Consistency of the planar rotor-routing action via the trinity definition

Lilla Tóthmérész

Abstract

The sandpile group of a plane graph has a canonical torsor structure on the spanning trees. This torsor can either be defined via rotor-routing or via the Bernardi process, or via trinities. It is most often called the rotor-routing torsor.

Klivans conjectured that for a plane graph, this torsor is in some sense the unique torsor of the sandpile group on the spanning trees. This conjecture was made precise by Ganguly and McDonough, who proposed the notion of conistency for a sandpile torsor. Consistency means that the torsor behaves well with respect to deletion and contraction of the edges of the graph. Then they showed that the rotor-routing torsor of a plane graph is consistent, moreover, it is the unique consistent sandpile torsor for plane graphs.

In their proofs, they used the rotor-routing definition for the torsor. In this note, we give an alternative, somewhat simpler proof for the consistency of the action using the trinity definition of the rotor-routing action. We also give an example highlighting that considering adjacent vertices in the definition of consistency is important.

1 Introduction

1.1 Basic definitions

Throughout this paper, we assume all graphs and directed graphs to be connected. We allow loops and multiple edges. For a graph G and an edge e, we denote by $G \setminus e$ the graph obtained by deleting the edge e, and by G/e the graph obtained by contracting the edge e (that is, gluing its two endpoints and deleting the edge).

A subgraph of an graph is called a spanning tree if it is connected and cycle-free. For a graph G, we denote the set of spanning trees by $\mathcal{T}(G)$.

For a graph, a ribbon structure is the choice of a cyclic ordering of the edges around each vertex. If a graph is embedded into an orientable surface, the embedding gives a ribbon structure using the positive orientation of the surface, and conversely, for any ribbon graph there exists a closed orientable surface of minimal genus so that the graph embeds into it, giving the particular ribbon structure. For us, the most important case is the case of graphs embedded into the plane (plane graphs). For an edge xy of the graph, we denote by xy^+ the edge following xy at x according to the ribbon structure.

Let us introduce notations for some special vectors in $\mathbb{Z}^{|V|}$. By $\mathbf{0}$, we denote the vector with all coordinates equal to zero, while by $\mathbf{1}$ the vector with all coordinates equal to one. For a set $S \subseteq V$, we let $\mathbf{1}_S$ denote the characteristic vector of S, i.e. $\mathbf{1}_S(v) = 1$ for $v \in S$ and $\mathbf{1}_S(v) = 0$ otherwise.

1.2 The sandpile group

In this subsection we give the definition of the sandpile group. As later on we will need the sandpile group of Eulerian digraphs, we give the definition for this broader case.

For an Eulerian digraph $D = (V, A)$, we denote by $\mathrm{Div}(D)$ the free Abelian group on V. For $x \in \mathrm{Div}(D)$ and $v \in V$, we use the notation $x(v)$ for the coefficient of v. We refer to x as a chip configuration, and to $x(v)$ as the number of chips on v. We use the notation $\deg(x) = \sum_{v \in V} x(v)$, and call $\deg(x)$ the *degree* of x. We also write $\mathrm{Div}^d(D) = \{x \in \mathrm{Div}(D) : \deg(x) = d\}$.

The Laplacian matrix of a digraph is the following matrix $L_D \in \mathbb{Z}^{V \times V}$:

$$L_D(u, v) = \begin{cases} -d^+(v) & \text{if } u = v, \\ d(v, u) & \text{if } u \neq v. \end{cases}$$

Here, $d^+(v)$ denotes the outdegree of node v, and $d(v, u)$ denotes the number of edges pointing from v to u.

We call two chip configurations x and y *linearly equivalent* if there exists $z \in \mathbb{Z}^V$ such that $y = x + L_D z$. We use the notation $x \sim y$ for linear equivalence. Notice that, as for Eulerian digraphs we have $L_D \mathbf{1} = \mathbf{0}$, we can suppose that z has nonnegative elements and $z(v) = 0$ for some $v \in V$. Note also that linearly equivalent chip configurations have equal degree. We denote the linear equivalence class of a chip configuration x by $[x]$.

There is an interpretation of linear equivalence using the so-called chip-firing game. In this game, a step consists of firing a node v. The firing of v decreases the number of chips on v by the outdegree of v, and increases the number of chips on each neighbor w of v by $d(v, w)$. It is easy to check that the firing of v changes x to $x + L_D \mathbf{1}_v$. Hence x is linearly equivalent to y if and only if there is a sequence of firings that transforms x to y.

The Picard group of a digraph is the group of chip configurations factorized by linear equivalence: $\mathrm{Pic}(D) = \mathrm{Div}(D)/_\sim$. This is an infinite group. We will be interested in the subgroup corresponding to zero-sum elements, which is called the sandpile group.

Definition 1 (Sandpile group)**.** For an Eulerian digraph D, the sandpile group is defined as $\mathrm{Pic}^0(D) = \mathrm{Div}^0(D)/\sim$.

It is easy to see that $\mathrm{Pic}(D) = \mathrm{Pic}^0(D) \times \mathbb{Z}$. The sandpile group is a finite group.

We will use the notation $\mathrm{Pic}^d(D)$ for the set of equivalence classes of $\mathrm{Pic}(D)$ consisting of chip configurations of degree d.

