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A fixed-point approach to stable matchings and some applications

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Abstract

We describe a fixed-point based approach to the theory of bipartite stable matchings. By this, we provide a common framework that links together seemingly distant results, like the stable marriage theorem of Gale and Shapley [11], the Menelsohn-Dulmage theorem [21], the Kundu-Lawler theorem [19], Tarski's fixed point theorem [32], the Cantor-Bernstein theorem, Pym's linking theorem [22, 23] or the monochromatic path theorem of Sands *et al.* [29]. In this framework, we formulate a matroid-generalization of the stable marriage theorem and study the lattice structure of generalized stable matchings. Based on the theory of lattice polyhedra and blocking polyhedra, we extend results of Vande Vate [33] and Rothblum [28] on the bipartite stable matching polytope.

Keywords: Stable matchings; Lattices; Matroids; TDI; Lattice polyhedra; Blocking polyhedra

1 Introduction

In 1962, Gale and Shapley published their pioneering paper [11] on the now called stable matching theorem, that asserts the existence of a bipartite matching with a nonstandard stability criterion. The result is described in a "marriage model" that turned out to be an extremely applicable one in describing certain two-sided economies, like job matching markets or auctions (see [27]). For this reason, there is a strong interest towards the theory of stable matchings from Game Theory and Mathematical Economics. But beyond this, stable matchings are also considered as a particular topic in the theory of bipartite matchings; the stable matching algorithm is studied by the Computer Science community (see [18, 13]), and the description of the stable matching polytope [33, 28] indicates a connection to Combinatorial Optimization. More recently, Galvin solved the Dinitz conjecture by proving the list

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colouring conjecture for bipartite graphs [12]. In the proof (in spite of the denial of the author), the stable matching theorem plays the key role.

The "marriage model" of Gale and Shapley is a most natural one. At least, somewhat after that Gale and Shapley have published their result, it turned out that already ten years before the appearance of the stable matching theorem, a centralized scheme was introduced (the then called National Intern Matching Program) that produced a stable assignment between American medical students and hospitals (see [24]). In this present paper, we date the history of the stable matching theorem even further back in time by showing that as early as 1927, a more general statement has been proved by Knaster and Tarski (see [17]), as a corollary of a set theoretical fixed point theorem. The Cantor-Bernstein theorem is a standard application of this fixed point theorem of Knaster and Tarski, and we are g oing to demonstrate that this fundamental result in set theory follows directly from an appropriate generalization of the stable matching theorem.

However, our treatment is not the first fixed point based approach to the theory of stable matchings. Concentrating on the algorithmic aspects of the nonbipartite stable matching problem, Feder [6] and Subramanian [31] used a network model to decide whether there exists a stable matching in a given graph-model. That is, they could decide the so called network stability problem for a certain class of logical networks. To do this, they reduced the problem to the decision of the existence of a fixed point of certain set-functions. Subramanian even observed that in case of the bipartite stable matching problem the crucial set-function is monotone so it has a fixed point because of the Knaster-Tarski fixed point theorem. A difference between our approach and those that are based on stable networks is that we use a fixed-point theorem and we cannot handle the nonbipartite problem, while the former methods decide whether a fixed point of a certain set function exists to solve a nonbipartite problem. In our approach, we deduce a generalization of the Gale-Shapley theorem directly from the Knaster-Tarski fixed point theorem. This generalization includes well-known generalizations of Kelso and Crawford [16] and of Roth [25, 26] on stable assignments of workers and firms. Further, in the comonotone framework other special properties of stable matchings (like the lattice structure) can be formulated and treated in a direct way.

The robustness of our fixed-point based approach also allows us to give interesting new generalizations of the Gale-Shapley theorem and to point out links between stable matchings and other seemingly distant results. An example is the matroid generalization of the stable matching theorem that answers a question of Roth [26], by providing a matroid-based explanation of several well-known properties of stable matchings. As a special case, this matroid generalization also contains the generalization of the Mendelsohn-Dulmage theorem [21] by Kundu and Lawler [19]. We point out some other unexpected links to Combinatorics by deriving the monochromatic path theorem of Sands *et al.* [29] and the linking theorem of Pym [22, 23] in our framework.

This paper is organized as follows. In Section 2, we survey some well-known facts and generalizations on bipartite stable matchings like the one of Kelso and Crawford [16] or of Roth [25, 26]. We point out that the monochromatic path theorem of Sands *et al.* is also a generalization of the Gale-Shapley theorem.

In Section 3, we describe our main tool, the lattice theoretic fixed-point theorem of Tarski [32]. It states that the fixed points of a monotone function on a complete lattice form a nontrivial lattice subset of the original lattice. (Actually, throughout the paper we only use Tarski's fixed point theorem for subset-lattices, and essentially, this special case is an earlier theorem of Knaster and Tarski [17]). Our reason for using the more general approach is that proofs become somewhat easier and the lattice subset property of fixed points (that we use later heavily) is explicitly stated in Tarski's formulation). We derive the Cantor-Bernstein theorem (a standard application of Tarski's fixed point theorem) from the infinite version of the Mendelsohn-Dulmage theorem that in turn follows from an infinite version of the Gale-Shapley theorem.

Through the definition of comonotone functions, we introduce our comonotone framework in Section 4, and we prove our main tool, Theorem 4.2 as a simple corollary of Tarski's fixed point theorem. We show that Roth's worker-firm assignment theorem [25, 26] is a straightforward consequence of Theorem 4.2 and we generalize the proposal algorithm of Gale and Shapley to the comonotone framework.

Section 5 is devoted to applications of the stable marriage theorem to graph paths. We prove Pym's theorem [22, 23] together with another result on edge-disjoint paths. In Section 6, with the help of the greedy algorithm of Edmonds, we formulate a matroid generalization of the bipartite stable matching theorem and deduce a matroid generalization of the Mendelsohn-Dulmage theorem: the Kundu-Lawler theorem [19].

After this, we focus on lattice properties of generalized stable matchings. For this reason, we recall some lattice-related notions. On a *lattice* we mean an a four-tuple $L = (X, <, \wedge, \vee)$ so that $<$ is a partial order on X in such a way that any two elements x and y of X have a unique greatest lower bound (the so called *meet* of x and y, denoted by $x \wedge y$) and a unique lowest upper bound (the so called *join* of x and y denoted by $x \vee y$). As an abuse of notation, we can say that $L = (X, <)$ is a lattice if $<$ is a *lattice order*, that is, operations $\wedge$ and $\vee$ are well-defined for any two elements x and y of X. On the other hand, if L is a lattice then we can reconstruct the underlying partial order $<$ from any of the lattice operations: $x \leq y$ if and only if $x \wedge y = x$ if and only if $x \vee y = y$. In this sense, we can consider a lattice as an algebraic structure $L = (X, \wedge, \vee)$, if both operations $\wedge$ and $\vee$ determine the same relation $<$ which is a partial order and $\wedge$ and $\vee$ are the lattice operations of $<$. The two different approaches to lattices give different meanings for the notion of substructure. We say that lattice $L' = (X', <')$ is a *lattice subset* of lattice $L = (X, <)$, if $X' \subseteq X$ and partial order $<'$ is a restriction of $<$ to X'. Lattice $L' = (X', \wedge', \vee')$ is a *sublattice* of lattice $L = (X, \wedge, \vee)$, if $X' \subseteq X$ and operations $\wedge'$ and $\vee'$ are restrictions of $\wedge$ and $\vee$ on X', respectively. It follows from the definition that any sublattice is a lattice subset, but the two notions are not the same. For example, in Tarski's fixed point theorem, it can happen that the lattice subset of fixed points does not form a sublattice of the original lattice.

So in Section 7, we return to Tarski's fixed point theorem and concentrate on the lattice structure of fixed points. With the help of the comonotone framework, we prove a related result of Blair that justifies the lattice structure of stable matchings in Roth's worker-firm assignment model.

A well-known observation, attributed to Conway, is that on stable marriage schemes there are natural operations that define a lattice structure on stable matchings. These operations can be defined in the comonotone framework as well, and we study whether the structure becomes a lattice or not. This problem is related to the question whether the lattice subset of fixed points in Tarski's theorem is a sublattice of the original lattice. We exhibit a property (the so called strong monotonicity) that ensures this, and we formulate the increasing property that corresponds to strong monotonicity in the comonotone framework. We verify that most of the interesting generalizations (in particular the matroid generalization) of the stable matching theorem share this increasing property, and we show consequences for the matroid model.

Based on the lattice structure of generalized stable matchings, we characterize in Section 8 certain polyhedra that naturally emerge in the comonotone framework. To do this, we apply the theory of lattice polyhedra of Hoffman and Schwartz [14] and the theory of blocking polyhedra by Fulkerson [8, 9, 10]. What we prove extends results of Vande Vate [33] and Rothblum [28] on the stable marriage polytope.

We end this section by recalling some notations that we will need later on. We denote the set of reals, nonnegative reals, nonpositive reals, natural numbers and positive integers by $\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-, \mathbb{N}$ and $\mathbb{N}_+$, respectively. The notation $[n]$ stands for the set of the first n positive integer. The *Minkowski sum* of subsets A and B of $\mathbb{R}^n$ is $A + B := \{a + b : a \in A, b \in B\}$. The cone and the convex hull of A is defined by

$$\begin{aligned} \operatorname{cone}(A) &:= \{\sum_{i=1}^{n} \lambda_i a_i : n \in \mathbb{N}_+, a_i \in A, \lambda_i \in \mathbb{R}_+\} \qquad \text{and} \\ \operatorname{conv}(A) &:= \{\sum_{i=1}^{n} \lambda_i a_i : n \in \mathbb{N}_+, a_i \in A, \lambda_i \in \mathbb{R}_+, \sum_{i=1}^{n} \lambda_i = 1\}, \end{aligned}$$

respectively. If A is a subset of a ground set X then the *characteristic vector*, χ^A of A is defined by $\chi^A(x) := 1$ if $x \in A$ and $\chi^A(x) := 0$ otherwise.

For a graph $G = (V, E)$ with vertex set V and edge set E, we denote by $d_G(v) := d(v)$ the degree of vertex v, that is the number of edges incident with v. The notation $D_G(v) := D(v)$ stands for the set of edges incident with v (that is, $d = |D|$) and $\Gamma(v)$ denotes the set of neighbours of v. For a function $b : V \to \mathbb{N}$ a *b-matching* is a subset E' of E such that $d_{G'} \leq b$, for subgraph $G' := (V, E')$ of G. A 1-matching is called a *matching*.

2 Bipartite stable matchings: some properties and generalizations

In 1962, Gale and Shapley published the following result [11]:

Theorem 2.1 (Gale-Shapley). *If, each of n men and n women ranks the members of the opposite sex as a marriage partner, then there is a so-called stable marriage scheme, that is a scheme of n marriages pairing the $2n$ persons in such a way that* *no man and woman can be found who mutually prefer each other to their marriage partner.* □

Gale and Shapley proved their result algorithmically, that is, they gave a method that produces a stable scheme in finite time. Their proposal (originally called "deferred acceptance") algorithm works in rounds as follows. In the beginning of each round, there is an underlying bipartite graph with one vertex corresponding to each person and the edges represent possible marriages. (In the very beginning, we have complete bipartite graph $K_{n,n}$.) In a round each man selects his most preferred partner from the graph, and proposes to her. Then, each women refuses all proposals but the one that arrived from the most preferred proposer. Those edges of the bipartite graph along which a proposal is refused get deleted, and the next round starts. If, in a round, no refusal takes place, then the proposals of the particular round describe a stable marriage scheme.

Gale and Shapley proved that although for the same instance, there might be more than one stable schemes possible, the above algorithm finds the so called *man-optimal* one. That is, in this particular matching, each man gets the best possible partner that he can have in any of the stable schemes. By exchanging the role of men and women in the proposal algorithm, one can also prove the existence of a *woman-optimal* matching, in which each woman receives the best possible partner that she can have in a stable scheme. It is also true that in the man-optimal scheme each woman gets the worst partner that she can have in a stable scheme, and the same is true for men in the woman-optimal matching. The existence of the above optimal schemes also follow from an observation attributed to John Conway on the lattice structure of stable matchings: if two stable marriage schemes are given and each man chooses the better partner from the two, then it results in a stable marriage scheme in which each woman receives the worse partner from the two matchings. By exchanging the role of men and women, we get another operation on stable schemes, and it is possible to show that with these operations, the set of stable schemes becomes a lattice.

