Globally rigid components in two-dimensional generic frameworks

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Abstract

We show that the maximal globally rigid subgraphs of a graph in $\mathbb{R}^2$ can be determined in polynomial time.

1 Introduction

A d-dimensional framework (G, p) is called *globally rigid* in $\mathbb{R}^d$ if every other d-dimensional framework (G, q), with the same graph and same edge lengths, is congruent to (G, p). We say that a graph G is *globally rigid* in $\mathbb{R}^d$ if every d-dimensional generic[1] framework with underlying graph G is globally rigid in $\mathbb{R}^d$. For $d = 1, 2$ combinatorial characterizations and corresponding deterministic polynomial time algorithms are known for (testing) global rigidity. The existence of such a characterization (or algorithm) for $d \geq 3$ is a major open question. For more details on globally rigid graphs and frameworks see e.g. [7].

A maximal (with respect to inclusion) globally rigid subgraph of G is called a *globally rigid component* of G in $\mathbb{R}^d$. The globally rigid components are induced subgraphs. Since K_2 is globally rigid, every edge of G belongs to some globally rigid component of G. It is a folklore result in rigidity theory that a graph on at least three vertices is globally rigid $\mathbb{R}^1$ if and only if it is 2-connected. Therefore the globally rigid components of G in $\mathbb{R}^1$ are the maximal 2-connected subgraphs of G, together with the (K_2 subgraphs of) the cut-edges of G. They are easy to find in linear time.

In this note we show that the globally rigid components in $\mathbb{R}^2$ can also be found in polynomial time[2]. This result has implications concerning the so-called *globally rigid subgraph number* of a graph G, denoted by $\mathrm{grn}^*(G)$, which is a new parameter that was introduced recently in [2]. By definition $\mathrm{grn}^*(G)$ is the maximum d so that G contains a subgraph H on at least $d+2$ vertices that is globally rigid in $\mathbb{R}^d$. Based on the list of globally rigid components in $\mathbb{R}^2$, we can decide whether $\mathrm{grn}^*(G) \geq 2$ holds for a given graph G. Note that $\mathrm{grn}^*(G) \geq 1$ if and only if G contains a cycle.

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[1] A framework (G, p) is said to be *generic* if the set of its $d|V(G)|$ vertex coordinates is algebraically independent over the rationals.

[2] There was an informal remark in [1] about the existence of such a polynomial time algorithm, without proof.

2 The algorithm for determining the globally rigid components

From now on we focus on the two-dimensional case $d = 2$. Recall the definition of the two-dimensional rigidity matroid $\mathcal{R}_2(G)$ of a graph, defined on the set of edges of G, and some basic notions (see e.g. [6] for more details). Let $r_2(G)$ denote the rank of (the edge set of) G in this matroid. An edge e of G is an *M-bridge* if $r_2(G-e) = r_2(G)-1$ holds. We say that a subgraph H of G is an *M-circuit* if the edge set of H is a minimal dependent set in $\mathcal{R}_2(G)$. Thus an edge e is an M-bridge if and only if it belongs to no M-circuit of G. A graph G with at least two edges is called *M-connected* if for every pair e, f of its edges there is an M-circuit H which contains both edges.

The following characterization of globally rigid graphs in $\mathbb{R}^2$ (the equivalence of (i), (ii), and (iii) below) is from [4]. As it was noted in [5], (iii) is in fact equivalent to (iv). In the analysis of our algorithm we shall rely on (iv).

Theorem 2.1. *[4] Let G be a graph on at least four vertices. The following assertions are equivalent.*
(i) G is globally rigid in $\mathbb{R}^2$,
(ii) G is 3-connected and redundantly rigid in $\mathbb{R}^2$,
(iii) G is 3-connected and M-connected in $\mathbb{R}^2$,
(iv) G is 3-connected and contains no M-bridges in $\mathbb{R}^2$.