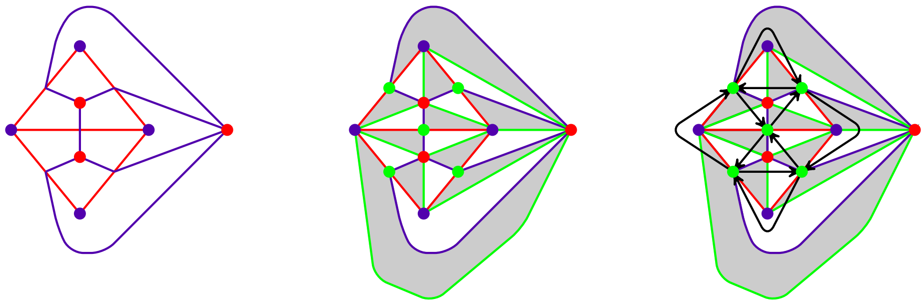

Figure 1: A plane graph and its dual (left panel), the corresponding trinity (middle panel), and the digraph D_E (right panel).

If we have an undirected graph, we can apply the above definitions to the bidirected version of the graph, that is, where we substitute each undirected edge by two oppositely directed edges.

2 The canonical action (planar rotor-routing action)

It is a well-known fact that for an undirected graph G, the order of $\mathrm{Pic}^0(G)$ is equal to the number of spanning trees of G.

There are many different ways to define a free transitive action of $\mathrm{Pic}^0(G)$ on $\mathcal{T}(G)$. One of the constructions is the rotor-routing action, defined by Holroyd et. al. [5]. Another one is the Bernardi action by Baker and Wang [1].

In this section, following [8], for a plane (ribbon) graph G, we give the definition of the canonical action, that is a free and transitive action of $\mathrm{Pic}^0(G)$ on $\mathcal{T}(G)$, and it agrees with the planar rotor-routing action and the planar Bernardi action.

First, we need to recall the notion of trinities.

Definition 2 (Trinity)**.** A *trinity* is a triangulation of the sphere S^2 together with a three-coloring of the 0-simplices. (I.e., 0-simplices joined by a 1-simplex have different colors.) According to dimension, we will refer to the simplices as *points, edges*, and *triangles*. (See Figure 1 for an example.)

Trinities will be important for us because embedded planar graphs naturally yield trinities. For a plane ribbon graph G, we construct the corresponding trinity in the following way. (See Figure 1 for an example.) Let V be the set of vertices of G, and color these vertices violet. Subdivide each edge of G by a new node. These new nodes are in one-to-one correspondence with the edges of G. Hence we denote the set of these nodes by E and color them emerald. Let us call the obtained bipartite graph G_R. Then place a red node in the interior of each region of G_R, and call the set of these nodes R (they correspond to the regions of the plane graph G). Traverse the boundary of each region of G_R and at each corner of the boundary, connect the emerald or violet node to the red node of the region. This way we get a three-colored triangulation of the surface. Let us color a triangle white if the violet, emerald and red vertices follow each other in positive cyclic order, and color it black otherwise.

Notice that the triangulation contains two further bipartite graphs (other than G_R). Let us call G_V the bipartite graph connecting vertices of R and E (this is the graph obtained by subdividing each edge of the planar dual G^* with a new node), and let us call G_E the bipartite graph connecting vertices of R and V.

We can also associate three directed graphs D_V, D_E and D_R to a trinity: The node set of D_V is V, and a directed edge points from $v_1 \in V$ to $v_2 \in V$ if a black triangle incident to v_1 and a white triangle incident to v_2 share their violet edge. D_E and D_R are defined analogously. Notice that if we obtained the trinity from a planar graph as described above, then D_V is the "bidirected" version of G (that is, each edge is substituted by two oppositely directed edges), and D_R is the bidirected version of G^*. D_E is generally not bidirected, but it is always Eulerian, as the indegree of any node $e \in E$ is the number of white triangles incident to it, and the outdegree is the number of black triangles incident to e, and these triangles alternate around e. We call D_E the medial (di)graph of G.

In this paper, we will always consider graphs G embedded into the plane, and their corresponding trinities. We will simultaneously think of edges of G in the ordinary sense (as curves connecting two vertices), and as an emerald node of the trinity. Moreover, we will think of edges of D_V as orientations of the edges of G.

Remark 3. We note that there exist trinities that do not correspond to plane graphs. In the general case, one can also associate 3 digraphs D_V, D_E and D_R to the trinity, and their roles are completely symmetric. Most results about the sandpile group of plane graphs can be generalized to this case, see [8].

2.1 The isomorphism between $\mathrm{Pic}^0(D_V)$, $\mathrm{Pic}^0(D_E)$ and $\mathrm{Pic}^0(D_R)$

It is well-known that for a planar graph G, $\mathrm{Pic}^0(G)$ and $\mathrm{Pic}^0(G^*)$ are canonically isomorphic [4]. That is, if we construct the above described trinity from the graph G, then $\mathrm{Pic}^0(D_V)$ and $\mathrm{Pic}^0(D_R)$ are canonically isomorphic. However, also $\mathrm{Pic}^0(D_E)$ is also canonically isomorphic to $\mathrm{Pic}^0(D_V)$ and $\mathrm{Pic}^0(D_R)$ [8]. Here, we describe this isomorphism. For this, we need some preparations.

