The globally rigid complete bipartite graphs

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Abstract

Let d be a positive integer. We prove that a complete bipartite graph $K_{m,n}$ on at least three vertices is globally rigid in $\mathbb{R}^d$ if and only if $m, n \geq d+1$ and $m+n \geq \binom{d+2}{2}+1$.

1 Introduction

The goal of this note is to give a short proof for the following theorem, which appeared in the handbook chapter [6] without proof. A different proof (of sufficiency) can be found in [3]. We follow the terminology and notation of [6].

Theorem 1.1. [6, Theorem 63.2.2] A complete bipartite graph $K_{m,n}$ on at least three vertices is globally rigid in $\mathbb{R}^d$ if and only if $m, n \geq d+1$ and $m+n \geq \binom{d+2}{2}+1$.

The condition requiring that the graph has at least three vertices, which was not part of the original statement in [6], is needed to exclude the trivial case when the graph is $K_2 = K_{1,1}$. This graph is globally rigid in $\mathbb{R}^d$ for all $d \geq 1$.

The proof relies on some central results of rigidity theory. To prove sufficiency, we need the following three theorems. The rigid complete bipartite graphs were characterized by W. Whiteley.

Theorem 1.2. [8] A complete bipartite graph $K_{m,n}$, with $m, n \geq 2$, is rigid in $\mathbb{R}^d$ if and only if $m, n \geq d+1$ and $m+n \geq \binom{d+2}{2}$.

We say that a graph G is *vertex-redundantly rigid* in $\mathbb{R}^d$ if $G - v$ is rigid in $\mathbb{R}^d$ for all $v \in V(G)$. The next result is due to S. Tanigawa.

Theorem 1.3. [7] Let G be a graph. If G is vertex-redundantly rigid in $\mathbb{R}^d$ then it is globally rigid in $\mathbb{R}^d$.

The *d-dimensional edge splitting* operation replaces an edge of graph G with a new vertex joined to the end vertices of the edge and to $d-1$ other vertices. R. Connelly proved that this operation preserves global rigidity in the following sense.

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Theorem 1.4. [2] Let G be a graph obtained from K_{d+2} by iteratively adding edges or performing d-dimensional edge splitting operations. Then G is globally rigid in $\mathbb{R}^d$.

In the proof of necessity we shall use the following theorems. We say that a graph G is *redundantly rigid* in $\mathbb{R}^d$ if $G - e$ is rigid in $\mathbb{R}^d$ for all $e \in E(G)$. The following two necessary conditions for global rigidity are due to B. Hendrickson.

Theorem 1.5. [5] Let G be globally rigid in $\mathbb{R}^d$. Then either G is a complete graph on at most $d+1$ vertices or G is $(d+1)$-connected and redundantly rigid in $\mathbb{R}^d$.

A further necessary condition, which is valid for complete bipartite graphs, was shown by R. Connelly, see also [4].

Theorem 1.6. [1] Let $d \geq 3$ be an integer. Then no complete bipartite graph $K_{m,n}$, with $m, n \geq d+2$ and $m + n = \binom{d+2}{2}$ is globally rigid in $\mathbb{R}^d$.

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1: We first prove sufficiency. Let us consider $K_{m,n}$ with $m, n \geq d+1$ and $m + n \geq \binom{d+2}{2} + 1$. If the stronger lower bound $m, n \geq d+2$ is also satisfied, then we can use Theorem 1.2 to deduce that the deletion of any vertex gives rise to a rigid graph in $\mathbb{R}^d$. Hence $K_{m,n}$ is globally rigid in $\mathbb{R}^d$ by Theorem 1.3, and we are done.

So we may assume that $m = d+1$, in which case $m+n \geq \binom{d+2}{2}+1$ gives $n \geq \binom{d+1}{2}+1$. First suppose that $n = \binom{d+1}{2} + 1$. In this case the graph can be obtained from K_{d+2} by a sequence of d-dimensional edge splitting operations as follows. Let us denote the vertices of K_{d+2} by $\{v_0, v_1, \ldots, v_{d+1}\}$. By splitting every edge $v_i v_j$ of K_{d+2} with $1 \leq i < j \leq d+1$, in any order, so that the new vertex created by the splitting is always connected to the vertices v_ℓ, $1 \leq \ell \leq d+1$, we obtain the graph $K_{d+1,\binom{d+1}{2}+1}$. Thus global rigidity in $\mathbb{R}^d$ follows from Theorem 1.4.

If n is greater than $\binom{d+1}{2} + 1$, then $K_{d+1,n}$ can be obtained from $K_{d+1,\binom{d+1}{2}+1}$ by repeatedly adding vertices of degree $d+1$. This operation can also be interpreted as adding a new edge and then performing an edge split. So by applying Theorem 1.4 again, we obtain that $K_{d+1,n}$ is globally rigid in $\mathbb{R}^d$. This proves sufficiency.

We next prove necessity. Let $K_{m,n}$ be a complete bipartite graph which is globally rigid in $\mathbb{R}^d$, with $m + n \geq 3$. Theorem 1.5 implies that either $K_{m,n}$ is a complete graph on at most $d+1$ vertices, or it is $(d+1)$-connected. In the former case we must have $m = n = 1$, which contradicts the assumption $m + n \geq 3$. In the latter case $m, n \geq d+1$ follows. Theorem 1.2 then gives $m + n \geq \binom{d+2}{2}$, for otherwise $K_{m,n}$ is not even rigid in $\mathbb{R}^d$. We are done if the inequality is strict, so it remains to consider the case when $m + n = \binom{d+2}{2}$, and to show that it is impossible. If $m, n \geq d+2$, then we must have $d \geq 3$. Since $K_{m,n}$ is globally rigid in $\mathbb{R}^d$, Theorem 1.6 implies that this cannot hold. Finally, if $m = d+1$, then a simple computation shows that $m \cdot n = d(n+m) - \binom{d+1}{2}$, which means that $K_{m,n}$ is minimally rigid in $\mathbb{R}^d$. But it is not possible by Theorem 1.5. This completes the proof.

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