A note on the existence of EFX allocations for negative additive valuations

Kristóf Bérczi* and Fekadu Tolessa Gedefa**

Abstract

We study the problem of fairly allocating a set S of m indivisible items among a set N of n agents with individual preferences. The notion of fairness considered here is envy-freeness up to any item (EFX), a well-studied relaxation of envy-freeness. In spite of the considerable efforts over the past years, the existence of EFX solutions is wide open. In particular, the problem of finding EFX solutions has been resolved only in very restricted cases of negative additive valuations.

In this paper, we show that there always exists an EFX solution for at most seven items and four agents, each having one of two possible negative additive valuations. Our proof is algorithmic and so provides an efficient procedure that determines an EFX allocation.

1 Introduction

In a fair division problem we are given a set S of m items and a set N of n agents with individual preferences over the subsets of items, and the goal is to allocate the items to agents in such a way that that each agent finds the allocation fair. The problem is motivated by numerous real-world applications, such as the division of taxi fare or rent, task distribution, division of inheritance, partnership dissolutions, divorce settlements, electronic frequency allocation, airport traffic management, and exploitation of Earth observation satellites. Due to its wide applicability, deciding the existence of fair allocations is an active research area in mathematics, social choice theory, dispute resolution, and computer science, see [3, 2, 5, 9, 12, 16, 18, 21, 15, 17, 19].

*MTA-ELTE Momentum Matroid Optimization Research Group and MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös Loránd University, Budapest, Hungary. Email: kristof.berczi@ttk.elte.hu.

**Department of Operations Research, Eötvös Loránd University, Budapest, Hungary. Email: gedefaft@math.elte.hu.

Previous work. The origins of problem goes back to the work of Steinhaus [22]. Following his work, several fairness concepts were introduced. Among them, the notion of *envy freeness (EF)*, introduced by Foley [14], is probably the most prominent and well-established one which requires that each agent prefers his own bundle over that of any other agent. The work on envy-free allocations mainly focused on divisible items [1, 4, 13].

When the items are indivisible, an envy-free allocation does not necessarily exist, which motivated several relaxations of envy-freeness [8, 9, 18]. One of these relaxations, introduced by Lipton [18] and Budish [8] for non-negative valuations is *envy-freeness up to one item (EF1)*. In this setting, an allocation of indivisible items is EF1 if any envy between agents can be eliminated by deleting some item from the envied agent's bundle. Note that no item is removed from the envied agent's bundle, this is just a thought experiment to quantify the envy that the envious agent has toward the envied agent. For additive valuations, a simple round-robin algorithm always gives an EF1 allocation [10], while a standard envy-graph based algorithm provides an EF1 allocation for more general (sub-additive) valuations [18].

In many situations, the most valuable item might be the main reason for the presence of envies. Therefore, stronger notions of fairness are inevitable. Caragiannis et al. [10] suggested another interesting relaxation of envy-freeness, called *envy-freeness up to any item (EFX)*, which attracted considerable attention. An allocation is said to be EFX if no agent envies another agent after the removal of any item from the other agent's bundle. Thus EFX, though strictly weaker than EF, is strictly stronger than EF1. As remarked in [9]: "*Arguably, EFX is the best fairness analog of envy-freeness of indivisible items*". In contrast to EF1 allocations, the existence of EFX allocations is open apart for some restricted cases. Plaut and Roughgarden [21] proved that EFX allocations exist for two agents with arbitrary valuations, and for any number of agents with identical valuation functions. Bérczi et al. [6] formalized variants of EFX for non-monotone instances with indivisible items, and showed that an EFX allocation may not exist for two agents with non-monotone, non-additive, identical valuation functions. They also proved the existence of an EFX allocation for non-positive instances with identical monotone valuation functions. Quite recently, for non-degenerate additive valuation functions, Berger et al. [7] verified the existence of an EFX allocation in the setting of four agents. Similarly, EFX allocations was shown to exist when the agents can be partitioned into two types based on their additive valuation functions [20], or when agents have dichotomous preferences [2].

The long list of results show that deciding the existence of EFX allocations is indeed a central problem. However, most works concentrated on non-negative valuations. The goal of this paper is to initiate a systematic study of negative additive valuations as well.