Definition 4 ($\mathcal{A}$)**.** Let $\mathcal{A}$ be the free Abelian group on the set $V \cup E \cup R$. We describe the elements of $\mathcal{A}$ by vector triples (x_V, x_E, x_R), where $x_V \in \mathbb{Z}^V$, $x_E \in \mathbb{Z}^E$, and $x_R \in \mathbb{Z}^R$.

Definition 5 (white triangle equivalence)**.** Two elements of $\mathcal{A}$ are said to be white triangle equivalent if their difference can be written as an integer linear combination of characteristic vectors of white triangles. We denote white triangle equivalence by $\approx_W$.

Note that $\approx_W$ is indeed an equivalence relation. Now one can define a group by factorizing with white triangle equivalence. This group was introduced by Cavenagh and Wanless, and we call it the trinity sandpile group.

Definition 6 ($\mathcal{A}_W$, [2])**.** $\mathcal{A}_W = \mathcal{A}/_{\approx_W}$.

The relationship of $\mathcal{A}_W$ to the sandpile group was first pointed out by Blackburn and McCourt. For our purposes, the following statement will be needed.

Theorem 7. *[8] The equivalence classes of $\mathcal{A}_W$ containing at least one element of the form $(x_V, \mathbf{0}, \mathbf{0})$ with $\deg(x_V) = 0$ form a group isomorphic to $\mathrm{Pic}^0(D_V)$.*

The equivalence classes of $\mathcal{A}_W$ containing at least one element of the form $(\mathbf{0}, x_E, \mathbf{0})$ with $\deg(x_E) = 0$ form a group isomorphic to $\mathrm{Pic}^0(D_E)$.

The equivalence classes of $\mathcal{A}_W$ containing at least one element of the form $(\mathbf{0}, \mathbf{0}, x_R)$ with $\deg(x_R) = 0$ form a group isomorphic to $\mathrm{Pic}^0(D_R)$.

Remark 8. Theorem 7 is true for any trinity, not only for ones coming from planar graphs. This is an example where concentrating on trinities obtained from graphs hides symmetries. The definition of (general) trinities is completely symmetric for the three color classes. However, if we consider trinities coming from planar graphs, then the color class E has special properties.

Definition 9. Let $\varphi_{V\to E} : \mathrm{Pic}^0(D_V) \to \mathrm{Pic}^0(D_E)$ be defined by $\varphi_{V\to E}([x]) = [y]$ where $(x, \mathbf{0}, \mathbf{0}) \approx_W (\mathbf{0}, -y, \mathbf{0})$.

Theorem 10. *[8] $\varphi_{V\to E}$ is well-defined and is an isomorphism between $\mathrm{Pic}^0(D_V)$ and $\mathrm{Pic}^0(D_E)$.*

Remark 11. Notice that $(x, \mathbf{0}, \mathbf{0}) \approx_W (\mathbf{0}, -y, \mathbf{0})$ is equivalent to $(x, y, \mathbf{0}) \approx_W (\mathbf{0}, \mathbf{0}, \mathbf{0})$. Hence $\varphi_{V\to E}([x]) = [y]$ can be witnessed by the linear combination of some white triangles, such that the linear combination of the characteristic vectors gives $(x, y, \mathbf{0})$.

2.2 Spanning trees are representatives of $\mathrm{Pic}^{|V|-1}(D_E)$

To any spanning tree T of G, one can associate the characteristic vector of T, that is a vector in $\mathbb{Z}^E$ with coordinate 1 on $e \in T$ and coordinate 0 on $e \notin T$. By a slight abuse of notation, we denote this characteristic vector also by T.

Hence with this convention, $T \in \mathrm{Pic}^{|V|-1}(D_E)$ for any spanning tree. In fact, more is true.

Theorem 12. *[8] Let G be a plane graph and D_E the medial graph of the trinity of G. Then the set of spanning trees of G gives a system of representatives of linear equivalence classes of $\mathrm{Pic}^{|V|-1}(D_E)$. In other words, for any chip configuration x_E on E with $\deg(x_E) = |V| - 1$, there is exactly one spanning tree $T \in \mathcal{T}(G)$ such that $T \sim x_E$ (where linear equivalence is meant for the graph D_E).*

2.3 The definition of the canonical action

Now we define an action of the sandpile group of a planar ribbon graph on the set of spanning trees. The definition uses only the embedding and needs no "base point" as auxiliary data. This action agrees with the planar rotor-routing action and with the planar Bernardi action.

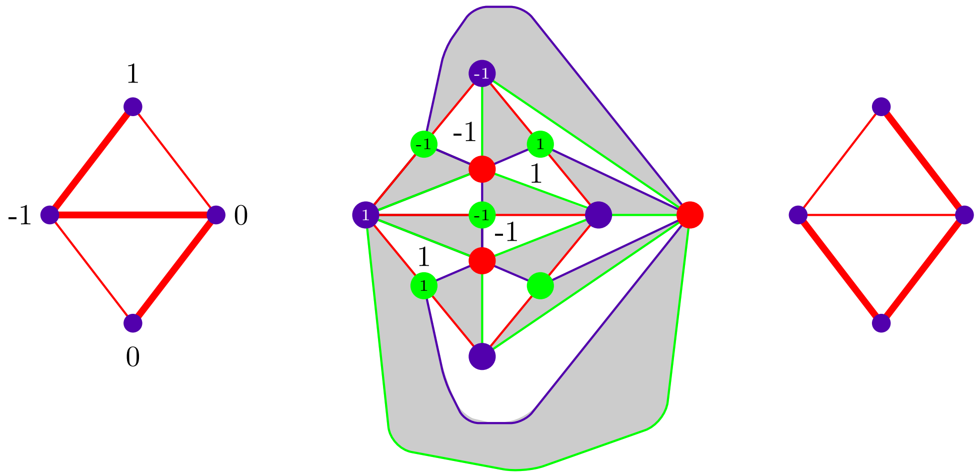

Figure 2: An example for the computation of the canonical action. See Example 14 for more details.

