Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2001-03. Published by the Egrerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

A note on the path-matching formula

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January 2001

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Abstract

As a common generalization of matchings and matroid intersections, W. H. Cunningham and J. F. Geelen introduced the notion of path-matchings. They proved a minmax formula for the maximum value of a path-matching. with the help of a linear algebraic method of Tutte and Lovász. Here we exibit a simplified version of their minmax theorem and provide a purely combinatorial proof.

1 Introduction

W. H. Cunningham and J. F. Geelen in [3] and [4] introduced the notion of path-matchings as a common generalization of the weighted matching problem and the weighted matroid intersection problem.

They proved that this problem is solvable in polinomial time via the ellipsoid method [6]. They also proved the total dual integrality of the corresponding linear system.

Cunningham and Geelen defined a path-matching as follows. Let $G = (V, E)$ be an undirected graph and T_1, T_2 disjoint stable sets of G, we call this two sets the terminal sets of G. We denote $V - (T_1 \cup T_2)$ by R. Let M_1 and M_2 be two rank r matroids on T_1 and T_2, respectively. A *basic path-matching* is a subset K of edges E such that the subgraph $G_K = (V, K)$ is a collection of r disjoint paths, all of whose internal nodes are in R, linking a basis of M_1 to a basis of M_2, together with a perfect matching of the nodes of R not in any of the paths. An *independent path-matching* with respect to M_1, M_2 is a set K of edges such that every component of the subgraph $G_K = (V, K)$ having at least one edge is a simple path from $T_1 \cup R$ to $T_2 \cup R$, all of whose internal nodes are in R, and such that the set of nodes of T_i in any of these paths is independent in M_i, for $i = 1$ and 2. The value of a path-matching is defined to be the number

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Research supported by the Hungarian National Foundation for Scientific Research Grant, OTKA T17580. The second author is supported by the Siemens-ZIB Fellowship Program.

of the edges contained in it plus the number of its one-edge-components in R, that is, each one of these edges counts twice (these edges are called *matching edges*. For example the value of a basic path-matching is $r + |R|$.

If M_1 and M_2 are free matroids, then we refer to a basic path-matching as a *perfect path-matching* and to an independent path-matching as a *path-matching*.

A pair of subsets $I_1 \subseteq T_1 \cup R$, $I_2 \subseteq T_2 \cup R$ is called *stable* if no edge of G joins a node in $I_1 - I_2$ to a node in I_2 or a node in $I_2 - I_1$ to a node in I_1. We denote by $c(G)$ the number of components of G having an odd number of nodes. For a subset S of nodes of G, $G[S]$ denotes the subgraph of G induced by S.

W. H. Cunningham and J. F. Geelen in [3] proved the following theorem with the help of a certain generalization of Tutte-matrix.

Theorem 1.1. (Maximum path-matching formula)

$$\max_{M \text{path-match.}} val(M) =$$

$$\min_{(I_1, I_2) \text{ stable pair}} |T_1 \cup R - I_1| + |T_2 \cup R - I_2| + |I_1 \cap I_2| - c(G[I_1 \cap I_2])$$

Corollary 1.2. $|T_1| = |T_2| = k$. *There exists a perfect path-maching if and only if*

$$|I_1 \cup I_2| + c(G[I_1 \cap I_2]) \leq n \quad \textit{for all stable pairs } (I_1, I_2).$$

As a consequence of the TDI-ness Cunningham and Geelen derived the following formula in the case of independent path-matchings in [4].

Theorem 1.3. (Maximum independent path-matching formula)

$$\max_{M \text{indep.p-m.}} val(M) = |R| +$$

$$\min_{(I_1, I_2) \text{ stable pair}} r_1(T_1 - I_1) + r_2(T_2 - I_2) + |R - (I_1 \cup I_2)| - c(G[I_1 \cap I_2])$$

Corollary 1.4. $r(M_1) = r(M_2) = r$. *There exists a basic path-maching if and only if*

$$r_1(T_1 - I_1) + r_2(T_2 - I_2) + |R - (I_1 \cup I_2)| \leq r + c(G[I_1 \cap I_2])$$

for all stable pairs (I_1, I_2).

In this note we provide a simplified characterization for the existence of a perfect path-matching. This form is a direct extension of Tutte's theorem on perfect matchings and permits us to provide a combinatorial proof by mimicking Anderson's simple proof on Tutte's theorem [1]. Then we prove a simplified form of the maximum path-matching formula. Our proofs can easily be extended to the case of basic and independent path-matchings.