In [11], Gale and Shapley also described a more general framework in which students and colleges play the role of men and women. In that model, each student has a preference list on colleges, each college has a ranking on students plus a quota for the places that it may fill up. Using a straightforward modification of the proposal algorithm, it was shown that there is a stable assignment in which no college C and student s can be found such that s prefers C to the college that s is assigned to and either the quota of C is not filled up, or C ranks s higher than some other student assigned to C. In this model, man- and woman-optimality can be generalized appropriately (here we speak about student- and college-optimality), the student-optimal assignment is worst for colleges, and vice versa. Moreover, we can define lattice operations on stable assignments similarly as in the marriage model. Further, it is also true that if a college can not fill up its quota in some stable scheme then in each stable assignment it receives the very same set of students.

In fact, what is claimed above is still true and can be proved with the appropriately modified proposal algorithm for the case where the underlying bipartite graph of the model is not complete, i.e., when by default, certain assignments are not allowed.

Here, the definition of stability must change of course, in such a way that we postulate that each agent prefers to be assigned any way rather than not at all. Formally, there is the following infinite version of the stable marriage theorem.

Theorem 2.2. *Let $G = (A \cup B, E)$ be a bipartite (multi)graph with colour classes A and B, and for each vertex v of G, let $<_v$ be a well-order on $D(v)$. Then there is a matching M of G such that for any edge e of G there is a vertex v incident with e and an edge f of M also incident with v such that $f \leq_v e$ holds.*

The matching described in Theorem 2.2 is called *stable.* If matching M is not stable then there is an edge e of G for which the conclusion of Theorem 2.2 does not hold. Such an edge is called a *blocking edge.*

A key observation in this paper is that all the linear orders of one colour class in Theorem 2.2 together define a partial order on the edge set E of G. That is, we can define partial order $<_A$ (and $<_B$) on E by $e <_A f$ (and $e <_B f$) if $e <_v f$ for some $v \in A$ (and $v \in B$, respectively). Then the conclusion of Theorem 2.2 is that for any edge e, there is an edge f such that $f \leq_A e$ or $f \leq_B e$ holds. It follows that there are subsets E_A and E_B of E such that $E_A \cup E_B = E$ and $M = E_A \cap E_B$ is the set of $<_A$-minima of E_A and also the set of $<_B$-minima of E_B.

The following theorem is a generalization of Theorem 2.2 along these lines. To formulate an infinite version, we call partial order $<$ on ground set X a *partial well-order* if any subset Y of X has a $<$-minimum. Or, equivalently, if any linearly ordered subset Y of X is well-ordered by $<$. For partial orders $<_1$ and $<_2$ on X, a subset S of X is a *stable antichain* if it is a common antichain of $<_1$ and $<_2$ and contains a lower bound for all other elements, i.e. if

$$\text{the elements of } S \text{ are pairwise both } <_1\text{- and } <_2\text{-incomparable, and} \tag{1}$$

$$\text{for each } x \in X \text{ there exists an } s \in S \text{ such that } s \leq_1 x \text{ or } s \leq_2 x. \tag{2}$$

Theorem 2.3.

A If $<_1$ and $<_2$ are partial orders on X, then there are subsets X_1 and X_2 of X such that

$$X_1 \cup X_2 = X \textit{ and} \tag{3}$$

$$X_1 \cap X_2 \textit{ is the set of } <_i \textit{-minima of } X_i \textit{ for } i \in \{1, 2\}. \tag{4}$$

B Moreover, if $<_1$ and $<_2$ are partial well-orders then there exists a stable antichain for these partial orders.

Part B of Theorem 2.3 is a special case of the following monochromatic path theorem of Sands *et al.* [29]. Here we state a slight generalization of that.

Theorem 2.4 (Sands *et al.* [29]). *Let A_1 and A_2 be arc-sets on vertex-set V, such that there is no $i \in \{1, 2\}$ and vertices v_j of V (for $j \in \mathbb{N}$) such that*

$$\begin{array}{l}\textit{there is a simple } A_i\textit{-path from } v_j \textit{ to } v_{j+1} \textit{ and} \\ \textit{there is no simple } A_i\textit{-path from } v_{j+1} \textit{ to } v_j.\end{array} \tag{5}$$

Then there is a subset K of V such that

$$\text{for each element } v \in V \text{ there is a simple path in } A_1 \text{ or in } A_2 \text{ from } v \text{ to } K, \text{ and} \tag{6}$$

$$\text{there is neither a simple } A_1\text{-, nor a simple } A_2\text{-path between different elements of } K. \tag{7}$$

To deduce Theorem 2.3 B from Theorem 2.4, define arc set A_i by $xy \in A_i$ if $y <_i x$ for $i \in \{1, 2\}$. Then (5) is a consequence of the partial well-ordered property, (6) is equivalent with (2) and (7) with (1). On the other hand, we can deduce Theorem 2.4 from Theorem 2.3 the following way.

Proof of Theorem 2.4. Let $<$ be a well-order on V, i.e. $<$ is a linear order and every subset of V has a $<$-minimal element. The existence of such a well-order follows from the axiom of choice; this is actually the only place in our treatment where we use this axiom. For $i \in \{1, 2\}$ define $<_i$ such that $u <_i v$ if and only if

$$\text{there is a simple } A_i\text{-path from } v \text{ to } u, \tag{8}$$

and

$$u < v \text{ or there is no simple } A_i\text{-path from } u \text{ to } v. \tag{9}$$

Relation $<_i$ is transitive because if $x <_i y <_i z$ and there is a zx-path of A_i, then x, y and z are in the same strong A_i-component, so $x < y < z$ must hold.

If $x \le_i y \le_i x$ then x and y are in the same strong component. Thus $x \le y \le x$, that is $x = y$. It means that $<_i$ is antisymmetric. As $<_i$ is trivially reflexive, it is a partial order, indeed.

Next we check that $<_i$ is pwo, i.e. any subset U of V has a $<_i$-minimal element, for $i \in \{1, 2\}$. From (5), there is an element u of U with the property that if there is a simple A_i-path from u to some u' then there is a simple A_i-path from u' to u. Consider $U' := \{x \in U : \text{there is a simple } A_i\text{-path from } u \text{ to } x\}$. By definition, orders $<$ and $<_i$ are the same on U', so the $<$-minimal element of U' is a $<_i$-minimal element of U as well.

As any stable antichain K of $<_1$ and $<_2$ satisfies (6, 7), Theorem 2.4 follows directly by applying Theorem 2.3 B to partial well-orders $<_1$ and $<_2$. □

The stable matching theorem of Gale and Shapley has been generalized by several authors. For more details than what we are going to present, the reader should consult especially Chapter 6 of the book of Roth and Sotomayor [27]. Here, we review those results that are closely connected to our topic.

Continuing on a paper of Crawford and Knoer [5], Kelso and Crawford [16] extended the college (or many-to-one) model to a model where workers are to be assigned to firms. Firms would like to have certain specific jobs to be done, and this is why they have a more sophisticated preference function on workers than plain ranking. Namely, each firm f has a so called choice function C_f that selects from any subset

W' of workers a subset $C_f(W')$ of W' that firm f would hire if on the labour-market only firm f and workers in W' would be present. Set-function $C : 2^W \to 2^W$ is a *choice function* if there is a well-order $<$ on 2^W such that $C(W')$ is the $<$-minimal subset of W', for any subset W' of W. In the model of Crawford and Knoer, each firm has a choice function and each worker has an ordinary preference ranking on firms.

An assignment of workers to firms is called *stable* if it is not blocked by a worker-firm pair. Worker-firm pair (w, f) *blocks* an assignment if w prefers f to his/her assignment and in the meanwhile firm f would take worker w if w would be available (that is $w \in C_f(W_f \cup \{w\})$, where W_f is the set of workers assigned to firm f).

Not surprisingly, in the above model there might be no stable assignment. However, if each choice function has the so-called substitutability property, then a stable assignment always exists. We say that choice function $C_f : 2^W \to 2^W$ of firm f has the *property of substitutability*, if

$$w \in C_f(W') \text{ implies } w \in C_f(W' \setminus \{w'\}) \tag{10}$$

for any subset W' of the set of workers W and for any two different workers w, w' of W'. This means that if a firm would like to employ a certain worker, then it still would like to hire him/her if some other worker leaves the labour-market.

Theorem 2.5 (Crawford-Kelso [5]). *If firms have substitutable preferences in the worker-firm assignment model, then there is a stable assignment.*

The proof of Crawford and Kelso is via the accordingly modified Gale-Shapley algorithm. They also observed that firm-proposing results in the firm-optimal assignment, and the worker-proposal based method leads to the worker-optimal situation. In [25, 26], Roth extended Theorem 2.5 to the many-to-many model.

Theorem 2.6 (Roth [25, 26]). *Let F and W be disjoint finite sets, and for each $f \in F$ and $w \in W$ let $C_w : 2^F \to 2^F$ and $C_f : 2^W \to 2^W$ set functions with substitutability property (10). Then there is bipartite assignment graph A with colour classes F and W, such that for any $w \in W$ and $f \in F$ we have that $wf \in E(A)$ if and only if $f \in C_w(\Gamma_A(w) \cup f)$ and $w \in C_f(\Gamma_A(f) \cup w)$.*

Clearly, the stable marriage theorem of Gale and Shapley is a special case of Theorem 2.6, where the choice functions simply select the highest ranked partner from the input. For the college model, the choice function of a college selects the best inputs that still fit with the quota.

In [26], Roth studies three models: the one-to-one, the many-to-one and the many-to-many with substitutable preferences. He shows that for all three models there is a firm-optimal, "worker-pessimal" and a worker-optimal, "firm-pessimal" stable assignment. The name "polarization of interests" refers to this property. Roth also observes the "opposition of common interests" of workers and firms, which means that if all workers prefer some stable outcome at least as much as some other, then for firms the opposite holds.

Further on, Roth introduced the notion of *consensus property*, by which he means the following. If each agent on one side of the market combines his/her most preferred assignment from a set of stable assignments, then this way another stable assignment is constructed. This is a generalization of the lattice property of stable schemes in the marriage model. Unfortunately, this property does not always hold in Theorem 2.6. In [26], Roth asked whether some lattice structure can still be defined on stable assignments. Blair answered this question positively [2]. His idea was that instead through lattice operations, he defined the stable assignment lattice by introducing a more or less natural partial order on stable assignments and it turned out that that order defines a lattice that generalizes the lattice property of bipartite stable matchings.

3 Tarski's fixed point theorem

In this section, we describe the lattice-theoretic fixed point theorem of Tarski, our main tool to handle stable assignment-related problems.

Lattice $L = (X, <, \wedge, \vee)$ is *complete* if there is both a meet (i.e. a greatest lower bound) and an join (that is, a lowest upper bound) for any subset Y of X. These generalized meet and join operations on Y are denoted by $\bigwedge Y$ and $\bigvee Y$, respectively. Clearly, $\bigwedge X = \mathbf{0} \in X$ is the zero-element and $\bigvee X = \mathbf{1} \in X$ is the unit element of L and let, by definition, $\bigwedge \emptyset := \mathbf{1}, \bigvee \emptyset := \mathbf{0}$. Function $f : X \to X$ is *monotone* if $x \leq y$ implies $f(x) \leq f(y)$ for any elements x, y of X. The following fixed-point theorem of Tarski is a most important result on complete lattices:

Theorem 3.1 (Tarski [32]). *If $L = (X, <, \wedge, \vee)$ is a complete lattice and $f : X \to X$ is a monotone function, then $L_f := (X_f, <)$ is a nonempty, complete lattice subset of L, where $X_f := \{x \in X : f(x) = x\}$ is the set of fixed points of f.*[1]

Proof. Let Y be a (possibly empty) subset of X_f. By monotonicity of f, $f(\bigwedge Y) \leq f(y) = y$ for any $y \in Y$, hence $f(\bigwedge Y) \leq \bigwedge Y$. Define

$$K := \{k \in X : k \leq f(k) \wedge \bigwedge Y\}$$

and $l := \bigvee K$. Clearly, if $x = f(x) \leq \bigwedge Y$ for a fixed point x of f, then $x \in K$ and $x \leq l$. Hence it is enough to show that $f(l) = l$.

