(G)$ is maximised if and only if G is (almost) k-regular. This proves the conjecture (when n is large compared to k), by noting that such (almost) regular graphs exist for $k \geq 2$.

A different version of this problem, when the graphs to be augmented are k-regular, was studied in [8]. It was conjectured there that if G is a k-regular k-connected graph on n vertices, and n is even and large compared to k, then G can be made $(k+1)$-connected by adding $n/2$ edges. If G is k-regular, $b(K) \leq k$ for any cut of size k. Thus if n is large enough, we have $\max\{b(G)-1, \lceil t(G)/2 \rceil\} = n/2$. Now the conjecture follows from Theorem 4.27.

A similar question is whether $a_k(T) = \lceil (\sum_{v \in V(T)} (k - d(v))^+)/2 \rceil$ holds when graph T is a tree, where $x^+ = \max\{0, x\}$ for some number x. It is known that the minimum number of edges needed to make a tree k-edge-connected (or an arborescence k-edge- or k-vertex-connected) is determined by the sum of the (out)degree-deficiencies of its vertices. As above, using the fact that $b^*(G) - 1 \leq n$, Theorem 4.34 implies (when n, and hence also $a_k(T)$, is large compared to k) that if $k \geq 3$ then $a_k(T) = \lceil t(T)/2 \rceil$. That is, $a_k(T)$ is determined by the total deficiency of a family of pairwise disjoint subsets of $V(T)$. Since T is a tree, each member X of this family induces a forest. This implies that there exists a vertex $v \in X$ with $k - d(v) \geq k - n(X)$. Therefore we can find a family consisting of singletons with the same total deficiency. This yields an affirmative answer to our question provided $k \geq 3$ and n is large compared to k. Note that the answer is negative for $k = 2$.

Frank and Jordán [7, Corollary 4.8] prove that every $(k-1)$-connected graph $G = (V, E)$ can be made k-connected by adding a set F of new edges such that (V, F) consists of vertex-disjoint paths. They conjectured that such an F can be found among the optimal augmenting sets as well. We can verify this, provided $a_k(G)$ is large enough. In this case we may use the min-max formula of Theorem 4.27. If $a_k(G) = \lceil t(G)/2 \rceil$ then an optimal augmenting set is obviously a collection of vertex-disjoint paths. If $a_k(G) = b(G) - 1$, then a careful analysis of the forest augmentation method shows that we can find an optimal augmenting set F satisfying $d_F(v) \leq 2$ for all $v \in V$. Since F is a forest, it induces vertex-disjoint paths, as claimed.

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