2 A simplified form of the maximum path- matching formula

We define a *cut* separating the tarminal sets T_1 and T_2 to be a subset $X \subseteq V$ such that there is no path between $T_1 - X$ and $T_2 - X$ in $G - X$. (See Figure 1.)

From now on we denote by $odd_G(X)$ the number of connected components of $G - X$ which are disjoint from $T_1 \cup T_2$ and have an odd number of nodes. Let $Odd_G(X)$ denote the union of these components. If it does not cause misunderstanding, then we omit the index.

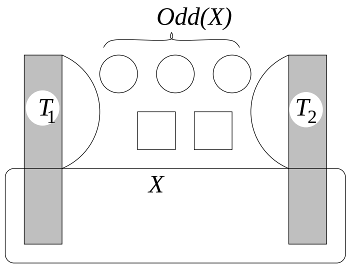

Figure 1: A cut X separating T_1 and T_2

Theorem 2.1. *In $G = (V, E)$ there exists a perfect path-matching if and only if $|T_1| = |T_2| = k$ and*

$$|X| \geq odd_G(X) + k \qquad \textit{for all cuts } X. \tag{1}$$

Theorem 2.2. (Maximum path-matching formula 2)

$$\max_{M \text{path-matching}} val(M) = |R| + \min_{X \text{cut}} (|X| - odd_G(X))$$

In this note we prove Theorem 2.1, then we derive Theorem 2.2. It is clear that, if $T_1 = T_2 = \emptyset$, then we get Tutte's theorem and Berge-Tutte-formula immediately. (Recall the definition of the value of a path-matching.) As Cunnigham and Geelen showed in [4], Menger's theorem on the number of node-disjoint paths can be proved through a simple construction from Corollary 1.2, and so from Theorem 2.1:

Suppose we are given a graph $(G' = V', E')$ whose nodeset is partitioned into sets T_1', T_2', R with $|T_1'| = |T_2'| = k$. We wish to find, if possible, k node-disjoint path from T_1' to T_2'. The construction is the following: form a new graph G by adding, for every $r \in R$ nodes r_1, r_2 and edges rr_1, rr_2, r_1r_2, and put $T_1 := R_1 \cup T_1', T_2 := R_2 \cup T_2'$, where R_i denotes $\{r_i : r \in R\}$. Then there exists a perfect path-matching of G with respect to terminal sets T_1, T_2 if and only if the desired paths exist in G'.

Menger's Theorem states that the disjoint paths exist if and only if there is no set S that separates T_1' from T_2' in G' and has cardinality less than k. The necessity of this condition is obvious. If G' does not contain the desired k paths, then there exists cut X separating T_1 and T_2 in G such that

$$|X| < (k + |R|) + odd_G(X).$$

$X \cap R$ is a separating set in G' with cardinality less than k.

Of course, Corollary 1.2 and Theorem 2.1 are equivalent. Theorem 2.1 implies Corollary 1.2 in this way: if (1) does not hold for the cut X, then a stable pair, which violates the condition of Corollary 1.2, can be found in the following way: $I_1 \cup I_2 := V - X, I_1 \cap I_2 :=$ the union of the components of $G - X$ which are entirely in R.

Our proof can be extended to prove the following theorems by using basic matroidal methods.

Theorem 2.3. *In $G = (V, E)$ there exists a basic path-matching if and only if $r(M_1) = r(M_2) = r$ and*

$$r_1(T_1 \cap X) + r_2(T_2 \cap X) + |R \cap X| \geq r + odd_G(X)$$

for all cuts X.

Theorem 2.4. (Maximum independent path-matching formula 2)

$$\max_{M \text{indep.path-m.}} val(M) = \min_{X \text{cut}} r_1(T_1 \cap X) + r_2(T_2 \cap X) + |R \cap X| - odd_G(X).$$

Theorem 2.3 implies Edmonds' theorem on the maximum cardinality of a common independent set of two matroids [5]. Theorem 2.4 contains Brualdi's theorem [2] as a special case.

A cut X is *trivial* if $X = T_1$ or $X = T_2$. A cut X is defined to be *tight* if $|X| = odd(X) + k$, that is, the condition (1) is satisfied by equality.

A graph $G = (V, E)$ is said to be *factorcritical* if it is connected and, for every $Y \subseteq V and |Y| \geq 2$, $G - Y$ has at most $|Y| - 1$ number of components with odd number of nodes.