Our results. There are various tools that proved to be helpful when dealing with non-negative valuations. Somewhat surprisingly, these standard techniques are not applicable for negative valuations, and settling the existence of EFX allocations requires new tools and ideas.

Motivated by the result of Berger et al. [7], it is natural to ask the following: Can one always find an EFX allocation for four agents with negative additive valuations? This problem already seems to be quite intricate in general, hence we concentrate on the special case when each agents's valuation is one of two negative additive valuation functions. Our main contribution is a proof showing that such an EFX allocation exists if the number of items is at most 7.

The paper is organized as follows. Basic notations and the precise definitions of the different fairness concepts are introduced in Section 2, together with an explanation why previous techniques are no longer applicable in our setting. Section 3.1 shows the existence of an EFX solution when all agents but one have the same negative additive valuation function. In Section 3.2, we prove that an EFX allocation always exists for at most seven items and four agents, each having one of two negative additive valuation functions.

2 Preliminaries

Notation. We denote the *set of non-positive reals* by $\mathbb{R}_-$. In our setting, a fair allocation problem consists of a set N of n agents and a set S of m items. Each agent $i \in N$ has a valuation function $v_i : 2^S \to \mathbb{R}$ over the subsets of S. By convention, we assume that $v_i(\emptyset) = 0$ for $i \in N$. Throughout the paper, we consider *additive valuations* where the value of a set of items for any agent is equal to the sum of the values of the individual items in the set, i.e., for any agent $i \in N$ and any set of items $X \subseteq S$ we have $v_i(X) = \sum_{s \in X} v_i(s)$. A valuation is *negative* if the value of any subset of items is non-positive. In our setting, we assume that each agent has one of two negative additive valuation functions, denoted by v_α and v_β.

An allocation of the items is a partition $\pi = \{\pi(1), \dots, \pi(n)\}$ of S, where $\pi(i) \subseteq S$ is the bundle of agent i. Agent i *envies* agent j if $v_i(\pi(i)) < v_i(\pi(j))$. In our setting, let $(v_\alpha(s_i), v_\beta(s_i))$ denote the values of the i^{th} item with respect to v_α and v_β. To any allocation π, we associate a directed graph $G_\pi = (N, E)$ called the *envy graph*, where there is a directed edge from i to j if agent i envies agent j for the given allocation π. Given an allocation π and a directed cycle C in G_π, the *cycle-swapped allocation* π^C is obtained by reallocating bundles backwards along the cycle. For each agent i in the cycle, define i^+ to be the agent for which i is in the cycle. Using this notation, $\pi(i)^C = \pi(i^+)$ if $i \in C$, otherwise $\pi(i)^C = \pi(i)$.

Envy-freeness for negative additive valuations. Assume that each agent has a negative additive valuation and consider an allocation $\pi = \{\pi(1), \dots, \pi(n)\}$. The allocation is *envy-free* if no agent envies another agent:

(EF) For any $i, j \in N$ inequality $v_i(\pi(i)) \geq v_i(\pi(j))$ holds.

The allocation is *envy-free up to one item* if envies might be present, but those can be eliminated by deleting an item from the envious agent's bundle:

(EF1) For any $i, j \in N$ at least one of the following holds:

(i) $v_i(\pi(i)) \geq v_i(\pi(j))$

(ii) $v_i(\pi(i) - s) \geq v_i(\pi(j))$ for some $s \in \pi(i)$.

In this paper, we are focusing on a less permissive relaxation of envy-freeness called *envy-freeness up to any item* in which envies might be present, but those can be eliminated by deleting any item from the envious agent's bundle:

(EFX) For any $i, j \in N$ at least one of the following holds:

(i) $v_i(\pi(i)) \geq v_i(\pi(j))$

(ii) $v_i(\pi(i) - s) \geq v_i(\pi(j))$ for every $s \in \pi(i)$.

If $v_i(\pi(i)) < v_i(\pi(j))$ for some allocation π, then we refer to this envy as an *EFX envy* if i and j satisfy (ii), otherwise it is called a *non-EFX envy*.