By definition, $k \leq l \leq y$ for any $k \in K$ and $y \in Y$. Thus by monotonicity, $k \leq f(k) \leq f(l) \leq f(y)$. This means that $l = \bigvee K \leq \bigvee\{f(k) : k \in K\} \leq f(l) \leq \bigwedge Y$, hence that $l \leq f(l) \leq \bigwedge Y$. Again, by monotonicity, $f(l) \leq f(f(l))$, that is $f(l) \in K$. We got that $l \leq f(l) \leq \bigvee K = l$. Thus l is indeed the meet of Y in X_f.

Obviously, $L^{-1} = (X, \geq)$ is a complete lattice as well, and f is monotone on L^{-1}. According to the above argument, any subset Y of X_f has a $\geq$-meet in X_f, that is a $\leq$-join in X_f.

We conclude that L_f is indeed a nonempty, complete lattice subset of L. □

[1]Theorem 3.1 seems to be proved first for lattice $(2^X, \subseteq)$ by Knaster and Tarski in as early as 1927. Birkhoff published a weaker form of Tarski's Theorem (cf. [1, p. 54]) where he proved only the existence of a fixed point and remarked later in an exercise that the set of fixed points is not necessarily a sublattice.

We remark that in case of finite lattices (that are clearly complete) there is an algorithmic proof for the existence of a minimal and a maximal fixed point in Theorem 3.1. The algorithm is based on the observation that by monotonicity, $\mathbf{0} \le f(\mathbf{0}) \le f(f(\mathbf{0})) \le \ldots$ holds. This increasing chain has to stabilize after some iterations at (say) $x := f^{(k)}(\mathbf{0}) = f^{(k+1)}(\mathbf{0}) = f(x)$, providing the zero-element of L_f. Similarly, if we start to iterate f form $\mathbf{1}$, then we get a decreasing chain, that stabilizes at the unit-element of L_f.

Next we recall a well-known set theoretical application of Theorem 3.1.

Theorem 3.2 (Cantor-Bernstein). *If $f : A \to B$ and $g : B \to A$ are injections between sets A and B then there is a bijection h between A and B.*[2]

Theorem 3.2 justifies the notion of cardinality, as it can be equivalently stated such that $|A| \le |B|$ and $|B| \le |A|$ implies $|A| = |B|$. Theorem 3.2 is a special case of the following well-known result from Graph Theory.

Theorem 3.3 (Mendelsohn-Dulmage [21]). *If $G = (U \cup V, E)$ is a bipartite graph with colour classes U and V, and M_1 and M_2 are matchings in G, then there is a matching M of G that covers all vertices U' of U that are covered by M_1 and all vertices V' of V that are covered by M_2.*

To see that Theorem 3.3 implies Theorem 3.2, we may assume that A and B are disjoint, and we can define matchings M_1 and M_2 as the underlying undirected graph of $(A \cup B, f)$ and $(A \cup B, g)$, respectively. (Remember that a function f is a set of ordered pairs, i.e. arcs.) As $M_1 + M_2$ is bipartite, by Theorem 3.3, there is a matching M of $M_1 + M_2$ covering all vertices of A covered by M_1 and all vertices of B covered by M_2. Hence M is a perfect matching between A and B, exhibiting a bijection between these sets.

On the other hand, we can deduce Theorem 3.3 from the infinite stable matching theorem as follows. Define linear order $<_u$ on $D(u)$ by $e <_u f$ for vertex u of U if e belongs to M_2 and f to M_1. Similarly, $e <_v f$ for vertex v of V if $e, f \in D(v)$ and $e \in M_1$ and $f \in M_2$. By Theorem 2.2, there is a stable matching M of G. As no edge of M_1 can be a blocking edge of M, each vertex of U that is covered by M_1 must be covered also by M. Similarly, no edge of M_2 blocks M, hence each vertex of V that is covered by M_2 must be covered by M. Thus M has the property required by Theorem 3.3.

4 Monotone and comonotone set-functions

In this section, we deduce a generalization of Theorem 2.6 from Theorem 3.1 and prove Theorems 2.3 and 2.6. By this, we justify what we have claimed in Section 2 without proof.

[2]Note that Theorem 3.2 has several names. Sometimes, it is called Schröder-Bernstein or Bernstein-Schröder. According to Levy's account [20], it has been proved by Dedekind in 1887, conjectured by Cantor in 1895 and proved again by Bernstein in 1898. Other sources talk about Schröder, giving a wrong proof in 1896.

A set function $f : 2^X \to 2^X$ is *monotone*, if $A \subseteq B \subseteq X$ implies $f(A) \subseteq f(B)$. We say that $\mathcal{F} : 2^X \to 2^X$ is *comonotone* if there is a monotone function $f : 2^X \to 2^X$ such that

$$\mathcal{F}(A) = A \setminus f(A) \quad \text{for } A \subseteq X. \tag{11}$$

In particular, if $\mathcal{F}$ is comonotone then $\overline{\mathcal{F}}$ is monotone, where

$$\overline{\mathcal{F}}(A) := A \setminus \mathcal{F}(A) = A \cap f(A), \text{ for } A \subseteq X. \tag{12}$$

It is easy to see that $\mathcal{F}$ is comonotone if and only if

$$\mathcal{F}(Y) \subseteq Y \text{ for any } Y \subseteq X, \tag{13}$$

and $\overline{\mathcal{F}}$ is monotone. Actually, checking these two properties is our standard way to decide comonotonicity of a set function.

The following simple statement gives an equivalent reformulation of the comonotone property. It implies for example that choice functions with substitutability property (10) (in Theorem 2.5 and 2.6) are comonotone.

Proposition 4.1. *Set function $\mathcal{F} : 2^X \to 2^X$ is comonotone if and only if (13) holds and*

$$\mathcal{F}(Y) \cap Y' \subseteq \mathcal{F}(Y') \qquad \textit{whenever } Y' \subseteq Y \subseteq X. \tag{14}$$

□

Proof. If (13) holds for $\mathcal{F}$ then monotonicity of $\overline{\mathcal{F}}$ is equivalent with (14). □

To formulate the main result of this section, a corollary of Theorem 3.1 for comonotone functions[3], we need further definitions. For $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$ we call (A, B) an $\mathcal{FG}$-*stable pair* if

$$A \cup B = X \text{ and} \tag{15}$$

$$\mathcal{F}(A) = A \cap B = \mathcal{G}(B). \tag{16}$$

We say that a subset K of X is an $\mathcal{FG}$-*kernel* if there is an $\mathcal{FG}$-stable pair (A, B) such that $K = A \cap B$. In such a situation, we say that $\mathcal{FG}$-stable pair (A, B) corresponds to $\mathcal{FG}$-kernel K. For set functions $\mathcal{F}$ and $\mathcal{G}$, the set of $\mathcal{FG}$-kernels is denoted by $\mathcal{K}_{\mathcal{FG}}$. We introduce partial order $\leq$ on $2^X \times 2^X$, by

$$(A, B) \leq (A', B') \text{ if } A \subseteq A' \text{ and } B \supseteq B'. \tag{17}$$

Note that $(2^X \times 2^X, \leq)$ is a complete lattice with lattice operations

$$(A, B) \wedge (A', B') = (A \cap A', B \cup B') \text{ and } (A, B) \vee (A', B') = (A \cup A', B \cap B'). \tag{18}$$

[3]Note that Tarski deduced a corollary in terms of Boolean algebras already in [32] that is more general than our Theorem 4.2.

Theorem 4.2. *If $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$ are comonotone functions then the set of $\mathcal{FG}$-stable pairs is a nonempty complete lattice subset of lattice $(2^X \times 2^X, \leq)$.*

Proof. Define $f : 2^X \times 2^X \to 2^X \times 2^X$ by

$$f(A, B) := (X \setminus \overline{\mathcal{G}}(B), X \setminus \overline{\mathcal{F}}(A)). \tag{19}$$

Clearly, $\mathcal{FG}$-stable pairs are exactly the fixed points of f.

If $(A, B) \leq (A', B')$ then $X \setminus \overline{\mathcal{F}}(A) \subseteq X \setminus \overline{\mathcal{F}}(A')$ and $X \setminus \overline{\mathcal{G}}(B) \supseteq X \setminus \overline{\mathcal{G}}(B')$, because $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ are monotone. Hence $f(A, B) \leq f(A', B')$, so f is monotone.

From Theorem 3.1, the set of fixed points of f (that is the set of $\mathcal{FG}$-stable pairs) is a nonempty lattice subset of $(2^X \times 2^X, \leq)$. □

We shall denote by $\wedge_{\mathcal{FG}}$ and $\vee_{\mathcal{FG}}$ the lattice operations of the lattice of $\mathcal{FG}$-stable pairs. To see Theorem 2.3 as a corollary of Theorem 4.2, we make the following observation.

Observation 4.3. Let $<$ be a partial order on X and for $A \subseteq X$, let $\mathcal{F}(A)$ be the set of $<$-minimal elements of A. Then $\mathcal{F}$ is a comonotone function on X.

Proof. $\overline{\mathcal{F}}(A)$ is the set of nonminimal elements of A, hence $\overline{\mathcal{F}}$ is a monotone function. □

Proof of Theorem 2.3. By constructing comonotone functions $\mathcal{F}$ and $\mathcal{G}$ from $<_1$ and $<_2$ according to Observation 4.3 and plugging them in Theorem 4.2, we get subsets X_1 and X_2 of X with properties (3,4). For part B of Theorem 2.3, define $S := X_1 \cap X_2$. By (4), S has property (1). Because of the partial well-ordered property of $<_1$ and $<_2$, for any elements x of X_1 and y of X_2, there are elements x' and y' of S such that $x' <_1 x$ and $y' <_2 y$. This proves property (2) of S, hence S is a stable antichain, indeed. □

We prove Theorem 2.6 with the idea of our "key observation" after Theorem 2.2. That is, from the choice functions of firms we define a joint choice function C_F on the edges of the bipartite graph between workers and firms, and for workers we construct a joint choice function similarly. Formally, for a set E' of edges between firms and workers, we define firm and worker choice functions by $C_F(E') := \{wf \in E' : w \in C_f(\Gamma_{E'}(f))\}$ and $C_W(E') := \{wf \in E' : f \in C_w(\Gamma_{E'}(w))\}$.

By induction, we see from substitutability property (10) that property (14) holds for any choice function C_f or C_w (for $f \in F$ and $w \in W$). So these choice functions are comonotone by Proposition 4.1. Hence joint choice functions C_F and C_W will be comonotone as well, because both of them are "direct sums" of comonotone functions. For the following proof, we also observe the important property of a choice function C that

$$C(A) \subseteq B \subseteq A \quad \Rightarrow \quad C(A) = C(B). \tag{20}$$

We remark without proof that property (20) for a comonotone function C is equivalent with the property that

$$C(A \cup B) = C(C(A) \cup C(B))$$

for any subsets A, B in the domain of C.

Proof of Theorem 2.6. Applying Theorem 4.2 to comonotone functions C_F and C_W provides edge-sets E_W and E_F such that $E_W \cup E_F = W \times F$. Define assignment graph A by $E(A) := E_W \cap E_F$. By property (20), $C_F(E(A)) = C_W(E(A)) = E(A)$. On the other hand, if $wf \in E_W \setminus E(A)$ then $f \notin \Gamma_A(w) = C_w(\Gamma_{E_F}(w)) = C_w(\Gamma_A(w) \cup \{f\})$, by the definition of joint choice functions, where the last equation follows from property (20) of C_w. Similarly, if $wf \in E_W \setminus E(A)$ then $f \notin C_w(\Gamma_A(w) \cup \{f\})$ holds. □

We recall that the original proof of Theorem 2.6 (and other stable matching related results) is via the appropriate modification of the proposal algorithm of Gale and Shapley. A related problem in the comonotone framework of Theorem 4.2 is the algorithmic construction of an $\mathcal{FG}$-kernel. That is, in case of a finite ground set X, we want to construct the $<$-minimum and the $<$-maximum $\mathcal{FG}$-stable pair for comonotone functions $\mathcal{F}$ and $\mathcal{G}$. To do this, we find the $<$-maximum and $<$-minimum fixed points of f in (19) according to the algorithm that we described in Section 3. That is, we iterate f starting from $(\emptyset, X)$ and $(X, \emptyset)$, respectively. This observation leads to the following method that generalizes the proposal algorithm of Gale and Shapley.