By Theorem 12, spanning trees correspond to $\mathrm{Pic}^{|V|-1}(D_E)$, hence we can define an action of $\mathrm{Pic}^0(D_E)$ on the spanning trees of G as the action of $\mathrm{Pic}^0(D_E)$ on its coset. In other words, for any spanning tree T, and $x \in \mathrm{Pic}^0(D_E)$, $x + T \in \mathrm{Pic}^{|V|-1}(D_E)$, hence there is exactly one spanning tree T' such that $x + T \sim T'$ (in D_E). Let us define $x \oplus T = T'$, and this is the action of x on T.

Now we can use the canonical isomorphism between $\mathrm{Pic}^0(G)$ and $\mathrm{Pic}^0(D_E)$ to pull this action to $\mathrm{Pic}^0(G)$.

Definition 13. Let G be a planar ribbon graph. For an element $x \in \mathrm{Pic}^0(G)$, and $T \in \mathcal{T}(G)$, we define $c(x, T) = \varphi_{V\to E}(-x) \oplus T$.

As the definition of the sandpile action and the isomorphism $\varphi_{V\to E}$ depended only on the embedding, $c : \mathrm{Pic}^0(G) \times \mathcal{T}(G) \to \mathcal{T}(G)$ is indeed well defined.

Example 14. Figure 2 shows the computation of $c(x, T)$ for a concrete example. The first panel shows a spanning tree T and a chip configuration x. The second panel shows $(-x, \varphi_{V\to E}(-x), 0)$, as well as the linear combination of white triangles witnessing this. For clarity of the picture, we omitted the 0 coordinates. Of course there are many representatives of $\varphi_{V\to E}(-x)$. We chose to draw the representative whose addition to T results in a spanning tree. The third panel shows this unique spanning tree T' in the linear equivalence class (in D_E) of $\varphi_{V\to E}(-x) + T$. Hence $c(x, T) = T'$.

Theorem 15. *[8] The canonical action agrees with the rotor-routing action, that is, for any plane (ribbon) graph G, $x \in \mathrm{Pic}^0(G)$ and $T \in \mathcal{T}(G)$, we have $c(x, T) = r(x, T)$ where r denotes the rotor-routing action.*

3 Consistency

Suppose that for each plane ribbon graph G, α_G is a torsor of $\mathrm{Pic}^0(G)$ on $\mathcal{T}(G)$. Note that if G is a plane ribbon graph and e is an edge of G, then a ribbon structure gets induced on $G \setminus e$ and also on G/e, moreover, these are also planar. When we write $G \setminus e$ or G/e for a plane graph G, we assume that the minors are equipped with this induced plane ribbon structure.

Using this observation, Ganguly and McDonough introduced the following property of sandpile torsors of ribbon graphs.

For a set of edges $A \subseteq E$, we denote $V(A) = \{v \in V : \exists e \in A \text{ s.t. } v \text{ is incident to } e\}$.

Definition 16. [6, Definition 4.3] A sandpile torsor algorithm α of plane ribbon graphs is consistent if for every plane graph G, every choice of $f \in E(G)$, and every choice of $T \in \mathcal{T}(G)$, the following three properties hold (where we suppose that $V(f) = \{c, s\}$):

(1) For any $e \in E(G)$ such that $V(e) \neq \{c, s\}$, if $e \in T \cap \alpha_G([c - s], T)$, then

$$\alpha_G([c - s], T) \setminus e = \alpha_{G/e}([c - s], T \setminus e).$$

(2) For any $e \in E(G)$, if $e \notin T \cup \alpha_G([c - s], T)$, then

$$\alpha_G([c - s], T) \setminus e = \alpha_{G\setminus e}([c - s], T).$$

(3) For any $e \in E(G) \setminus f$, if there is a cut vertex x such that all paths from e to f pass through x, then

$$e \in T \Leftrightarrow e \in \alpha_G([c - s], T).$$

Proposition 17. *Let G be a plane graph. Then $c_G(x, T) = \varphi_{V\to E}(-x) \oplus T$ satisfies the three above conditions.*

Proof. Let $T' = c_G([c - s], T)$. Then $\varphi_{V\to E}([s - c]) \sim T' - T$. By definition, this means that $(s - c, T' - T, 0) \approx_W (0, 0, 0)$. The above white triangle equivalence is very special, since there are only 2 nonzero violet coordinates, and T and T' are both spanning trees. We will use the following structural result.