Applicability of old techniques. Among the techniques used in the earlier works to show existence of EFX solution for positive value items, *cycle-elimination* and *champions* are the most prominent ones. Both techniques are primarily based on graph-theoretical concepts, namely the *envy-graph* and the *champions-graph*.

The notion of champions was introduced in [11] for positive valuations. To understand this concept, we first need the notion of a *most envious agent*, first appeared in [12]. Consider an allocation π of the items, and a set $Z \subseteq S$ of items that is envied by at least one agent i. Let $Z_i \subseteq Z$ be a subset of smallest size satisfying $v_i(\pi(i)) < v_i(Z_i)$. That is, for any $T_i \subsetneq Z_i$, we have $v_i(\pi(i)) \geq v_i(T_i)$. The agent i with the smallest value of $|Z_i|$ is called the most envious agent for set Z. Now let π' be a partial EFX allocation, and let s be an unallocated item. We say that i champions j with respect to s if i is a most envious agent for $\pi'(j) + s$. The champions graph $M_{\pi'}$ is a directed graph in which each vertex corresponds to an agent, and there is a directed edge $(i, j) \in M_{\pi'}$ if and only if i champions j.

By relying on the aforementioned tools, Berger et al. [7] showed the existence of EFX allocations for four agents with positive additive valuation functions. They iteratively constructed a sequence of partial EFX allocations in which each allocation Pareto dominates its predecessor to obtain an EFX allocation that leaves at most one item unallocated. Moreover, their results extend beyond additive valuations to all nice cancelable valuations. Using similar techniques, Mahara [20] showed the existence of EFX allocations when every agent's valuation function is one of two positive additive valuations. The idea of their proof is to reshuffle the current partial allocation based on the envy-graph or the champions-graph to obtain a new allocation with higher potential, while preserving the EFX property.

At first glance, an instance with negative valuations seems to be just the opposite of one with positive valuations, and thus one might intuitively expect that the natural adaptation of algorithms designed to construct a sequence of EFX allocations for positive valuations would also work for this setting. However, it turns out that this is not the case, as we explain below.

Example 1. Consider the following instances with eight items $s_1, \ldots, s_8$ and four agents with valuation function v_α and v_β, where v_α is the valuation of agents 1 and 2, and v_β is the valuation of agents 3 and 4; see Table 1.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
v_α	-1	-2	-3	-3	-5	-6	-6	-6
v_β	-5	-2	-2	-5	-1	-4	-5	-6

Table 1: The values of v_α and v_β.

Let π denote the allocation $\pi(1) = \{s_5, s_6\}$, $\pi(2) = \{s_3, s_8\}$, $\pi(3) = \{s_2, s_4\}$, $\pi(4) = \{s_1, s_7\}$. It is not difficult to check that π is an EFX allocation. Still, the directed cycle $1 \to 4 \to 2 \to 3 \to 1$ is contained in the envy-graph. In the case of positive valuations, a standard step is to remove this cycle by applying a cycle-elimination step when for each edge ij of the cycle, the bundle of agent i is replaced with that of agent j. In our example, such a step results in the allocation π' where $\pi'(1) = \{s_1, s_7\}$, $\pi'(2) = \{s_2, s_4\}$, $\pi'(3) = \{s_5, s_6\}$, $\pi'(4) = \{s_3, s_8\}$. Then, for the allocation π', agent 1 envies agent 2, and this is a non-EFX envy as deleting s_1 from the bundle of agent 1 does not eliminate it.

The above example shows that, unlike for positive valuation functions, the standard cycle-swapping step is no longer applicable for negative valuation functions. As for the champions-graph, it is not clear how to extend the notion of champions to the case of negative additive valuations since if $v_i(\pi(i)) < v_i(Z)$ for some agent i and subset Z of items, then $v_i(\pi(i)) < v_i(Z')$ holds for every subset $Z' \subseteq Z$ by $v_i(Z - Z') \leq 0$.

3 EFX allocations for two negative additive valuations

In this section, we discuss the existence of EFX allocations for two negative additive valuation functions. When there is at most one agent whose valuation differs from the others', the existence of such a solution follows from previous results. However, the case when there are two such agents is significantly more difficult. As a starting step toward understanding the general case, here we concentrate on the case of four agents with at most seven items. In this setting, the proof is based on a careful analysis of the relation of the two valuations.