Proof of Theorem 2.1

Necessity of (1). Let us consider a perfect path-matching M. Let $P_1, P_2, \ldots\ P_k$ be denote the k paths, and let α be the number of the $Odd(X)$ components which are traversed by some P_i, and let β be the number of $Odd(X)$ components which are not traversed by any P_i. For a path P_i, let t_i denote the number of $Odd(X)$ components which are traversed by P_i.

It is clear, that

$$k + \alpha + \beta \leq \sum_{i=1}^{k} (t_i + 1) + \beta \leq |X|,$$

for all cuts X. (Remark: if cut X is tight, then an odd component K is traversed either by one path in a perfect path-matching M, or there is only one matching edge leaving K in M.)

The proof of sufficiency goes by induction on $|R|+|E|$. When $|R|=0, |E| \leq 1$ the theorem is obviously true.

CASE 1: There does not exist any nontrivial tight cut.

If $k=0$, then every cut which has cardinality one is nontrivial and tight, hence $k>0$. Let us consider an edge $e=uv$ with $u \in T_1$. Let G' denote $G-e$. If the condition (1) is satisfied in G', then we are done by induction. Suppose now that G' does not satisfy (1), that is there is a cut X in G' so that $|X| < odd_{G'}(X)+k$. Since $|X| \geq odd_G(X)+k$, v is in an odd component of $G-X$ or is in a path from T_1-X to T_2-X and in both cases $u \in T_1-X$. In the first case $odd_G(X)+k \leq |X| < odd_{G'}(X)+k = odd_G(X)+1+k$, so $|X| = odd_G(X)+k$, X is tight and nontrivial. (Figure 2a.) In the second case $|X| < odd_{G'}(X)+k$. $X+u$ and $X+v$ is a cut in G, so $|X+u| \geq odd_G(X+u)+k = odd_{G'}(X)+k$, and the same is true for $X+v$. (Figure 2b.) We get $|X+u| = odd_G(X+u)+k$ and $|X+v| = odd_G(X+v)+k$, so they are tight cuts. If none of them is nontrivial, then $k=1, X=\emptyset$, and this case can be checked easily.

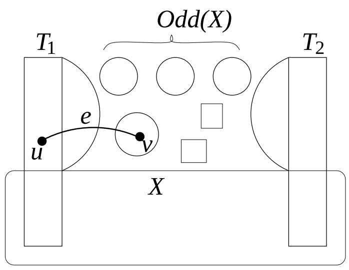

Figure 2

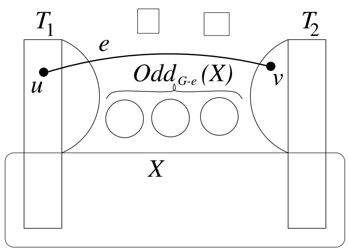

Figure 3

CASE 2: There exists a nontrivial tight cut.

Let us consider a maximal nontrivial tight cut X. It is clear that every component of $G-X$, which are in entirely in R, is factorcritical (specially odd). Indeed, in an odd component is not factorcritical, then let us put its maximal cut with maximum deficiency into X. Furthermore, if there exists a component with even number of nodes, then let us put a single node of it into X, we get a bigger nontrivial tight cut in G.

Let us contract each component of $Odd_G(X)$ to a node. It will not cause misunderstanding if we denote these nodes by $Odd_G(X)$ as well.

The *left-hand side* of G is the induced subgraph by the nodes: $Odd(X) \cup (X - T_1) \cup (T_1 - X) \cup \{$the nodes that can be reached along a path from $T_1 - X$ in $G - X\}$. Similarly the *right-hand side* of G is induced by $Odd(X) \cup (X - T_2) \cup (T_2 - X) \cup \{$the nodes that can be reached along a path from $T_2 - X$ in $G - X\}$. (Figure 3.) Note that these two graphs have common nodes.

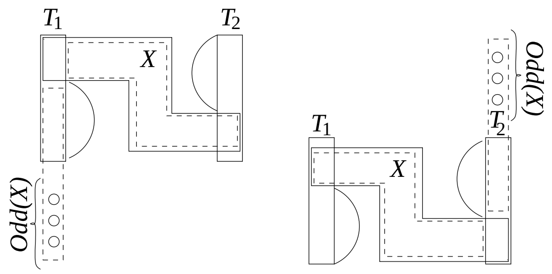

Figure 4

Claim 2.5. *On the left-hand side there exists a perfect path-matching respect to terminal sets $T_1' = (T_1 - X) \cup Odd(X), T_2' = X - T_1$.*

Proof. $|T_1'| = |T_2'|$ because of the tightness of X. If $X \cap R \neq \emptyset$, then we can apply the inductive hypothesis. If $X \cap R = \emptyset$, then $T_1 \cap X \neq \emptyset$ and $T_2 \cap X \neq \emptyset$ (X is nontrivial!) hence we can apply the inductive hypothesis.