Lemma 18. *If $(s - c, T' - T, 0) \approx_W (0, 0, 0)$ where T and T' are spanning trees, then there is a path P in G_E leading from s to c such that $\Delta_0, \Delta_1, \ldots, \Delta_{2t-1}$ are the white triangles incident to P (in this order) and $(s - c, T' - T, 0) = \sum_{i=0}^{t-1}(\mathbf{1}_{\Delta_i} - \mathbf{1}_{\Delta_{i+1}})$.*

For an example, see Figure 3.

Proof. Take an integer linear combination $\sum a_i \mathbf{1}_{\Delta_i} = (s - c, T' - T, 0)$ of white triangles witnessing $(s - c, T' - T, 0) \approx_W (0, 0, 0)$. As $s := v_0$ has value 1, at least 1 triangle Δ_0 is incident to s with a positive coefficient. Take the red node r_1 of the triangle Δ_0. As r_1 is red, it has value 0 in $\sum a_i \mathbf{1}_{\Delta_i}$, hence at least one triangle Δ_1 is incident to it with negative coefficient. Let v_2 be the violet node of Δ_1. Either $v_2 = c$, or v_2 has value 0 in $s - c$, and then there is at least one triangle incident to it with positive coefficient.

Continuing with this argument, we find a series of triangles $\Delta_0, \Delta_1, \dots \Delta_{2t+1}$ with alternating coefficient signs, and such that the emerald edges of these triangles form a walk from s to c in G_E. By subtracting $\sum_{i=0}^{t}(\mathbf{1}_{\Delta_{2i}} - \mathbf{1}_{\Delta_{2i+1}})$ from $\sum a_i \mathbf{1}_{\Delta_i}$, we get a linear combination of white triangles where the sum of absolute values of the coefficients decreased, and now the coefficient of each violet and red node is 0.

If there is still any triangle with nonzero (say, positive) coefficient, then, for both its red and violet vertex, another triangle is incident with negative coordinate. This way, we can find a cycle $C = \{v'_0, r'_1, \dots v'_{2k-2}, r'_{2k-1}\}$ in G_E such that the incident white triangles have alternately positive and negative coefficients. Subtracting the sum $\mathbf{1}_{\Delta'_0} - \mathbf{1}_{\Delta'_0} + \dots - \mathbf{1}_{\Delta'_{2k-1}}$ from the linear combination (where Δ'_{2i} is the triangle incident to v'_{2i} and r'_{2i+1} and Δ'_{2i+1} is the triangle incident to r'_{2i+1} and v'_{2i+2}) further reduces the absolute sum of coefficients of triangles.

Altogether, we conclude that the sum $\sum a_i \mathbf{1}_{\Delta_i} = (c - s, T' - T, 0)$ can be written as the sum of alternating signed triangles along a path P in G_E from s to c and some cycles $C_1, \dots C_t$ in G_E. Also, note that if a triangle has positive coefficient for one of the cycles or for P, then it has a nonnegative coefficient for each of them, and similarly for negative coefficients. We will show that $t = 0$, that is, in fact we do not have any additional cycles, only the path P.

Any cycle C in G_E divides the plane into two connected components. If we take the triangles along C with alternately $+1$ and -1 coefficients, then all the triangles with coefficient $+1$ fall into one of the components, and all the triangles with coefficient -1 fall into the other component. Let us call the component with the $+1$ coefficients the *interior* of the cycle C, and call the component with the -1 coefficients the *exterior* of the cycle C.

Now let us take the path P in G_E connecting c and s mentioned above. Let us add the edges (c, f) and (f, s) to P, creating a cycle P_c. As (c, f) and (f, s) are edges of G_R, this is now a cycle in $G_E \cup G_R$. Note however, that it is a proper cycle (i.e. it does not contain vertex repetitions), as f is not a vertex of G_E, and hence it is also not a vertex of P. Hence P_c also divides the plane into two connected components, and once again, one of these components contains all the triangles with coefficient $+1$ along P (we call this component the *interior* of P_c), and the other component contains all the triangles with coefficient -1 along P (we call this component the *exterior* of P_c).

Now take the union of the cycles $P_c, C_1, \dots C_t$ as a subgraph of $G_R \cup G_E$. (An edge might occur in multiple cycles, but we take it only once.) We claim that if $t \geq 1$, then the graph $P_c \cup C_1 \cup \dots \cup C_t$ has a cycle C that is a subgraph of G_E, moreover, one of the regions bounded by C is a facet of $P_c \cup C_1 \cup \dots \cup C_t$, and the triangles incident to C within this facet have all the same coordinate signs. (We call this region the good region of C.)

Let us show the existence of C by induction on t. If $t = 1$, then if either the interior or the exterior of C_1 is disjoint from P_c, then (since there are no cancellations in the triangle coefficients) $C = C_1$ is suitable with the region that is disjoint from P_c. If both regions of C_1 intersect P_c then take the intersection of the interiors and the intersection of the exteriors of C_1 and P_c. Both of the intersections will by nonempty by our assumption. The boundary of these intersections will be a set of cycles (maybe more than 2), and only one of them can contain (c, f) and (f, s) on its boundary, hence at least one of the cycles will be good. If we know the statement for $t-1$, then by adding the cycle C_t, either one of its regions is entirely within the good region of C, in which case C_t is a suitable cycle (once again, because there is no cancellation in the triangle coefficients). If one of the regions of C_t is entirely in the complement of the good region of C, then C continues to be good. Finally, suppose that C_t intersects both regions of C. If the good region of C had positive coefficients, take a component of the intersection of the interior of C_t and the good region of C. If the good region of C had negative coefficients, take a component of the intersection of the exterior of C_t with the good region of C. These will be suitable.