We call a vertex set $S \subseteq V$ in a graph $G = (V, E)$ a *vertex separator* if $G - S$ is disconnected. Let G be a graph and let S be a vertex separator in G. Let $C_1, C_2, \dots C_t$ denote the connected components of $G - S$. We say that the subgraph of G on vertex set $C_i \cup S$, for some $1 \le i \le t$, is an *extended component* of S in G. Observe that if S is a minimal separator then every extended component is connected. We shall only consider minimum (and hence minimal) separators of size at most two.

It will be convenient to call a graph *large* if it has at least four vertices and *small* otherwise. The small globally rigid graphs are K_2 and K_3.

Algorithm GRC

The input is a connected graph $G = (V, E)$ on at least four vertices. Let $\mathcal{G} = \{G\}$ be the initial collection of large subgraphs of G and let $\mathcal{E} = \emptyset$ be the initial collection of selected edges.

Repeat the following as long as possible:

(a) If there is a graph $H \in \mathcal{G}$ which contains a non-empty set F of M-bridges then replace H in $\mathcal{G}$ by the graph $H - F$ and add the edges of F to $\mathcal{E}$.

(b) If there is a graph $H \in \mathcal{G}$ which is not 3-connected then choose a minimum vertex separator S in H and replace H in $\mathcal{G}$ by the large extended components of S in H. Furthermore, add the edges of the small extended components to $\mathcal{E}$.

If no graph in $\mathcal{G}$ satisfies the properties required by (a) or (b), then terminate.

Theorem 2.2. *The subgraphs of G in $\mathcal{G}$ at the termination of Algorithm GRC are the large globally rigid components of G. The number of iterations is bounded by a polynomial function of the size of G.*

Proof. The graphs in $\mathcal{G}$ during the execution of the algorithm are large. When the algorithm terminates, every graph in $\mathcal{G}$ is large, 3-connected, and contains no M-bridges. Thus they are all globally rigid, by the implication (iv) $\rightarrow$ (i) in Theorem 2.1. Furthermore, for every large globally rigid component K of G we have that there exists a graph H in the final $\mathcal{G}$ which contains K as a subgraph, and no proper subgraph of K ever occurs in $\mathcal{G}$. These properties follow from the fact that $\mathcal{G}$ contains G at the beginning, and that no edge or vertex of K can be removed from K by the steps of the algorithm, since – by the implication (i) $\rightarrow$ (iv) of Theorem 2.1 – in the subgraph H that contains K as a subgraph no edge of K is an M-bridge and for every separator S of size at most two of H the subgraph K is contained by one of the extended components of S. These observations imply that the subgraphs of G in $\mathcal{G}$ at the termination of Algorithm GRC are exactly the large globally rigid components of G.

In order to bound the number of iterations first we claim that (although the graphs in $\mathcal{G}$ may not be edge-disjoint), the number of new copies of an edge $e = uv \in E$ created by the algorithm is at most $2|V| - 6$. These copies are created if and only if step (b) is applied to a separator $S = \{u, v\}$ of size two of some graph $H \in \mathcal{G}$. In this case each large extended component C of S in H, which is added to $\mathcal{G}$, inherits a new copy of e. For a fixed edge e consider the collection $\mathcal{C}_e$ of the sets $V(C) - S$ over all such extended components over all those steps that involve S. The key point is that $\mathcal{C}_e$ is a so-called *laminar* family: for each pair $X, Y \in \mathcal{C}_e$ we have $X \cap Y = \emptyset$, or $Y \subset X$, or $X \subset Y$. It is well known (and easy to show) that a laminar family of sets on some ground-set K of size k, which does not contain the empty-set and the whole set K, has at most $2k - 2$ members. Now the size of the ground-set of $\mathcal{C}_e$ is at most $|V| - 2$, which gives the desired inequality

$$|\mathcal{C}_e| \leq 2|V| - 6. \tag{1}$$

To finish the proof we first determine an upper bound on the size of the final collection $\mathcal{G} \cup \mathcal{E}$, where we consider $\mathcal{E}$ as a multiset (i.e. it may contain several copies of the same edge). Since each member of this collection has at least one edge, (1) implies $|\mathcal{G} \cup \mathcal{E}| \leq |E|(2|V| - 5)$. Since every iteration creates at least two smaller graphs out of some graph $H \in \mathcal{G}$, the number of iterations (the number of internal vertices of the corresponding rooted tree) is at most $|E|(2|V| - 5)$. This completes the proof. □

If the set $\mathcal{G}$ of the large globally rigid components is available then it is straightforward to find the small globally rigid components. The components on three vertices correspond to those K_3 subgraphs of G which are not subgraphs of some graph in $\mathcal{G}$. The components on two vertices are those K_2's (i.e. edges) of G which do not belong to some globally rigid component of size at least three.