Consequently if $|Y| \geq odd_{\text{leftside}}(Y) + (k - |T_1 \cap X| + odd_G(X))$ for every cut Y on the left-hand side, then there exists a perfect path-matching on the left-hand side with respect to the new terminal sets T_1' and T_2'. (We denote by odd_{leftside} the *odd* operator on the left.)

Let us suppose that there exists a cut Y such that $|Y| < odd_{\text{leftside}}(Y) + k - |T_1 \cap X| + odd(X)$. (See Figure 4.) We get

$$|Y| + |T_1 \cap X| - |Odd(X) \cap Y| < odd_{\text{leftside}}(Y) + odd(X) - |Odd(X) \cap Y| + k,$$

that is, $Z = (T_1 \cap X) \cup (Y - Odd(X))$ is a cut in G after replacing $Odd(X)$ components for which (1) does not hold. It is trivial that Z is indeed a cut in G. □

We can see similarly that on the right-hand side there exists a perfect path-matching respect to terminal sets $T_2'' = (T_2 - X) \cup Odd(X), T_1'' = X - T_2$.

It is easy to get a perfect path-matching of G from the perfect path-matching M_1 on the left and the perfect path-matching M_2 on the right. Recall that the components of $Odd(X)$ were factorcritical so we can complete $M_1 \cup M_2$ suitably (by the two facts about factorcritical graphs mentioned after this proof) and we are able to replace even circuits by matching easily. And now we finished the proof of Theorem 2.1 (See Figure 5.) □

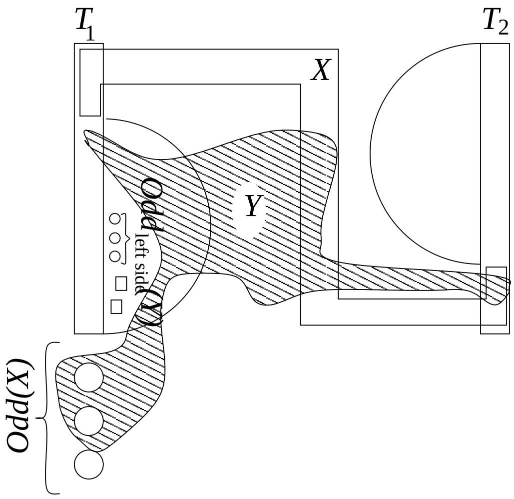

Figure 5

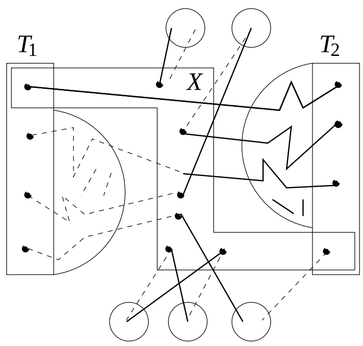

Figure 6

At the last step of the above proof we used the following two facts about factorcritical graphs. For every node, there exists a matching covering all the nodes but one. For every two nodes, there exists a path between them such that there exists a perfect matching on the nodes not in the path. These facts followed from the induction hypothesis.

Proof of Theorem 2.2 Let us suppose that $|T_1 \cup R| = l \geq k = |T_2 \cup R|$, $\min_{X \text{cut}}(|X| - odd(X)) = m$. Let us add $l-k$ nodes to T_2, and let us put an edge between all these nodes and every node in $T_1 \cup R$. Let us add $k-m$ nodes to R and put an edge between all these nodes and every node in $V \cup \{\text{all the new nodes}\}$. We added $(l-k)+(k-m) = l-m$ new nodes to G. Let G' denote the obtained graph. If a cut Y of G' does not contain at least one new node, then it must contain T_1' or T_2', thus: $|Y| - odd_{G'}(Y) \geq l$.

A cut X of G together with the new nodes form a cut of G'. So $\min_{Y \text{cut}} |Y| - odd(Y)$ $= m + (l-m) = l$, consequently there exists a perfect path-matching M in G' and the

value of such a path-matching: $val(M) = |R \cup \{\text{new nodes in } R'\}| + l = |R| + k - m + l$.
$E \cap M$ is trivially a maximal path-matching in G. □

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