Let $A_0 := X$, $B_0 := \emptyset$ and define

$$B_{i+1} := X \setminus \overline{\mathcal{F}}(A_i) \text{ and } A_{i+1} := X \setminus \overline{\mathcal{G}}(B_i). \tag{21}$$

Then $(A^{max} := A_{|X|}, B^{min} := B_{|X|})$ is the $\leq$-maximal $\mathcal{FG}$-stable pair. If we start the recursion with $A_0 := \emptyset$ and $B_0 := X$, then (21) will produce the $\leq$-minimal $\mathcal{FG}$-stable pair (A^{min}, B^{max}). Note that this algorithm (just like the iterative method for monotone functions) can be extended to a transfinite induction proof of Theorem 4.2. The advantage of the method we have followed is that it does not lean on the axiom of choice and indicates an unexpected connection with lattice theory. Here I acknowledge András Biró for drawing my attention to the fixed-point theorem of Knaster and Tarski.

The interested reader can find a detailed analysis of the original proposal algorithm of Gale and Shapley in the book of Knuth [18].

5 Paths and stability

In Theorem 2.4, we have already seen a corollary on graph-paths. In what follows, we deduce the so called "linking theorem" of Pym [22, 23] as a special case of Theorem 2.3. Although, formally we prove an extension of Pym's result by showing extra property (22), the proof that we give is essentially Pym's [23]. Our aim here is only to indicate that this result can also be viewed in the comonotone framework. For a family $\mathcal{P}$ of paths, $In(\mathcal{P})$ and $End(\mathcal{P})$ denotes the set of initial and terminal vertices of paths in $\mathcal{P}$; ,$V(\mathcal{P})$ and $A(\mathcal{P})$ stands for the set vertices and arcs that occur in a path of $\mathcal{P}$, respectively.

Theorem 5.1 (Pym [22, 23]). *Let $D = (V, A)$ be a directed graph and X, Y be subsets of V. Let moreover $\mathcal{P}$ and $\mathcal{Q}$ be families of vertex-disjoint simple XY-paths.*

Then there exists a family $\mathcal{R}$ of vertex-disjoint simple XY-paths, such that

$$\text{any path of } \mathcal{R} \text{ consists of a (possibly empty) initial segment of a path of } \mathcal{P} \text{ and of a (possibly empty) end segment of a path of } \mathcal{Q}, \text{ moreover} \tag{22}$$

$$In(\mathcal{P}) \subseteq In(\mathcal{R}) \subseteq In(\mathcal{P} \cup \mathcal{Q}) \tag{23}$$

$$End(\mathcal{Q}) \subseteq End(\mathcal{R}) \subseteq End(\mathcal{P} \cup \mathcal{Q}). \tag{24}$$

Proof. To prove Theorem 5.1, it suffices to find a set S of switching vertices. Knowing S, we can construct vertex-disjoint path family $\mathcal{R}$ the following way. Define vertex-disjoint path-family $\mathcal{P}'$ as the set of paths of $\mathcal{P}$ disjoint from S together with the set of initial segments of paths of $\mathcal{P}$ that end in S. Similarly, we define $\mathcal{Q}'$ as the set of paths of $\mathcal{Q}$ disjoint from S and the end segment of $\mathcal{Q}$-paths starting from S. To obtain $\mathcal{R}$, we merge paths in $\mathcal{P}' \cup \mathcal{Q}'$ that start and end in the same vertex of S.

To make this construction work, subset S of V must have the following properties:

1. any path p of $\mathcal{P} \cup \mathcal{Q}$ contains at most one vertex from S, and

2. if v is a common vertex of path p of $\mathcal{P}$ and of path q of $\mathcal{Q}$ then either $v \in S$ or there is a vertex s of S before v on p or after v on q.

Define $<_{\mathcal{P}}$ on $V(\mathcal{P}) \cap V(\mathcal{Q})$ such that $u <_{\mathcal{P}} v$ if there is a uv-subpath of some path of $\mathcal{P}$. Define $<_{\mathcal{Q}}$ also on $V(\mathcal{P}) \cap V(\mathcal{Q})$ by $u <_{\mathcal{Q}} v$ if there is a vu-subpath of some path of $\mathcal{Q}$. Observe that properties 1. and 2. above describe exactly a stable antichain of $<_{\mathcal{P}}$ and $<_{\mathcal{Q}}$. As both relations are partial well-orders, Theorem 5.1 follows from Theorem 2.3 B. □

Note that in the above proof we did not use Theorem 2.3 in full generality. For finite vertex-set V, what we actually need is the Gale-Shapley theorem for multigraphs. In that model, paths of $\mathcal{P}$ correspond to men, paths in $\mathcal{Q}$ are women, and each common vertex of a $\mathcal{P}$-path and $\mathcal{Q}$-path yields a possible marriage. Each man would like to switch to a woman-path from his path as soon as possible and each woman would like to receive a man-path as late possible. (So everybody strives to minimize the part of his/her path that is used in $\mathcal{R}$.) A stable marriage scheme in this model is exactly a set of switching vertices of a family $\mathcal{R}$ as in Theorem 5.1.

Brualdi and Pym proved a modified version of the original linking theorem of Pym (Theorem 5.1 without (22)) where they require condition (26) but allow generalized paths [3]. A *generalized path* is either a circular path or an infinite path. A *circular path* is a sequence $v_0, a_1, v_1, \dots v_{t-1}, a_t, v_t$, where a_i is a $v_{i-1}v_i$ arc, $v_t = v_0$, otherwise all other vertices are different in the sequence. An *infinite path* is an infinite sequence $v_0, a_1, v_1, a_2, \dots$ or $\dots, a_{-1}, v_{-1}, a_0, v_0$ or $\dots, a_{-1}, v_{-1}, a_0, v_0, a_1, v_1, a_2, \dots$, in such a way that a_i is a $v_{i-1}v_i$ arc and all vertices v_i are different in the sequence. The above circular path have initial and terminal vertex v_0, the first type of infinite path has initial vertex v_0, and has no terminal vertex, the second infinite path has no initial vertex, but v_0 is its terminal vertex, and the third infinite path has neither initial nor terminal vertex.

Theorem 5.2 (Brualdi-Pym[3]). *In digraph $D = (V, A)$, let $\mathcal{P}$ and $\mathcal{Q}$ be families of vertex-disjoint generalized paths. There exists a family $\mathcal{R}$ of vertex-disjoint general paths of D such that*

$$In(\mathcal{P}) \subseteq In(\mathcal{R}) \subseteq In(\mathcal{P} \cup \mathcal{Q}) \qquad End(\mathcal{Q}) \subseteq End(\mathcal{R}) \subseteq End(\mathcal{P} \cup \mathcal{Q}) \tag{25}$$
$$V(\mathcal{P}) \cap V(\mathcal{Q}) \subseteq V(\mathcal{R}) \subseteq V(\mathcal{P} \cup \mathcal{Q}) \qquad A(\mathcal{P}) \cap A(\mathcal{Q}) \subseteq A(\mathcal{R}) \subseteq A(\mathcal{P} \cup \mathcal{Q}). \tag{26}$$

Note that although this theorem sounds similar to Theorem 5.1, it seems to be substantially different. To be able to prove condition (26), we *must* drop condition (22), as even if both $\mathcal{P}$ and $\mathcal{Q}$ consist of finite simple paths, it might be necessary to use both circular and infinite paths in $\mathcal{R}$ (see [3]). For a simple proof of Theorem 5.2, based on node-splitting, see Ingleton and Piff [15].

The following corollary is also observed by others (see e.g. [4]) and provides an interesting application of Theorem 5.1 on families of edge-disjoint (rather than vertex-disjoint) paths. In [4], by Conforti *et al.*, Corollary 5.3 is deduced directly from the stable matching theorem on bipartite multigraphs, using the framework we described after the proof of Theorem 5.1.

Corollary 5.3. *Let $G = (V, E)$ be an undirected graph and x, y, z be different vertices of V. Let $\mathcal{P}$ be a set of k edge-disjoint xy-paths and $\mathcal{Q}$ be a set of k edge-disjoint yz-paths. Then there exist a set $\mathcal{R}$ of k edge-disjoint xz-paths such that each path of $\mathcal{R}$ is the union of a (possibly empty) initial segment of a path of $\mathcal{P}$ and of a (possibly empty) end segment of a path of $\mathcal{Q}$.*

To prove the above result, we apply Theorem 5.1 on the line-graphs of paths of $\mathcal{P}$ and $\mathcal{Q}$. (A line-graph of a path is a path again.) There still remain some tiny details to take care of. This is done in the following.

Proof. Let vertex-disjoint path-families $\mathcal{P}', \mathcal{Q}'$ be the collection of the line-graphs of the paths in $\mathcal{P}$ and in $\mathcal{Q}$, respectively. By applying Theorem 5.1 on $\mathcal{P}'$ and $\mathcal{Q}'$ we get a vertex-disjoint path collection $\mathcal{R}'$. Family $\mathcal{R}'$ is the set of line-graphs of a set $\mathcal{R}^*$ of edge-disjoint walks. (These walks are not necessarily paths). Clearly, $|\mathcal{R}^* \cap \mathcal{P}| = |\mathcal{R}^* \cap \mathcal{Q}|$, so we can pair those paths and merge them via y. By this operation, $\mathcal{R}^*$ becomes a collection of edge-disjoint xz-walks. To obtain $\mathcal{R}$ as described in the corollary, we have to shortcut the possible circles on each element of $\mathcal{R}^*$. When no more shortcut is possible, we get edge-disjoint xz-paths switching exactly once, as stated. $\square$

Using Corollary 5.3 in [4], Conforti *et al.* described a Gomory-Hu based maxflow-representing structure. For each edge uv of a Gomory-Hu tree of a graph G, they store a list of $\lambda_G(u, v)$ edge disjoint uv paths. They also do it for some other $|V(G)|$ pairs uv of vertices of G. Then, by applying the stable marriage algorithm $O(\alpha(n))$ times as in Corollary 5.3, they construct a collection of $\lambda_G(x, y)$ edge-disjoint xy-paths of G for any two vertices x and y of G (where $\alpha(n)$ is the inverse Ackerman-function of n that is regarded almost as good as a constant function).

6 Matroid-kernels

There is a well-known matroid generalization of the Mendelsohn-Dulmage theorem.

Theorem 6.1 (Kundu-Lawler [19]). *Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ be two matroids on the same ground set, and let $I_1, I_2 \in \mathcal{I}_1 \cap \mathcal{I}_2$ be two common independent sets. Then there is a common independent set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ such that $\mathrm{span}_{\mathcal{M}_1}(I_1) \subseteq \mathrm{span}_{\mathcal{M}_1}(I)$ and $\mathrm{span}_{\mathcal{M}_2}(I_2) \subseteq \mathrm{span}_{\mathcal{M}_2}(I)$.*

While in case of matchings, it was more or less natural to prove the Mendelsohn-Dulmage theorem in the comonotone framework, here it is not at all that clear how the fixed point theorem of Tarski can be applied. However, if we approach matroids from the greedy property, then a comonotone function emerges immediately. For this reason, we review some properties of the greedy algorithm of Edmonds for the deletion minor of a matroid.