3.1 Almost identical valuations

As a warm up, we show that an EFX allocation always exists when all agents but at most one have identical negative additive valuations, and the possibly remaining agent has a different negative addditive valuation. The proof relies on a more general result of [6] on EFX allocations of chores for monotone (not necessarily additive) valuation functions.

Theorem 1. *There always exists an EFX allocation when all agents but at most one have the same negative additive valuation v_α, and the possibly remaining agent has a different negative additive valuation v_β.*

Proof. If all agents have the same negative additive valuations, then an EFX allocation exists by [6, Theorem 3]. Assume now that one of the agents has a different negative additive valuation v_β. In this case, find an EFX allocation using [6, Theorem 3] as if all agents Shared the same valuation function v_α. Then give the bundle with highest v_β value to the agent with valuation v_β. We claim that the allocation thus obtained is EFX. Indeed, if an agent with valuation v_α envies another agent, then this envy is EFX due to the construction, while the agent with valuation v_β received the best bundle with respect to v_β, hence she does not envy anyone else. □

3.2 Four agents with at most seven items

Now we turn to the proof of the existence of an EFX allocation for at most seven items and four agents with at most two different valuation functions.

Theorem 2. *There always exists an EFX allocation for at most seven items and four agents, each having one of two negative additive valuations v_α and v_β.*

Proof. If the number of agents having valuation v_α is at most one, then an EFX allocation exists by Theorem 1. The same holds for v_β, too. Hence we may assume that agents 1 and 2 have valuation v_α, while agents 3 and 4 have valuation v_β. Let $\alpha_1 \geq \cdots \geq \alpha_m$ and $\beta_1 \geq \cdots \geq \beta_m$ denote the values of v_α and v_β in a non-increasing order, respectively. We denote the items by $S = \{s_1, \ldots, s_m\}$, where $v_\alpha(s_k) = \alpha_k$ for $1 \leq k \leq m$. As the ordering of the items with respect to v_α and v_β might be different, $v_\beta(s_k) = \beta_k$ does not necessarily hold. Note that by the negativity of the valuations and by $v_\alpha(\emptyset) = v_\beta(\emptyset) = 0$, the envy of an agent i can be non-EFX only if $|\pi(i)| \geq 2$.

If $m = 4$, then allocating at most one item to each agent arbitrarily results in an EFX allocation.

If $m = 5$, then let $\pi(1) \coloneqq \{s_1, s_2\}$, $\pi(2) \coloneqq \{s_3\}$, $\pi(3) \coloneqq \{s_4\}$ and $\pi(4) \coloneqq \{s_5\}$. It suffices to verify that if agent 1 envies another agent, then this envy is EFX. However, this holds by the fact that $v_\alpha(\pi(1) - s_1) = \alpha_2 \geq \alpha_k$ for $2 \leq k \leq 5$.

If $m = 6$, we consider two cases. If $\alpha_2 + \alpha_3 \geq \alpha_4$, then $\pi(1) \coloneqq \{s_1, s_2, s_3\}$, $\pi(2) \coloneqq \{s_4\}$, $\pi(3) \coloneqq \{s_5\}$ and $\pi(4) \coloneqq \{s_6\}$ is an EFX solution. If $\alpha_2 + \alpha_3 < \alpha_4$, then the allocation $\pi(1) \coloneqq \{s_1, s_4\}$, $\pi(2) \coloneqq \{s_2, s_3\}$, $\pi(3) \coloneqq \{s_5\}$ and $\pi(4) \coloneqq \{s_6\}$ is an EFX solution. To see any of these, it suffices to verify that $v_\alpha(\pi(1) - s_1) \geq v_\alpha(\pi(j))$ for $2 \leq j \leq 4$, which clearly holds in both cases.

Finally, we consider the case when $m = 7$. Observe that the second valuation v_β has not played any role until now, and an EFX allocation could always be found by relying on v_α only. The difficulties appear when the number of items reaches seven, when considering only the item values with respect to v_α is not enough to provide an EFX solution to the problem. Instead, we have to consider v_α and v_β simultaneously.