We claim that the existence of C contradicts the fact that T and T' are both spanning trees. Suppose that the good region of C had positive coefficients. Let U be the set of emerald nodes inside the good region of C. Then $|T' \cap U| - |T \cap U| \geq d_{D_E}(E-U, U)$, as in $T'-T$ each emerald node of U receives at least one chip for each white triangle incident to it such that the emerald edge of the triangle belongs to C, and these white triangles correspond to edges of D_E leading from $E-U$ to U. As explained in the proof of [7, Theorem 3.12], such an inequality implies that $|T' \cap U| \geq |V(U)|$, which is impossible for a spanning tree T'. Very similarly, one can prove that if the good region of C had negative coefficients, that implies that for the emerald nodes U lying inside the good region of C, we have $|T \cap U| \geq |V(U)|$, which is again a contradiction. We conclude that $t = 0$. ■

Notice that this implies that if for some emerald node $e \in E$, $f \neq e$, we have $T'(e) - T(e) = 0$, then in $\sum a_i \mathbf{1}_{\Delta_i} = (s-c, T'-T, 0)$, no white triangle with nonzero coefficient is incident to e. Indeed, all the triangles with positive coefficients are in the interior of P_c, and all the triangles with negative coefficients are in the exterior of P_c, with f being the only emerald node on P_c, hence the positive and negative triangles cannot cancel out for a node $e \neq f$.

Let us first suppose $e \neq f$. Both cases (1) and (2) imply that $T'(e) - T(e) = 0$. Hence for both cases, there is a linear combination $\sum a_i \mathbf{1}_{\Delta_i} = (s-c, T'-T, 0)$ witnessing $\alpha_{(G,\chi)}([s-c], T) = T'$ such that no triangle with nonzero coefficient is incident to e.

Let us examine what happens to a trinity corresponding to G upon deleting or contracting an edge e. Let v_1 and v_2 be the two violet neighbors of e in the trinity (that is, $V(e) = \{v_1, v_2\}$), and let r_1 and r_2 be the two red neighbors of e in the trinity. In $(G-e, \chi-e)$, the red nodes r_1 and r_2 are merged to a point, e is deleted, and also the the two black and the two white triangles incident to e disappear. The rest of the triangles of the trinity remain intact (with the exception that the red nodes r_1 and r_2 are merged). In $(G/e, \chi/e)$, the two violet nodes v_1 and v_2 are merged to a point, e is deleted, and the two black and the two white triangles incident to e disappear. The rest of the triangles of the trinity remain intact (with the exception that the violet nodes v_1 and v_2 are merged). In both cases, the linear combination $\sum a_i \mathbf{1}_{\Delta_i}$ remains valid, as it only contains triangles not incident to e, and those are also triangles of the deleted/contracted trinity. Hence indeed, for $G/e, \chi/e$, we have $(s-c, T'/e - T/e, 0) = \sum a_i \mathbf{1}_{\Delta_i}$ witnessing $c_{(G/e,\chi/e)}([c-s], T/e) = T'/e$ and for $G-e, \chi-e$, we have $(s-c, T'-T, 0) = \sum a_i \mathbf{1}_{\Delta_i}$ witnessing $c_{(G-e,\chi-e)}([c-s], T) = T'$.

Now we prove (2) for $e = f$. Hence $T(f) = 0$ and $T'(f) = 0$. If we have no white triangles with nonzero coefficient incident to f, then we can repeat the above argument. Suppose for a contradiction that we have such triangles. That can only happen so that the white triangle incident to (c, f) has coefficient $+1$, and the white triangle incident to (s, f) has coefficient -1. But then, for the trinity corresponding to $G - e$, we would obtain a linear combination of white triangles that sums to $(0, T' - T, 0)$. Indeed, by the disappearance of the two white triangles incident to f, the coefficients of each violet node go to zero. On the other hand, since the two red nodes incident to f merge, their coefficients continue to be zero, and the coefficients of the emerald nodes other than f remain the same. But this implies that T' and T are the same spanning tree, which in turn implies $c - s \sim 0$ in G. This can only happen if f is a bridge [1, Lemma 5.3]. But then we cannot have $T(f) = 0$ for a spanning tree, contradiction.

Now let us address part (3). Take the subgraph G_e consisting of the edges h such that there is a path from e to h not passing through x. Take also the subgraph G_f consisting of the edges h such that there is a path from f to h not passing through x. Then these subgraphs are glued together at the vertex x, and the rest of G is also glued to them at x. Note that any spanning tree T contains $|V(G_f)| - 1$ edges from G_f.

Let r be the region of the embedding of G_e that contains G_f. Then in the trinity of G, x and r form a cycle of length 2 that has G_e in its exterior, and G_f in its interior. Hence the cycle P_c is inside the 2-cycle $\{r, x\}$ (it might also contain the vertices r and/or x). Hence either the interior or the exterior of P_c is entirely in the interior of the 2-cycle $\{r, x\}$. By symmetry, we can suppose that the interior of P_c is in the interior of the 2-cycle. But then the emerald nodes in the interior of P_c are all edges of G_f, hence all the triangles with positive coefficient are incident to emerald nodes from G_f. As T and T' contains equal number of edges from G_f, this means that all white triangles with nonzero coefficients contain emerald nodes from G_f.