Let $\mathcal{M} = (E, \mathcal{C})$ be a matroid on ground set E and let $c : E \to \mathbb{R}_+$ be a cost function on $E = \{e_1, e_2, \ldots, e_n\}$ such that $c(e_i) \leq c(e_{i+1})$ for $1 \leq i < n$. Define set $K_n(E')$ recursively for any subset E' of E by $K_0(E') = \emptyset$ and for $0 \leq i \leq n$

$$K_i(E') = \begin{cases} K_{i-1}(E') & \text{if } e_i \notin E' \text{ or} \\ & \text{if there is a subset } C \text{ of } K_{i-1}(E') \\ & \text{such that } \{e_i\} \cup C \in \mathcal{C} \\ K_{i-1}(E') \cup \{e_i\} & \text{else.} \end{cases} \tag{27}$$

Claim 6.2. *The above defined function $K_n : 2^E \to 2^E$ is comonotone, and set $K_n(E')$ is a minimum cost subset of E' that spans E'. Moreover, if c is injective then this minimum cost spanning set is unique for any subset E' of E.*

Proof. Property (13) holds for K_n and by definition,

$$\overline{K_n}(E') := E' \setminus K_n(E') = \{e_i \in E' : \exists C \subseteq \{e_1, e_2, \ldots, e_{i-1}\} \text{ with } C \cup \{e_i\} \in \mathcal{C}\}$$

is a monotone function. Hence K_n is comonotone. The other facts are well-known. $\square$

For matroids $\mathcal{M}_1 = (E, \mathcal{C}_1)$ and $\mathcal{M}_2 = (E, \mathcal{C}_2)$ and cost functions $c_1, c_2 : E \to \mathbb{R}$, we say that (E_1, E_2) is an *$\mathcal{M}_1\mathcal{M}_2$-stable pair of E* if $E_1 \cup E_2 = E$ and $E_1 \cap E_2$ is a minimum c_i-cost spanning set of E_i in $\mathcal{M}_i$ for $i \in \{1, 2\}$. We call subset K of E an *$\mathcal{M}_1\mathcal{M}_2$-kernel* if it is a common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$ and if for every $e \in E \setminus K$ there is an $i \in \{1, 2\}$ and a subset C_e of K such that $\{e\} \cup C_e \in \mathcal{C}_i$ and $c_i(c) \leq c_i(e)$ for every $c \in C_e$. Set K is called a *dual $\mathcal{M}_1\mathcal{M}_2$-kernel* if it spans both $\mathcal{M}_1$ and $\mathcal{M}_2$ and for every element k of K there exists an $i \in \{1, 2\}$ and a subset C_k^* of $E \setminus K$ such that $C_k^* \cup \{k\}$ is a cocircuit of $\mathcal{M}_i$ with $c_i(k) \leq c_i(c)$ for all $c \in C_k$. Observe that if $\mathcal{M}_1 = \mathcal{M}_2$ and $c_1 = c_2$ then both an $\mathcal{M}_1\mathcal{M}_2$-kernel and a dual $\mathcal{M}_1\mathcal{M}_2$-kernel is a minimum cost basis of $\mathcal{M}$, so it can be constructed with the above greedy algorithm as $K_n(E)$. In this sense, we can regard matroid kernels and dual kernels as generalizations of minimum cost spanning sets.

Theorem 6.3. *Let $\mathcal{M}_1 = (E, \mathcal{C}_1)$ and $\mathcal{M}_2 = (E, \mathcal{C}_2)$ be matroids and $c_1, c_2 : E \to \mathbb{R}_+$ be cost functions on their common ground set. Then there is an $\mathcal{M}_1\mathcal{M}_2$-stable pair (E_1, E_2) of E and an $\mathcal{M}_1\mathcal{M}_2$-kernel K.*

Proof. For $A \subseteq E$ let $\mathcal{F}(A)$ be the minimum c_1-cost $\mathcal{M}_1$-spanning set $K_n(A)$ of A, constructed according to (27), and $\mathcal{G}(A)$ be the similarly constructed minimum c_2-cost $\mathcal{M}_2$-spanning set of A. From Claim 6.2, $\mathcal{F}$ and $\mathcal{G}$ are comonotone. So by Theorem 4.2, we have subsets E_1 and E_2 of E such that (E_1, E_2) is an $\mathcal{FG}$-stable pair. Define $K := E_1 \cap E_2$. By the mincost spanning property, for each $i \in \{1, 2\}$ and for each $e \in E_i \setminus K$, there exists a subset C_e of K such that $\{e\} \cup C_e \in \mathcal{C}_i$ and $c_i(e) \geq c_i(c)$ if $c \in C_e$. As minimum cost spanning sets are independent, K is indeed an $\mathcal{M}_1\mathcal{M}_2$-kernel. □

Proof of Theorem 6.1. By Theorem 6.3, there is a $\mathcal{M}_1\mathcal{M}_2$-kernel I corresponding to some $\mathcal{M}_1\mathcal{M}_2$-stable pair (A, B) for cost-functions $c_1 := \chi^{E \setminus I_2}$ and $c_2 := \chi^{E \setminus I_1}$. As I_1 is independent in $\mathcal{M}_2$, the 0-cost elements of $\mathcal{M}_2$ cannot span any element of $I_1 \cap (B \setminus A)$. Thus $I_1 \subseteq A \subseteq \mathrm{span}_{\mathcal{M}_1}(I)$, and by symmetry $I_2 \subseteq B \subseteq \mathrm{span}_{\mathcal{M}_2}(I)$. Theorem 6.1 follows. □

As another application of Theorem 6.3, we prove the existence of a dual $\mathcal{M}_1\mathcal{M}_2$-kernel for two matroids on the same ground set.

Theorem 6.4. *Let $\mathcal{M}_1 = (E, \mathcal{C}_1)$ and $\mathcal{M}_2 = (E, \mathcal{C}_2)$ be matroids and $c_1, c_2 : E \to \mathbb{R}_+$ be cost functions on their common ground set. Then there is a dual $\mathcal{M}_1\mathcal{M}_2$-kernel K.*

Proof. Let K^* be a $\mathcal{M}_1^*\mathcal{M}_2^*$-kernel with respect to cost functions $M - c_1$ and $M - c_2$, where we choose constant function $M > 0$ such that $M - c_i \geq 0$ for $i \in \{1, 2\}$. Define $K := E \setminus K^*$. As K^* is independent in both $\mathcal{M}_1^*$ and $\mathcal{M}_2^*$, K spans both $\mathcal{M}_1$ and $\mathcal{M}_2$. The kernel property of K^* implies the dual kernel property of K. □

We can deduce the stable matching theorem (the finite version of Theorem 2.2) as a special case of Theorem 6.3, by applying it to partition matroids defined by the stars in one colour class of the bipartite graph. The following generalization, the stable b-matching theorem can be proved similarly by applying Theorem 6.3 to the direct sum of uniform matroids.

Theorem 6.5. *Let $G = (A \cup B, E)$ be a finite bipartite (multi)graph with colour classes A and B, let $b : A \cup B \to \mathbb{N}$ be an arbitrary function and $<_v$ be a linear order on $D(v)$ for any vertex v of G. Then there is a subset M_b of E such that any vertex v is incident with at most $b(v)$ edges of M_b (that is $d_{M_b} \leq b$) and for any edge $e \in E \setminus M_b$ there is a vertex $v = v_e$ of G such that v is incident with $b(v)$ edges of M_b and each of these edges precedes e in order $<_v$.*

Proof. Define matroids $\mathcal{M}_A = (E, \mathcal{C}_A)$ and $\mathcal{M}_B = (E, \mathcal{C}_B)$ by

$$\begin{aligned}
\mathcal{C}_A &:= \{C : C \subseteq D(v) \text{ for some } v \in A \text{ and } |C| = b(v) + 1\} \quad \text{and} \\
\mathcal{C}_B &:= \{C : C \subseteq D(v) \text{ for some } v \in B \text{ and } |C| = b(v) + 1\},
\end{aligned}$$

cost functions $c_A, c_B : E \to \mathbb{N}$ by $c_A(e) = n$, $c_B(e) = m$ for any edge $e = uv$ of G, where $u \in A$, $v \in B$, and n is the height of e in $<_u$, and m is the height of e in $<_v$. Apply Theorem 6.3 on $\mathcal{M}_A, \mathcal{M}_B, c_A$ and c_B. The resulted matroid-kernel $K =: M_b$ will be a common independent set, that is $d_{M_b} \le b$, and the optimal spanning property shows that for any element $e \in E \setminus M_b$ there is a vertex v of G such that v is incident with $b(v)$ edges of M_b each preceding e in $<_v$. $\square$

The matching M_b described in Theorem 6.5 is called a *stable b-matching.*

7 The kernel lattice

In what follows, we focus on two well-studied aspects of the stable matching problem: first, we show a generalization of the so called "lattice structure" of stable matchings, and then, in Section 8, we deduce linear descriptions of kernel-related polyhedra from it. By this, we characterize among others the matroid generalization of the stable matching polytope described by Vande Vate [33] and Rothblum [28].

As we mentioned earlier, Blair in [2] proved that stable assignments in the many-to-many model of Theorem 2.6 form a lattice. To come over the fact that it is not true that stable configurations form a lattice for the natural meet and join operations on assignments, he introduced a partial order on stable assignments that turned out to be a lattice order. Namely, he defined $A \le_F B$ for assignments A and B if each firm in the model of Theorem 2.6 would choose assignment A if all choices in A and B would be offered. That is, if for each firm f we have $W_f(A) = C_f(W_f(A) \cup W_f(B))$, where sets of workers $W_f(A)$ and $W_f(B)$ are assigned to firm f in assignments A and B, respectively. Below, we show how Blair's theorem follows from the lattice subset property of fixed points in Tarski's theorem.

Lemma 7.1. *Let $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$ be comonotone set functions with property (20). If (A, B) and (A', B') are $\mathcal{FG}$-stable pairs with $\mathcal{FG}$-kernels $\mathcal{F}(A) = \mathcal{G}(B) = K$, $\mathcal{F}(A') = \mathcal{G}(B') = K'$ and if $\mathcal{F}(K \cup K') = K$ then $(A \cup A', B \cap B')$ is an $\mathcal{FG}$-stable pair that corresponds to K.*

Proof. As $A \cup B = X = A' \cup B'$, we have $(A \cup A') \cup (B \cap B') = X$. On the other hand,

$$\begin{aligned} A \cap \mathcal{F}(A \cup A') &\subseteq \mathcal{F}(A) = K \\ A' \cap \mathcal{F}(A \cup A') &\subseteq \mathcal{F}(A') = K' \end{aligned} \tag{28}$$

by property (14) of $\mathcal{F}$, as $A \subseteq A \cup A'$ and $A' \subseteq A \cup A'$. This means that $\mathcal{F}(A \cup A') = \mathcal{F}(A \cup A') \cap (A \cup A') \subseteq K \cup K' \subseteq A \cup A'$, hence

$$K = \mathcal{F}(K \cup K') = \mathcal{F}(A \cup A') \tag{29}$$

by property (20) of $\mathcal{F}$. From (28, 29), it follows that $A' \cap K \subseteq K'$, thus $K \subset B'$ holds because $K \subseteq A' \cup B'$. So we have that $\mathcal{G}(B) = K \subseteq B \cap B' \subseteq B$, and

$$\mathcal{G}(B \cap B') = \mathcal{G}(B) = K, \tag{30}$$

by property (20) of $\mathcal{G}$.

Finally, we show that $(A \cup A') \cap (B \cap B') = K$. From (29, 30), it is clear that $K \subseteq (A \cup A') \cap (B \cap B')$. For the opposite inclusion, we use relation

$$K' \cap B = \mathcal{G}(B') \cap (B \cap B') \subseteq \mathcal{G}(B \cap B') = K,$$

where the inclusion holds by property (14) of $\mathcal{G}$, and the last equation by (30). This means that

$$(A \cup A') \cap (B \cap B') = [A \cap (B \cap B')] \cup [A' \cap (B \cap B')] = (K \cap B') \cup (K' \cap B) \subseteq K.$$

□

From Lemma 7.1, we can give a simple explanation for the "opposition of common interest", a property observed by Roth in [26]. If, for $\mathcal{FG}$-kernels K and K' we have $\mathcal{F}(K \cup K') = K$, then, according to Lemma 7.1, there are $\mathcal{FG}$-stable pairs $(A', B') \leq (A^*, B^*)$ that correspond to K' and K, respectively. In particular, $\mathcal{G}(B^*) = K$ and $\mathcal{G}(B') = K'$ and $B^* \subseteq B'$. By property (20), from $\mathcal{G}(B') = K' \subseteq K \cup K' \subseteq B'$ we get that $\mathcal{G}(K \cup K') = K'$. In Roth's model it means that if each firm unanimously prefers stable assignment K to K' , then each worker prefers K' to K.

To prove Blair's theorem, we introduce a binary relation on $\mathcal{FG}$-kernels for comonotone functions $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$. For $\mathcal{FG}$-kernels K and K' we say that $K' \leq_{\mathcal{FG}} K$ if $\mathcal{F}(K \cup K') = K$. Recall that we have defined on $2^X \times 2^X$ a lattice order $<$ by (17) and lattice operations $\wedge$ and $\vee$ by (18).