Case 1. $v_\beta(s_1) = \beta_7$.
Let $S_1 \coloneqq \{s_1, s_2\}$. Furthermore, find an EFX allocation of the items $\{s_3, \ldots, s_7\}$ with respect to v_β into three bundles S_2, S_3 and S_4. We may assume that $1 = |S_2| \leq |S_3| \leq |S_4|$. Now define $\pi(i) \coloneqq S_i$ for $1 \leq i \leq 4$.

We claim that the allocation π thus obtained is EFX. Indeed, the bundle of agent 1 consists of the two highest valued items with respect to v_α, thus $v_\alpha(\pi(1) - s_1) = v_\alpha(s_2) \geq v_\alpha(S_i)$ for $2 \leq i \leq 4$. Agent 2 has no non-EFX envy towards any other agent by $|\pi(2)| = 1$. Finally, if any of agents 3 or 4 envies agent 2, then this envy is EFX by the construction of S_2, S_3 and S_4. By the assumption of the case $v_\beta(S_1) \leq v_\beta(S_2)$ holds, implying that agents 3 and 4 have no non-EFX envy towards agent 1 either.

Case 2. $v_\beta(s_1) \neq \beta_7$.
Let s_ℓ denote the item with $v_\beta(s_\ell) = \beta_7$. By the assumption of the case, $\ell \neq 1$. Find an EFX allocation of the items $\{s_2, \ldots, s_7\} - s_\ell$ with respect to valuation v_β into three bundles S_1, S_2 and S_3. Further, set $S_4 \coloneqq \{s_\ell\}$. We may assume that $|S_1| \geq |S_2| \geq |S_3| \geq |S_4| = 1$. Observe that $|S_3| = 1$ also holds.

Now we define an allocation as follows. Give the bundle with the highest v_α value among S_1, S_2, S_3 and S_4 to agent 1, together with the item s_1. From the remaining sets, allocate a bundle of size one to agent 2; note that such a bundle exists. Finally, assign the remaining two bundles to agents 3 and 4.

We claim that the allocation π thus obtained is EFX. Indeed, if agent 1 envies another agent, then this envy is EFX as the set $\pi(1) - s_1$ is one of the bundles $S_1, \ldots, S_4$ with the highest v_α value. If agent 2 envies another agent, then this envy is EFX by $|\pi(2)| = 1$. Assume now that agent 3 or 4 envies another agent. If the envious agent's bundle is S_4, then this envy is EFX by $|S_4| = 1$. Otherwise, as the bundles S_1, S_2 and S_3 correspond to an EFX allocation of the items $\{s_2, \ldots, s_7\} - s_\ell$ with respect to v_β, the envy is certainly EFX unless the envied agent's bundle contains s_ℓ, that is, the envied bundle is either $\{s_\ell\}$ itself or $\{s_1, s_\ell\}$. In either case, the envy must be EFX as, by $|S_3| = 1$, we have $v_\beta(\{s_1, s_\ell\}) \leq v_\beta(s_\ell) = \beta_7 \leq v_\beta(S_3) \leq v_\beta(\pi(i) - s)$ for $i = 3, 4$ and any $s \in \pi(i)$, where the last inequality follows from $\pi(i) \in \{S_1, S_2, S_3\}$ for $i = 3, 4$ and from S_1, S_2 and S_3 being EFX with respect to v_β. $\square$

4 Conclusion

In this paper, we considered the problem of fairly allocating a set S of m indivisible items among a set N of n agents, all having one of two negative additive valuation functions. We showed the existence of an EFX solution for at most seven items and four agents using a case-by-case analysis. Deciding the existence of an EFX allocation remains wide open for higher number of items.

Funding The work was supported by the Lendület Programme of the Hungarian Academy of Sciences – grant number LP2021-1/2021 and by the Hungarian National Research, Development and Innovation Office – NKFIH, grant numbers FK128673 and TKP2020-NKA-06.

References

[1] N. Alon. Splitting Necklaces. *Advances in Mathematics*, 63(3):247–253, 1987.