As e is not in G_f, we conclude that all white triangles incident to it have coefficient zero. Hence indeed, $T(e) = T'(e)$. ■

Remark 19. In Lemma 18 it is important that c and s are the endpoints of an edge. Indeed, if c and s are farther apart, then a cycle C_i can intersect the cycle P_c such that both the intersection of the interiors and the intersection of the exteriors contain red edges. Hence in this case there might not be any good region. Indeed, we show that if c and s are not neighbors, then Lemma 18 is not true. See the graph on the two left panels of Figure 5. The divisor on the first panel sends the spanning tree on the first panel to the spanning tree of the second panel. The witness for this can be seen on Figure 4. Notice that this witness induces a path and a cycle in G_E, hence it does not satisfy Lemma 18. We show that we did not choose the white triangles in a clumsy way, but there are indeed no white triangles satisfying the condition of Lemma 18. Indeed, notice that there are two neighboring violet-emerald node pairs such that one of them has coefficient 1 and the other coefficient -1. In this case the white triangle incident to the two nodes must have coefficient 0, otherwise the (only) other white triangle incident to the emerald node would need to have a coefficient with absolute

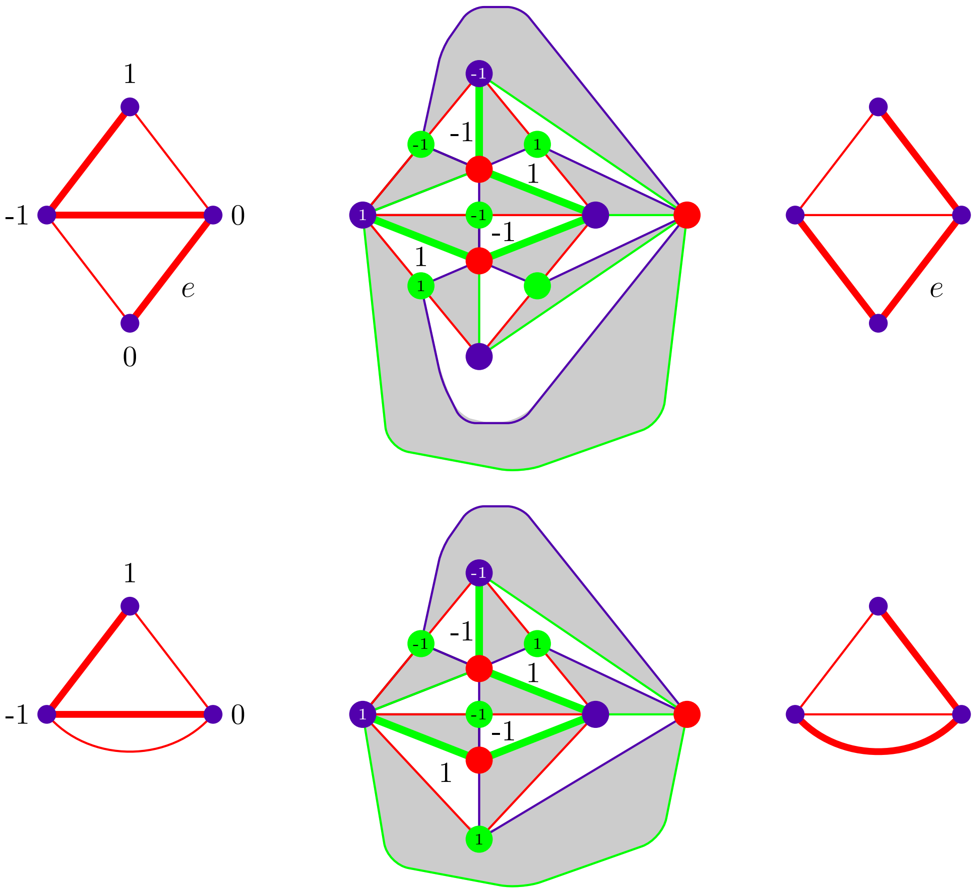

Figure 3: Computing the action in G and in G/e. The left panels show a divisor and a spanning tree, and the right panel shows the resulting spanning tree. The middle panel shows the white triangle equivalence witnessing the action. The path P indicated in Lemma 18 is drawn with thick lines.