Theorem 7.2 (Blair [2]). *Let $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$ be comonotone set-functions with property (20). Then relation $<_{\mathcal{FG}}$ is a lattice order on $\mathcal{FG}$-kernels.*

Proof. To prove that $<_{\mathcal{FG}}$ is a lattice order, it is enough to show by Lemma 7.1 that equivalence relation on $\mathcal{FG}$-stable pairs defined by

$$(A, B) \sim (A', B') \qquad \text{if} \qquad \mathcal{F}(A) = \mathcal{F}(A') \tag{31}$$

(that is if the corresponding $\mathcal{FG}$-kernels are the same) is compatible with lattice operations $\wedge_{\mathcal{FG}}, \vee_{\mathcal{FG}}$ of the lattice of $\mathcal{FG}$-stable pairs. Let $\mathcal{FG}$-stable pairs $(A_1', B_1') \sim (A_1'', B_1'')$ correspond to $\mathcal{FG}$-kernel K_1 and $\mathcal{FG}$-stable pairs $(A_2', B_2') \sim (A_2'', B_2'')$ to $\mathcal{FG}$-kernel K_2. As $\mathcal{F}$ and $\mathcal{G}$ have symmetric role and $\wedge_{\mathcal{FG}} = \vee_{\mathcal{GF}}$, it is enough to prove compatibility for the join, that is

$$(A', B') := (A_1', B_1') \vee_{\mathcal{FG}} (A_2', B_2') \sim (A_1'', B_1'') \vee_{\mathcal{FG}} (A_2'', B_2'') =: (A'', B'').$$

By applying Lemma 7.1 on $\mathcal{GF}$-stable pairs (B_1', A_1') and (B_1'', A_1'') and on (B_2', A_2') and (B_2'', A_2''), we find $\mathcal{GF}$-stable pairs

$$\begin{aligned}
(B_1, A_1) &:= (B_1' \cup B_1'', A_1' \cap A_1'') = (B_1', A_1') \vee_{\mathcal{GF}} (B_1'', A_1'') \sim (B_1', A_1') \text{ and} \\
(B_2, A_2) &:= (B_2' \cup B_2'', A_2' \cap A_2'') = (B_2', A_2') \vee_{\mathcal{GF}} (B_2'', A_2'') \sim (B_2', A_2').
\end{aligned}$$

Define $\mathcal{FG}$-stable pair $(A, B) := (A_1, B_1) \vee_{\mathcal{FG}} (A_2, B_2)$ and corresponding $\mathcal{FG}$-kernel $K := \mathcal{F}(A)$. By definition, $(A, B) \leq (A', B')$ and $(A, B) \leq (A'', B'')$.

Using property (20), we get from $\mathcal{F}(A) = K \subseteq K_1 \cup K \subset A$ and $\mathcal{F}(A) = K \subseteq K_2 \cup K \subset A$ that $K = \mathcal{F}(K \cup K_1)$ and $K = \mathcal{F}(K \cup K_2)$. Hence by Lemma 7.1, we see that both

$$(A, B) \vee (A'_1, B'_1) \vee (A'_2, B'_2) = (A \cup A'_1 \cup A'_2, B \cap B'_1 \cap B'_2) \text{ and} \tag{32}$$

$$(A, B) \vee (A''_1, B''_1) \vee (A''_2, B''_2) = (A \cup A''_1 \cup A''_2, B \cap B''_1 \cap B''_2) \tag{33}$$

are $\mathcal{FG}$-stable pairs with corresponding $\mathcal{FG}$-kernel K. This means that (32) and (33) describe (A', B') and (A'', B''), respectively. □

Next we generalize the sublattice property of bipartite stable matchings. Our aims are conditions that imply that the lattice subset of $\mathcal{FG}$-stable pairs in Theorem 4.2 and the lattice subset of fixed points in Theorem 3.1 is a sublattice. For a finite ground set X, we call function $f : 2^X \to 2^X$ *strongly monotone* if f is monotone and f has the subcardinal property of rank functions of matroids. Recall, that f is subcardinal if

$$|f(B) \setminus f(A)| \leq |B \setminus A| \tag{34}$$

for any $A \subseteq B \subseteq X$. Function f is *increasing* if

$$A \subseteq B \subseteq X \text{ implies } |f(A)| \leq |f(B)| \,. \tag{35}$$

Note that if comonotone function $\mathcal{F} = K_n$ is coming from (27) then $|\mathcal{F}(A)| = \mathrm{rank}(A)$, hence $\mathcal{F}$ is increasing. Also, the increasing property implies (20) for comonotone functions. We shall exhibit a link between strongly monotone functions, the sublattice structure of fixed points of monotone functions and increasing comonotone functions.

First we give a sufficient condition for a monotone function on subset-lattices so that the lattice subset of its fixed points is a sublattice.

Theorem 7.3. *If $f : 2^X \to 2^X$ is a strongly monotone function for a finite set X, then fixed points of f form a nonempty sublattice of* $(2^X, \cap, \cup)$.

Proof. By Theorem 3.1, the set of fixed points is nonempty. Assume that $f(A) = A$ and $f(B) = B$. By monotonicity, $A \cap B = f(A) \cap f(B) \supseteq f(A \cap B)$ and $A \cup B = f(A) \cup f(B) \subseteq f(A \cup B)$. By property (34),

$$|A \setminus (A \cap B)| \geq |f(A) \setminus f(A \cap B)| \geq |A \setminus A \cap B| \qquad \text{and}$$
$$|(A \cup B) \setminus A| \geq |f(A \cup B) \setminus f(A)| \geq |(A \cup B) \setminus A|,$$

hence there must be equality throughout. In particular, $f(A \cap B) = A \cap B$ and $f(A \cup B) = A \cup B$. □

The following link between strongly monotone and increasing comonotone functions is crucial for the lattice property of $\mathcal{FG}$-kernels.

Lemma 7.4. *If function $\mathcal{F} : 2^X \to 2^X$ is increasing and comonotone then $\overline{\mathcal{F}}$ is strongly monotone.*

Proof. We have seen in (12) that function $\overline{\mathcal{F}}$ is monotone. If $A \subseteq B$ then

$$|\overline{\mathcal{F}}(B) \setminus \overline{\mathcal{F}}(A)| = |\overline{\mathcal{F}}(B)| - |\overline{\mathcal{F}}(A)| = |B \setminus \mathcal{F}(B)| - |A \setminus \mathcal{F}(A)| = \\ |B| - |\mathcal{F}(B)| - |A| + |\mathcal{F}(A)| \leq |B| - |A| = |B \setminus A|. \qquad \square$$

Based on Lemma 7.4, we give a sufficient condition for the property that stable pairs in Theorem 4.2 form a sublattice. Recall that we have defined lattice order $<$ by (17) and lattice operations $\wedge$ and $\vee$ by (18) on $2^X \times 2^X$.

Theorem 7.5. *If X is a finite ground set, $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$ are increasing comonotone functions then $\mathcal{F}\mathcal{G}$-stable pairs form a nonempty, complete sublattice of* $(2^X \times 2^X, \wedge, \vee)$.

Proof. We use the construction in the proof of Theorem 4.2. There we saw that $\mathcal{F}\mathcal{G}$-stable pairs are exactly the fixed points of $f(A, B) := (X \setminus \overline{\mathcal{G}}(B), X \setminus \overline{\mathcal{F}}(A))$, defined in (19). It means that (A, B) is $\mathcal{F}\mathcal{G}$-stable if and only if $B = X \setminus \overline{\mathcal{F}}(A)$ and $A = f'(A) := X \setminus \overline{\mathcal{G}}(X \setminus \overline{\mathcal{F}}(A))$. Hence it is enough to prove that the fixed points of f' form a nonempty sublattice of $(2^X, \cap, \cup)$.

If $A \subseteq B \subseteq X$, then $\overline{\mathcal{F}}(A) \subseteq \overline{\mathcal{F}}(B)$ by monotonicity of $\overline{\mathcal{F}}$. Hence $X \setminus \overline{\mathcal{G}}(X \setminus \overline{\mathcal{F}}(A)) \subseteq X \setminus \overline{\mathcal{G}}(X \setminus \overline{\mathcal{F}}(B))$, by monotonicity of $\overline{\mathcal{G}}$. So f' is monotone.

From the subcardinal property (34) of $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$

$$\begin{aligned} |f'(B) \setminus f'(A)| &= |[X \setminus \overline{\mathcal{G}}(X \setminus \overline{\mathcal{F}}(B))] \setminus [X \setminus \overline{\mathcal{G}}(X \setminus \overline{\mathcal{F}}(A))]| = \\ &= |\overline{\mathcal{G}}(X \setminus \overline{\mathcal{F}}(A)) \setminus \overline{\mathcal{G}}(X \setminus \overline{\mathcal{F}}(B))| \leq \\ &\leq |[X \setminus \overline{\mathcal{F}}(A)] \setminus [X \setminus \overline{\mathcal{F}}(B)]| = \\ &= |\overline{\mathcal{F}}(B) \setminus \overline{\mathcal{F}}(A)| \leq |B \setminus A|. \end{aligned}$$

Hence f' is strongly monotone, and its fixed points form a nonempty, complete sublattice of $(2^X, \cap, \cup)$. That is, $\mathcal{F}\mathcal{G}$-stable pairs determine a nonempty, complete sublattice of $(2^X \times 2^X, <)$. $\square$

Theorem 7.5 is equivalent with saying that if $\mathcal{F}$ and $\mathcal{G}$ are increasing comonotone functions, then $<_{\mathcal{F}\mathcal{G}}$ defines a lattice $L_{\mathcal{F}\mathcal{G}}$ on $\mathcal{F}\mathcal{G}$-kernels with lattice operations given by $K_1 \vee_{\mathcal{F}\mathcal{G}} K_2 := (A_1 \cup A_2) \cap (B_1 \cap B_2)$ and $K_1 \wedge_{\mathcal{F}\mathcal{G}} K_2 := (A_1 \cap A_2) \cap (B_1 \cup B_2)$. Identity $\chi^{A_1 \cap B_1} + \chi^{A_2 \cap B_2} = \chi^{(A_1 \cup A_2) \cap (B_1 \cap B_2)} + \chi^{(A_1 \cap A_2) \cap (B_1 \cup B_2)}$ is easy to check, so we see that

$$\chi^K + \chi^L = \chi^{K \vee L} + \chi^{K \wedge L} \tag{36}$$

holds for any $\mathcal{F}\mathcal{G}$-kernels K_1 and K_2. Observe, that K is an $\mathcal{F}\mathcal{G}$-kernel if and only if it is a $\mathcal{G}\mathcal{F}$-kernel, and note that $<_{\mathcal{G}\mathcal{F}} = <_{\mathcal{F}\mathcal{G}}^{-1}$, thus $K_1 \vee_{\mathcal{F}\mathcal{G}} K_2 = K_1 \wedge_{\mathcal{G}\mathcal{F}} K_2$. In what follows, we might omit the subscript in the lattice operations or in the partial order when it does not cause ambiguity and clearly comes from $\mathcal{F}\mathcal{G}$-stability.

Corollary 7.6. *Let $\mathcal{F}$ and $\mathcal{G}$ be increasing comonotone functions. If $K_1, K_2, \ldots, K_n$ are $\mathcal{F}\mathcal{G}$-kernels then $|K_i| = |K_j|$ and $\bigvee_{i \in [n]} K_i = \mathcal{F}(\bigcup_{i \in [n]} K_i)$ and $\bigwedge_{i \in [n]} K_i = \mathcal{G}(\bigcup_{i \in [n]} K_i)$.*

Proof. Assume that $\mathcal{FG}$-stable pairs (A_i, B_i) and (A_j, B_j) correspond to $\mathcal{FG}$-kernels K_i and K_j, respectively. From the increasing property (35) of $\mathcal{F}$ and $\mathcal{G}$, we get that $|K_i| = |\mathcal{F}(A_i)| \leq |\mathcal{F}(A_i \cup A_j)| = |K_i \vee K_j| = \mathcal{F}(B_i \cap B_j)| \leq |\mathcal{G}(B_i)| = |K_i|$. Hence $|K_i| = |K_i \vee K_j| = |K_j|$.