Since (a) and (b) can be executed in polynomial time by searching for the M-bridges (for a quadratic algorithm see e.g. [1]) and small separators (for a linear time algorithm see e.g. [3]), respectively, it follows that the globally rigid components can be found in polynomial time. The running time can be improved in various ways. For example, just like the M-bridges, the triconected components of a graph can be found simultaneously, see [3]. Theorem 2.2 also implies a polynomial upper bound for the number of globally rigid components of a graph, which is probably not best possible. We do not consider these improvements in this note.

2.1 Examples

Let us analyse the steps of Algorithm GRC on two specific input graphs. First consider the 3-connected graph G_1 of Figure 1. In the first iteration the set F of the three M-bridges of G_1 (in the middle) are added to $\mathcal{E}$ and G_1 is replaced by $G_1 - F$. Since $G - F$ is disconnected (i.e. the smallest vertex separator is the empty-set), the next iteration replaces $G_1 - F$ by its connected components, each of which is an M-circuit on six vertices. The next iterations decompose these M-circuits along their vertex separators of size two, and hence they will be replaced in $\mathcal{G}$ by the corresponding large extended components, i.e. four copies of $K_4 - e$ in total. Since each edge of $K_4 - e$ is an M-bridge, all the edges in these graphs will then be added to $\mathcal{E}$. So the final collections are $\mathcal{G} = \emptyset$ and $\mathcal{E} = E(G_1)$.

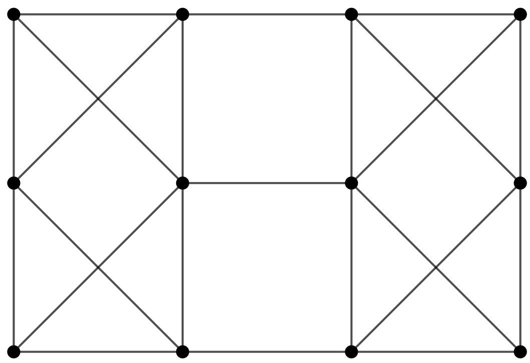

Figure 1: The graph G_1 with no large globally rigid components.

Next consider the M-connected and 2-connected graph G_2 of Figure 2. It has four vertex separators of size two, and hence after the next four iterations $\mathcal{G}$ may consist of three copies of $K_4 - e$, one copy of K_4, and one copy of $K_{3,3} + e$. Since the latter two graphs are 3-connected and M-connected, the remaining iterations will remove the three copies of $K_4 - e$ from $\mathcal{G}$ and add their edges to $\mathcal{E}$. Thus collection $\mathcal{G}$ at termination consists of K_4 and $K_{3,3}$, the two large globally rigid components of G_2.

3 Acknowledgements

This work was supported by the Hungarian Scientific Research Fund grant no. K135421.

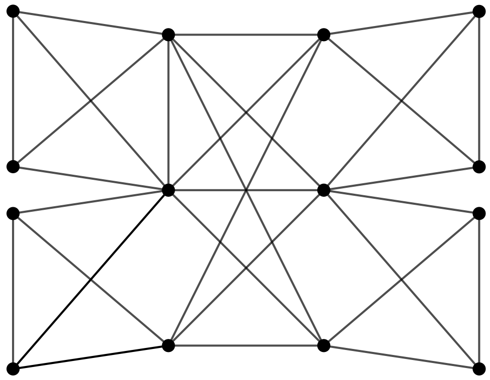

Figure 2: The graph G_2 with two large globally rigid components.

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