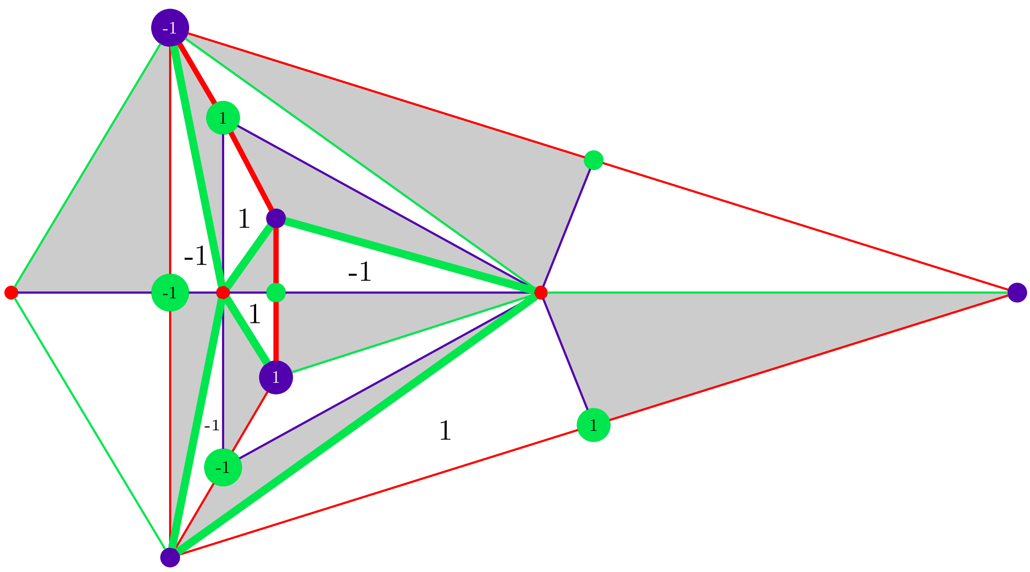

Figure 4: The action of $[c - s]$ does not induce a flat witness. (Some of the lines and triangles of the trinity are not drawn, but they do not matter in this construction.)

value more than 1. Now these two triangles with coefficient zero uniquely determine the coefficient of the rest of the trianles. (We have not drawn some of the triangles, but they would receive coefficient 0.)

Remark 20. The above example also shows us that c and s being endpoints of an edge is an important criterion in the definition of consistency. Indeed, Figure 5 shows us an example with a c and s that are not neighbors. The analogue of consistency does not hold there.

Notice that the example on Figure 5 is created using the counterexample for Lemma 18 (for c and s not neighbors). This suggests that Lemma 18 is a phenomenon behind consistency, and not just an arbitrary technical tool.

Acknowledgements

Supported by the National Research, Development and Innovation Office of Hungary – NKFIH, grant no. 132488, by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences, and by the ÚNKP-22-5 New National Excellence Program of the Ministry for Innovation and Technology, Hungary. Partially supported by the Counting in Sparse Graphs Lendület Research Group of the Alfréd Rényi Institute of Mathematics.

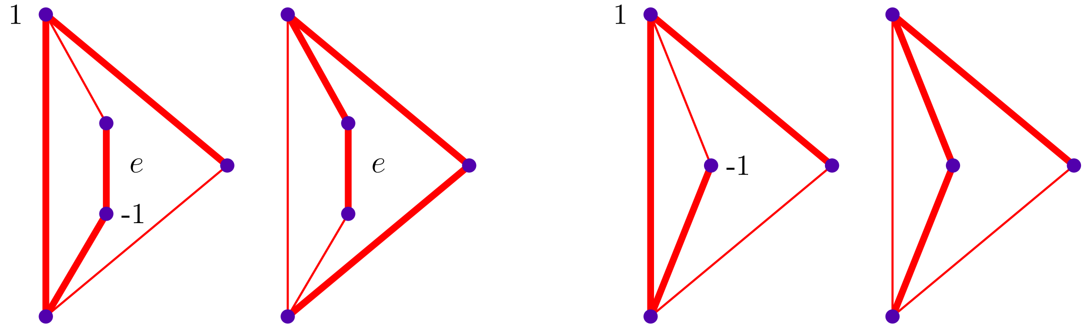

Figure 5: An example showing that c and s being neighbors is important. In the two left panels we can see the effect of $c - s$ where c and s are not neighbors. On the two right panels, the edge $e \in T \cap T'$ is contracted. We can see that $c - s$ does not send T/e to T'/e in the resulting graph G/e.

References

[1] M. Baker and Y. Wang, *The Bernardi process and torsor structures on spanning trees*, Int. Math. Res. Not. 2017, doi: 10.1093/imrn/rnx037

[2] N. J. Cavenagh and I. M. Wanless, *Latin trades in groups defined on planar triangulations*, J. Algebr. Comb. 30 (2009), no. 3, 323–347.

[3] M. Chan, T. Church, and J. Grochow. *Rotor-routing and spanning trees on planar graphs*, Int. Math. Res. Not. 11 (2015) 3225–3244.

[4] R. Cori and D. Rossin, *On the Sandpile Group of Dual Graphs*, Europ. J. Combinatorics 21 (2000), 447–459.

[5] A. E. Holroyd, L. Levine, K. Mészáros, Y. Peres, J. Propp, and D. B. Wilson. *Chip-firing and rotor-routing on directed graphs.* In V. Sidoravicius and M. E. Vares, editors, In and Out of Equilibrium 2, volume 60 of Progress in Probability, pp. 331–364. Birkhäuser Basel, 2008.

[6] Ankan Ganguly and Alex McDonough, *Rotor-routing induces the only consistent sandpile torsor structure on plane graphs*, arXiv:2203.15079, 2022.

[7] Tamás Kálmán, Seunghun Lee, Lilla Tóthmérész, *A canonical definition for the planar Bernardi/rotor routing action*

[8] Tamás Kálmán, Seunghun Lee, Lilla Tóthmérész, *The sandpile group of a trinity and a canonical definition for the planar Bernardi action*, accepted to Combinatorica.

[9] Caroline Klivans. *The Mathematics of Chip firing.* Chapman & Hall, 2018.