Let $A := \bigcup_{i\in[n]} A_i$ and $K := \bigcup_{i\in[n]} K_i$. Clearly, $\bigvee_{i\in[n]} K_i \subseteq K \subseteq A$. Property (14) of $\mathcal{F}$ yields $\bigvee_{i\in[n]} K_i = \mathcal{F}(A) \cap K \subseteq \mathcal{F}(K)$. On the other hand, the increasing property of $\mathcal{F}$ implies that $|\mathcal{F}(K)| \leq |\mathcal{F}(A)| = |\bigvee_{i\in[n]} K_i|$. Thus $\mathcal{F}(\bigcup_{i\in[n]} K_i) = \bigvee_{i\in[n]} K_i$. Similarly, it follows that $\mathcal{G}(\bigcup_{i\in[n]} K_i) = \bigwedge_{i\in[n]} K_i$. □

Corollary 7.6 can be regarded as a generalization of the "consensus property" observed by Roth in [26]. The fact that $\mathcal{F}(\bigcup K_i)$ is an $\mathcal{FG}$-kernel can be translated into the language of his model the following way. If we fix a set S of stable assignment and each firm f freely selects its employees from those workers who are assigned to f in at least one stable assignment of S then a stable assignment is defined.

From now on, we use k to denote the common size of $\mathcal{FG}$-kernels for increasing comonotone functions $\mathcal{F}$ and $\mathcal{G}$. Theorem 7.5 implies the following observation on matroids:

Corollary 7.7. *If $\mathcal{M}_1, \mathcal{M}_2$ are matroids on the same ground set, c_1, c_2 are injective cost functions, and K_1, K_2 are $\mathcal{M}_1\mathcal{M}_2$-kernels, then* $\mathrm{span}_{\mathcal{M}_i}(K_1) = \mathrm{span}_{\mathcal{M}_i}(K_2)$ *for $i \in \{1, 2\}$.*

Proof. Choose $\mathcal{M}_1\mathcal{M}_2$-stable pairs (A_1, B_1) and (A_2, B_2) such that $K_i = A_i \cap B_i$ for $i \in \{1, 2\}$. By Theorem 7.5, $K_1 \vee K_2 = (A_1 \cup A_2) \cap (B_1 \cap B_2)$ is both a minimal c_1-cost independent set spanning $A_1 \cup A_2$ and a minimal c_2-cost independent set spanning $B_1 \cap B_2$. Clearly, for $i, j \in \{1, 2\}$

$$\mathrm{span}_{\mathcal{M}_i}(K_j) = \mathrm{span}_{\mathcal{M}_i}(A_j) \subseteq \mathrm{span}_{\mathcal{M}_i}(A_1 \cup A_2) = \mathrm{span}_{\mathcal{M}_i}(K_1 \vee K_2)$$

hence $\mathrm{span}_{\mathcal{M}_i}(K_1) \supseteq \mathrm{span}_{\mathcal{M}_i}(K_1 \vee K_2) \subseteq \mathrm{span}_{\mathcal{M}_i}(K_2)$. From $|K_1| = |K_1 \vee K_2| = |K_2|$ there must be equality all way through. □

Corollary 7.7 explains a well-known fact in the one-to-many assignment model. We mentioned in Section 2 that those colleges in the student-to-college assignment model that cannot fill up their quota in some stable assignment receive the very same set of students in any stable assignment. To prove this, recall that we remarked earlier that the existence of a stable scheme is a special case of Theorem 6.3 for partition matroid $\mathcal{M}_1$ and matroid $\mathcal{M}_2$ that is a direct sum of uniform matroids. In $\mathcal{M}_1$, we partition the edge set along the stars of students, for $\mathcal{M}_2$ the partition is given by the stars of colleges. A student star has rank 1 in $\mathcal{M}_1$, and the rank of a college star is the same as the b-value of the particular college. So, if for stable b-matching M_b we have $b(u) > d_{M_b}(u)$, then in the star of u only the edges of M_b are spanned in $\mathcal{M}_2$ by M_b, hence $D(u) \cap M_b = D(u) \cap M'_b$ for any other stable b-matching M'_b. As a special case, we also proved that in the original marriage model no matter which stable marriage scheme is chosen, always the same persons get married.

8 Kernel polyhedra

For comonotone functions $\mathcal{F}$ and $\mathcal{G}$ on the same ground set X and for subset M of X, let us denote by

$$\begin{aligned}
\mathcal{B}_{\mathcal{FG}} &:= \{B \subseteq X : B \cap K \neq \emptyset \text{ for any } K \in \mathcal{K}_{\mathcal{FG}}\} \\
\mathcal{A}_{\mathcal{FG}} &:= \{A \subseteq X : |A \cap K| \leq 1 \text{ for any member } K \text{ of } \mathcal{K}_{\mathcal{FG}}\} \\
M_{\mathcal{FG}} &:= X \setminus \bigcup \mathcal{K}_{\mathcal{FG}} \\
\mathcal{C}_M &:= \text{cone}\{\chi^{\{e\}} : e \in M\} = \{x \in \mathbb{R}^X_+ : x_e = 0 \text{ for } e \in X \setminus M\}
\end{aligned}$$

the *blocker* and *antiblocker* of $\mathcal{K}_{\mathcal{FG}}$, the set of non-kernel elements and the projection of the positive orthant to $\mathbb{R}^M$, respectively. (Recall that $\mathcal{K}_{\mathcal{FG}}$ denotes the set of $\mathcal{FG}$-kernels.)

Define further

$$\begin{aligned}
\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}} &:= \text{conv}\{\chi^K : K \in \mathcal{K}_{\mathcal{FG}}\} && (37)\\
\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}} &:= \text{cone}\{\chi^K : K \in \mathcal{K}_{\mathcal{FG}}\} = \{\lambda \cdot x : \lambda \in \mathbb{R}_+, x \in \mathcal{P}_{\mathcal{K}_{\mathcal{FG}}}\} \\
\mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}} &:= \mathcal{P}_{\mathcal{K}_{\mathcal{FG}}} + \mathbb{R}^X_+ = \{x + y : x \in \mathcal{P}_{\mathcal{K}_{\mathcal{FG}}}, y \geq 0\} && (38)\\
\mathcal{P}^{\downarrow}_{\mathcal{K}_{\mathcal{FG}}} &:= (\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}} - \mathbb{R}^X_+) \cap \mathbb{R}^X_+ = \{x - y : x \in \mathcal{P}_{\mathcal{K}_{\mathcal{FG}}}, y \geq 0\} \cap \mathbb{R}^{|X|}_+ && (39)\\
\mathcal{P}^{\uparrow}_{\mathcal{B}_{\mathcal{FG}}} &:= \{\chi^B : B \in \mathcal{B}_{\mathcal{FG}}\}^{\uparrow} = \{x + y : x \in \text{conv}\{\chi^B : B \in \mathcal{B}_{\mathcal{FG}}\}, y \geq 0\} && (40)\\
\mathcal{P}^{\downarrow}_{\mathcal{A}_{\mathcal{FG}}} &:= \{\chi^A : A \in \mathcal{A}_{\mathcal{FG}}\}^{\downarrow} + \mathcal{C}_{M_{\mathcal{FG}}} = \\
&= \mathcal{C}_{M_{\mathcal{FG}}} + \{x - y : x \in \text{conv}\{\chi^A : A \in \mathcal{A}_{\mathcal{FG}}\}, y \geq 0\} \cap \mathbb{R}^X_+ && (41)
\end{aligned}$$

the *$\mathcal{FG}$-kernel polytope*, the *$\mathcal{FG}$-kernel cone*, the *dominant* of the $\mathcal{FG}$-kernel polytope, the *submissive* of the kernel polytope, the *$\mathcal{FG}$-blocker polyhedron* and the *$\mathcal{FG}$-antiblocker polyhedron*, respectively. We are going to characterize these polyhedra in terms of linear constraints. For $\mathcal{P}^{\uparrow}_{\mathcal{B}_{\mathcal{FG}}}$ and $\mathcal{P}^{\downarrow}_{\mathcal{A}_{\mathcal{FG}}}$ we apply the theory of lattice polyhedra and for the rest we use the theory of (anti)blocking polyhedra.

To state the Hoffman-Schwartz theorem, a basic result on lattice polyhedra, we need to formulate some assumptions. Fix a ground set X and a family $\mathcal{L}$ of subsets of X. A partial order $<$ on $\mathcal{L}$ is called *consistent* if $A \cap C \subseteq B$ holds for any members A, B, C of $\mathcal{L}$ with $A < B < C$. Family $\mathcal{L}$ is a *clutter* if there is a consistent lattice order $<$ on $\mathcal{L}$ with lattice operations $\wedge$ and $\vee$ such that

$$\chi^A + \chi^B = \chi^{A \wedge B} + \chi^{A \vee B}$$

holds for any members A, B of $\mathcal{L}$.

Theorem 8.1 (Hoffman-Schwartz [14]). *Let $\mathcal{L} \subseteq 2^X$ be a clutter for consistent lattice order $<$ and lattice operations $\wedge, \vee$ and let $d : X \to \mathbb{N} \cup \{\infty\}$ be and arbitrary function. If $r : \mathcal{L} \to \mathbb{N}$ is submodular then system*

$$\{x \in \mathbb{R}^X : 0 \leq x \leq d,\ x(A) \leq r(A) \text{ for any } A \in \mathcal{L}\} \tag{42}$$

is TDI.

If $r : \mathcal{L} \to \mathbb{N}$ *is supermodular then system*

$$\{x \in \mathbb{R}^X : 0 \le x \le d,\ x(A) \ge r(A) \text{ for any } A \in \mathcal{L}\} \tag{43}$$

is TDI. □

(Here, $r : \mathcal{L} \to \mathbb{N}$ is *submodular* if $r(A) + r(B) \ge r(A \wedge B) + r(A \vee B)$ holds for any $A, B \in \mathcal{L}$; r is *supermodular* if the reverse inequality is true.)

Next we observe that Theorem 8.1 is relevant in our setting.

Observation 8.2. If $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$ are increasing comonotone functions then family $\mathcal{K}_{\mathcal{FG}}$ of $\mathcal{FG}$-kernels is a clutter for lattice order $<_{\mathcal{FG}}$.

Proof. If $K_1 <_{\mathcal{FG}} K_2 <_{\mathcal{FG}} K_3$ for $K_1, K_2, K_3 \in \mathcal{K}_{\mathcal{FG}}$ then by Lemma 7.1, there are corresponding $\mathcal{FG}$-stable pairs $(A_1, b_1) <_{\mathcal{FG}} (A_2, B_2) <_{\mathcal{FG}} (A_3, B_3)$. By (13) and (14), $K_1 \cap K_3 = \mathcal{F}(A_1) \cap \mathcal{F}(A_3) \subseteq A_1 \cap \mathcal{F}(A_3) \subseteq A_2 \cap \mathcal{F}(A_3) \subseteq \mathcal{F}(A_2) = K_2$, hence lattice order $<_{\mathcal{FG}}$ is consistent. By (36), $\mathcal{K}_{\mathcal{FG}}$ is a clutter. □

By applying the Hoffman-Schwartz theorem on $\mathcal{K}_{\mathcal{FG}}$, we get the following.

Theorem 8.3. *If $\mathcal{FG}$ are increasing comonotone functions on ground set X then*

$$\mathcal{P}^{\uparrow}_{\mathcal{B}_{\mathcal{FG}}} = \{x \in \mathbb{R}^X : x \ge \mathbf{0} \text{ and } x(K) \ge 1 \text{ for any } K \in \mathcal{K}_{\mathcal{FG}}\} \text{ and} \tag{44}$$

$$\mathcal{P}^{\downarrow}_{\mathcal{A}_{\mathcal{FG}}} = \{x \in \mathbb{R}^X : x \ge \mathbf{0} \text{ and } x(K) \le 1 \text{ for any } K \in \mathcal{K}_{\mathcal{FG}}\}. \tag{45}$$

Proof. Obviously, the polyhedra on the left hand side of (44,45) are the integer hulls of the polyhedra described by right hand sides.

By Observation 8.2, $\mathcal{K}_{\mathcal{FG}}$ is a clutter. Let $d(v) := \infty$ and $r(K) := 1$ for all $v \in X$ and $K \in \mathcal{K}_{\mathcal{FG}}$. Clearly, r is sub- and supermodular. By Theorem 8.1, linear systems in (44,45) are TDI, hence the polyhedra on the right hand sides are integer. □

We introduce some basic notions from the theory of blocking and antiblocking polyhedra to be able to describe other kernel-related polyhedra.