[2] G. Amanatidis, G. Birmpas, A. Filos-Ratsikas, A. Hollender, and A. A. Voudouris. Maximum Nash Welfare and other stories about EFX. *Theoretical Computer Science*, 863:69–85, 2021.

[3] H. Aziz, I. Caragiannis, A. Igarashi, and T. Walsh. Fair Allocation of Indivisible Goods and Chores. *Autonomous Agents and Multi-Agent Systems*, 36(1):1–21, 2022.

[4] H. Aziz and S. Mackenzie. A Discrete and Bounded Envy-free Cake Cutting Protocol for any Number of Agents. In *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 416–427. IEEE, 2016.

[5] S. Barman and S. K. Krishnamurthy. Approximation Algorithms for Maximin Fair Division. *ACM Transactions on Economics and Computation (TEAC)*, 8(1):1–28, 2020.

[6] K. Bérczi, E. R. Bérczi-Kovács, E. Boros, F. T. Gedefa, N. Kamiyama, T. Kavitha, Y. Kobayashi, and K. Makino. Envy-free Relaxations for Goods, Chores, and Mixed Items. *arXiv preprint arXiv:2006.04428*, 2020.

[7] B. Berger, A. Cohen, M. Feldman, and A. Fiat. (Almost) EFX Exists for Four Agents (and Beyond). *arXiv preprint arXiv:2102.10654*, 2021.

[8] E. Budish. The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes. *Journal of Political Economy*, 119(6):1061–1103, 2011.

[9] I. Caragiannis, N. Gravin, and X. Huang. Envy-Freeness Up to Any Item with High Nash Welfare: The Virtue of Donating Items. *Proceedings of the 2019 ACM Conference on Economics and Computation*, Jun 2019.

[10] I. Caragiannis, D. Kurokawa, H. Moulin, A. D. Procaccia, N. Shah, and J. Wang. The Unreasonable Fairness of Maximum Nash Welfare. *Proceedings of the 2016 ACM Conference on Economics and Computation - EC '16*, 2016.

[11] B. R. Chaudhury, J. Garg, and K. Mehlhorn. EFX Exists for Three Agents. *Proceedings of the 21st ACM Conference on Economics and Computation*, Jul 2020.

[12] B. R. Chaudhury, T. Kavitha, K. Mehlhorn, and A. Sgouritsa. A Little Charity Guarantees Almost Envy-freeness. *SIAM Journal on Computing*, 50(4):1336–1358, 2021.

[13] F. Edward Su. Rental harmony: Sperner's Lemma in Fair Division. *The American Mathematical Monthly*, 106(10):930–942, 1999.

[14] D. K. Foley. *Resource Allocation and the Public Sector.* Yale University, 1966.

[15] P. Gorantla, K. Marwaha, and S. Velusamy. Fair Allocation of a Multiset of Indivisible Items. *arXiv preprint arXiv:2202.05186*, 2022.

[16] M. Hajiaghayi. Almost Envy-freeness, Envy-rank, and Nash Social Welfare Matchings. In *Proceedings of the AAAI Conference on Artificial Intelligence*, 2021.

[17] H. Hosseini, S. Sikdar, R. Vaish, and L. Xia. Fair and Efficient Allocations under Lexicographic Preferences. In *Proceedings of the AAAI Conference on Artificial Intelligence*, 2021.

[18] R. J. Lipton, E. Markakis, E. Mossel, and A. Saberi. On Approximately Fair Allocations of Indivisible Goods. *Proceedings of the 5th ACM Conference on Electronic commerce - EC '04*, 2004.

[19] V. Livanos, R. Mehta, and A. Murhekar. (Almost) Envy-Free, Proportional and Efficient Allocations of an Indivisible Mixed Manna. *arXiv preprint arXiv:2202.02672*, 2022.

[20] R. Mahara. Existence of EFX for Two Additive Valuations. *arXiv preprint arXiv:2008.08798*, 2020.

[21] B. Plaut and T. Roughgarden. Almost Envy-freeness with General Valuations. *SIAM Journal on Discrete Mathematics*, 34(2):1039–1068, 2020.

[22] H. Steihaus. The Problem of Fair Division. *Econometrica*, 16:101–104, 1948.