Polyhedron $P \subseteq \mathbb{R}^d_+$ is a *blocking type polyhedron* if $P = P + \mathbb{R}^d_+$, and it is an *antiblocking type polyhedron* if $P = \mathbb{R}^d_+ \cap (P + \mathbb{R}^d_-)$. Any finite subset H of $\mathbb{R}^d_+$ defines a blocking and an antiblocking polyhedron by

$$H^{\uparrow} := \mathrm{conv}(H) + \mathbb{R}^d_+ \qquad \text{and} \qquad H^{\downarrow} := \mathbb{R}^d_+ \cap (\mathrm{conv}(H) + \mathbb{R}^d_-),$$

respectively. For a polyhedron P

$$B(P) := \{x \in \mathbb{R}^d_+ : x^T y \ge 1 \text{ for all } y \in P\} \text{ and}$$
$$A(P) := \{x \in \mathbb{R}^d_+ : x^T y \le 1 \text{ for all } y \in P\}$$

are the *blocking* and *antiblocking polyhedron* of P, respectively. As suggested by the name, if P is a polyhedron then both $A(P)$ and $B(P)$ are polyhedra.

Theorem 8.4 (Fulkerson [8, 9, 10]). *If P is a blocking type polyhedron then $B(P)$ is a blocking type polyhedron and $P = B(B(P))$. If P is an antiblocking type polyhedron then $A(P)$ is an antiblocking type polyhedron and $P = A(A(P))$. Furthermore,*

$$\begin{aligned} B(\{x_1, x_2, \dots, x_n\}^{\uparrow}) &= \{y \in \mathbb{R}^d_+ : \quad y^T x_i \ge 1 \text{ for } i \in [n]\} \qquad (46) \\ A(\{x_1, x_2, \dots, x_n\}^{\downarrow}) + \mathcal{C}_M &= \{y \in \mathbb{R}^d_+ : \quad y^T x_i \le 1 \text{ for } i \in [n] \text{ and} \\ & \qquad y(m) = 0 \text{ for } m \in M\} \qquad (47) \end{aligned}$$

for any $n \in \mathbb{N}$, subset M of $[d]$ and elements x_i ($i \in [n]$) of $\mathbb{R}^d_+$. □

With these tools, we can give the following descriptions for our kernel polyhedra.

Theorem 8.5. *If $\mathcal{FG}$ are increasing comonotone functions on ground set X and k is the common size of $\mathcal{FG}$-kernels then*

$$\begin{aligned} \mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}} &= \{x \in \mathbb{R}^E : x \ge \mathbf{0},\ x(B) \ge 1 \ \text{ for } B \in \mathcal{B}_{\mathcal{FG}}\}\ , & (48) \\ \mathcal{P}^{\downarrow}_{\mathcal{K}_{\mathcal{FG}}} &= \{x \in \mathbb{R}^E : x \ge \mathbf{0},\ x(M_{\mathcal{FG}}) = 0 \text{ and } x(A) \le 1 \text{ for any } A \in \mathcal{K}_{\mathcal{FG}}\}\ , & (49) \\ \mathcal{P}_{\mathcal{K}_{\mathcal{FG}}} &= \{x \in \mathbb{R}^E : x \ge \mathbf{0},\ \mathbf{1}^T x \le k,\ x(B) \ge 1 \ \text{ for } B \in \mathcal{B}_{\mathcal{FG}}\}\ , & (50) \\ \mathcal{P}_{\mathcal{K}_{\mathcal{FG}}} &= \{x \in \mathbb{R}^E : x \ge \mathbf{0},\ x(M_{\mathcal{FG}}) = 0,\ \mathbf{1}^T x \ge k,\ x(A) \le 1 \ \text{ for } A \in \mathcal{A}_{\mathcal{FG}}\}\ , & (51) \\ \mathcal{C}_{\mathcal{K}_{\mathcal{FG}}} &= \{x \in \mathbb{R}^E : x \ge \mathbf{0},\ k \cdot x(B) \ge \mathbf{1}^T x \text{ for } B \in \mathcal{B}_{\mathcal{FG}}\}\ , \text{ and} & (52) \\ \mathcal{C}_{\mathcal{K}_{\mathcal{FG}}} &= \{x \in \mathbb{R}^E : x \ge \mathbf{0},\ x(M_{\mathcal{FG}}) = 0,\ k \cdot x(A) \le \mathbf{1}^T x \text{ for } A \in \mathcal{A}_{\mathcal{FG}}\}\ . & (53) \end{aligned}$$

Proof. By (44) and (46), $\mathcal{P}^{\uparrow}_{\mathcal{B}_{\mathcal{FG}}} = B(\mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}})$. From Theorem 8.4, we get that $\mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}} = B(\mathcal{P}^{\uparrow}_{\mathcal{B}_{\mathcal{FG}}})$, and (48) follows from (46). Similarly, $\mathcal{P}^{\downarrow}_{\mathcal{A}_{\mathcal{FG}}} = A(\mathcal{P}^{\downarrow}_{\mathcal{K}_{\mathcal{FG}}})$ from (45) and (47). Theorem 8.4 implies that $\mathcal{P}^{\downarrow}_{\mathcal{K}_{\mathcal{FG}}} = A(\mathcal{P}^{\downarrow}_{\mathcal{A}_{\mathcal{FG}}})$, so (49) follows from (47). As each $\mathcal{FG}$-kernel has the same size k, (50) follows directly from (48), and (51) from (49).

Clearly, both cones $\mathcal{C}$ and $\mathcal{C}'$ described on the right hand sides of (52) and (53) contain $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$. Let $x \ge \mathbf{0}$ be a vector outside $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$, and $\lambda = \frac{k}{\mathbf{1}^T x}$. Then $\mathbf{1}^T(\lambda \cdot x) = k$ and $\lambda \cdot x \notin \mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}} \cup \mathcal{P}^{\downarrow}_{\mathcal{K}_{\mathcal{FG}}}$, hence there is a member B of $\mathcal{B}_{\mathcal{FG}}$ such that $\lambda \cdot x(B) < 1$ and if $x(M_{\mathcal{FG}}) = 0$ then there is a member A of $\mathcal{A}_{\mathcal{FG}}$ with $\lambda \cdot x(A) > 1$. This means that $k \cdot x(B) < \frac{k}{\lambda} = \mathbf{1}^T x$ and $k \cdot x(A) > \frac{k}{\lambda} = \mathbf{1}^T x$. Thus $x \notin \mathcal{C}$ and $x \notin \mathcal{C}'$, justifying (52) and (53). □

Note that we gave two different descriptions for both $\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}}$ and $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$. Apart from the nonnegativity conditions, there is no constraint that appears in both of the description. In particular, this means that $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$ can neither be full dimensional nor 1-codimensional. It is also interesting to observe that the linear description by Vande Vate [33] and Rothblum [28] for the convex hull of bipartite stable matchings is related to (50).

Theorem 8.6 (Rothblum [28], see also Vande Vate [33]). *Let $G = (V, E)$ be a finite bipartite graph and for each $v \in V$ let $<_v$ be a linear order on $D(v)$. Define $\phi(e) := \{f \in E : f \le_u e \text{ or } f \le_v e\}$ for edge $e = uv \in E$. Then*

$$\begin{aligned} &\operatorname{conv}\{\chi^F : F \subseteq E \text{ is a stable matching of } G\} = & (54) \\ &\{x : \mathbf{0} \le x \in \mathbb{R}^E, x(D(v)) \le 1 \text{ for } v \in V, x(\phi(e)) \ge 1 \text{ for } e \in E\}. \end{aligned}$$

□

In (54), conditions $x(D(v)) \leq 1$ are special cases of conditions of type $x(A) \leq 1$ of (51) and together with nonnegativity of x, these are responsible for any solution x is a convex combination of bipartite matchings. Constraints $x(\phi(e)) \geq 1$ are special cases of $x(B) \geq 1$ type constraints in (50). In fact, as $\mathbf{1}^T x \geq k$ for any $x \in \mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}}$ and $\mathbf{1}^T x \leq k$ for any $x \in \mathcal{P}^{\downarrow}_{\mathcal{K}_{\mathcal{FG}}}$, we have that $\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}} = \mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}} \cap \mathcal{P}^{\downarrow}_{\mathcal{K}_{\mathcal{FG}}}$. It follows that

$$\begin{aligned}\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}} = \{x \in \mathbb{R}^E : x \geq \mathbf{0},\ & x(B) \geq 1 \text{ for } B \in \mathcal{B}_{\mathcal{FG}}, \\ & x(A) \leq 1 \text{ for } A \in \mathcal{A}_{\mathcal{FG}}\}\ ,\end{aligned} \tag{55}$$

because for a vector x of the right hand side, define x' by zeroing the coordinates of x that correspond to elements $e \in M_{\mathcal{FG}}$ and add $x(M_{\mathcal{FG}})$ to some other coordinate of x. It is easy to check that $x' \in \mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}} \cap \mathcal{P}^{\downarrow}_{\mathcal{K}_{\mathcal{FG}}}$, hence $\mathbf{1}^T x' = k$. But then $x \in \mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}}$ can only hold if $x = x'$, that is, condition $x(M_{\mathcal{FG}}) = 0$ automatically holds in (55). Note that characterization (55) resembles very much to (54).

Another interesting question whether linear descriptions (48-53) are good characterizations, that is, whether the separation problem over those polyhedra can be solved efficiently. The answer is yes, and a possible way for $\mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}}$ is explained in [7].

Finally, to contrast Theorem 8.5, we prove that it is NP-complete to decide whether a particular element of the ground set can belong to some stable antichain or not. It means that unless P=NP, it is necessary to have some extra assumption (like the increasing property) on the comonotone functions to hope for a good characterization of the corresponding $\mathcal{FG}$-kernel polytope, $\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}}$. We will use Observation 4.3, the only example of non-increasing comonotone function we have seen so far.

Theorem 8.7. *If undirected graph $G = (V, E)$ and $k \in \mathbb{N}$ are given then it is possible to construct partial orders $<$ and $<'$ and an element s of their common ground-set X in time polynomial in $|V|$, such that s belongs to a stable antichain of $<$ and $<'$ if and only if G contains an independent set of size k.*

Proof. We may assume $k \leq |V|$, otherwise the theorem is trivial. Otherwise let

$$X := \{s\} \cup \{a_j, a'_j : 1 \leq j \leq k\} \cup \{v_j, v'_j : v \in V, 1 \leq j \leq k\}.$$

Partial orders $<$ and $<'$ are determined by

$$a_j < s, \qquad u_j < v'_j, \qquad w_l < v'_j, \qquad v_j < a'_j$$

$$a'_j <' s, \qquad u'_j <' v_j, \qquad w'_l <' v_j, \qquad v'_j <' a_j$$

for $1 \leq j \leq k, 1 \leq l \leq k$, $j \neq l$, $u, v, w \in V$, $u \neq v$ and $vw \in E$ or $v = w$.

If G has an independent set $I = \{i^1, i^2, \ldots, i^k\} \subseteq V$ of size k, then $S := \{s\} \cup \{i^j_j, i^{j'}_j : 1 \leq j \leq k\}$ is a stable antichain of $<$ and $<'$. On the other hand, if s belongs to a stable antichain S then neither a_j, nor a'_j can belong to S. Thus for every j there must exist elements i^j and e^j of V such that $i^j_j, e^{j'}_j \in S$. By stability $i^j = e^j \neq i^l$ and $i^j i^l \notin E$ for $j \neq l$, in other words $I := \{i^1, i^2, \ldots, i^k\}$ is an independent set of G of size k. $\square$

As the decision problem whether there exists an independent set of size k in a graph is NP-complete, it is also NP-complete to solve the kernel-problem in Theorem 8.7. We give another reason why it is NP-complete to optimize kernels. For an undirected graph $G = (V, E)$, function $f : 2^E \to 2^E$, defined by $f(E') = \{e = uv \in E' : D(u) \cup D(v) \subseteq E'\}$ for $E' \subseteq E$ is monotone. Consider comonotone function $\mathcal{F}$, defined by $\mathcal{F}(E') := E' \setminus f(E')$. It is easy to see that $\mathcal{F}\mathcal{F}$-stable pairs are $(E(U), E(V \setminus U))$ where $E(U) := \bigcup_{v \in U} D(v)$ (for $U \subseteq V$), and $\mathcal{F}\mathcal{F}$-kernels are exactly the edge-sets of the form $D(U)$, for some $U \subseteq V$. If it would be possible to separate over the kernel polytope $P_{\mathcal{F}\mathcal{F}}$, then (by the ellipsoid method) it would be possible to find a maximum cut of G in polynomial time. But this latter problem is NP-complete.

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