Facility location functions are deep submodular functions

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Abstract

Deep submodular functions form a subclass of submodular functions that share many of the properties and advantages of deep neural networks. In [1], Bilmes and Bai showed that facility location functions can be approximated with deep submodular functions. We prove that facility location functions are in fact contained in the class of deep submodular functions.

1 Introduction

Greedy algorithms work well in practice for several machine learning related problems. The effectiveness of this approach is often due to a (nearly) submodular underlying structure [2], which motivates the development of submodular function learning techniques.

Dolhansky and Bilmes [3] and Bilmes and Bai [1] introduced a class of submodular functions called *deep submodular functions* in the form of special neural networks, which offers a general framework for describing and learning parameterized submodular function classes.

In order to apply the tools of deep submodular functions for learning certain submodular function classes, we need to know how to construct them. Now we focus on the class of facility location functions. In [1] it is mentioned that facility location functions can be approximated with deep submodular functions. The aim of this note is to show that facility location functions are in fact deep submodular.

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2 Preliminaries

For a finite set S, a set function $f : 2^S \to \mathbb{R}$ is **submodular** if $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$ for every pair of subsets $A, B \subseteq S$. If in addition f is non-decreasing and $f(\emptyset) = 0$ then it is called a **polymatroid function**.

We define a new class of submodular functions called **deep polymatroid functions** in a recursive way. Let S be a finite set. Every non-negative modular function $m : 2^S \to \mathbb{R}_+$ is a deep polymatroid function. Let $f_i : 2^S \to \mathbb{R}$ be deep polymatroid functions ($i = 1, \ldots, k$), $m : 2^S \to \mathbb{R}_+$ be a non-negative modular function and $\lambda_i \in \mathbb{R}_+$ for $i = 1, \ldots, k$. Then

$$f(X) = \Phi\left(\sum_{i=1}^{k} \lambda_i f_i(X) + m(X)\right)$$

is also a deep polymatroid function, where $\Phi : \mathbb{R} \to \mathbb{R}$ is a monotone increasing, normalized, convex function.

A **deep submodular function** f is the sum of a deep polymatroid function p and a modular function u, that is, $f(X) = p(X) + u(X)$ for $X \subseteq S$. Note that u might take negative values as well. In [3] it was shown that deep submodular and polymatroid functions are in the class of submodular functions.

Given a ground set S and a subset $X \subseteq S$, let M_X denote the matroid with rank function

$$r_X(Z) = \begin{cases} 1 & \text{if } Z \cap X \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, M_X is a matroid as it is the direct sum of a uniform matroid and a matroid consisting of loops. We will refer to such matroids as **elementary**.

Note that an elementary matroid is a special case of laminar matroids. Bilmes and Bai [1] implicitly proved the following.

Theorem 1 (Bilmes and Bai [1]). *The rank function of a laminar matroid is a deep polymatroid function.*

We will also use the following observation.

Proposition 1. The non-negative linear combination of deep polymatroid functions is again a deep polymatroid function.

3 Facility location functions

Let $w : S \to \mathbb{R}$ be a weight function and define $h(Z) = \max_{i \in Z} w(i)$.

Lemma 2. *If the weights are non-negative, then h is a deep polymatroid function.*

Proof. We first show that h can be written as the non-negative linear combination of rank functions of elementary matroids. Let $w(v_1) \leq w(v_2) \leq \cdots \leq w(v_n)$ denote the weights of the elements in S and let $S_i = \{v_i, \ldots, v_n\}$. Define $w(v_0)$ to be 0. Then it is easy to verify that

$$h(Z) = \max_{i \in Z} w(i) = \sum_{i=1}^{n} (w(v_i) - w(v_{i-1}))\, r_{S_i}(Z).$$

By Theorem 1, r_{S_i} is a deep polymatroid function for $i = 1, \ldots, n$. By Proposition 1, the sum of such functions is again deep polymatroid, thus concluding the proof of the lemma. □

Let $u : 2^S \to \mathbb{R}$ be a modular function, and let $w_d : S \to \mathbb{R}$ be weight functions for $d = 1, \ldots, D$. Then it is easy to see that the function

$$f(X) = u(X) + \sum_{d=1}^{D} \max_{i \in X} w_d(i)$$

is submodular. Functions of this form are called **facility location functions**. Now we show that such functions are deep submodular.

Theorem 3. *Facility location functions are deep submodular functions.*

Proof. It is enough to prove the theorem for non-negative weights since the weights can be translated by a constant.

By Lemma 2, $\max_{i \in X} w_d(i)$ is a deep polymatroid function for $d = 1, \ldots, D$. The sum of such terms $h(X) = \sum_{d=1}^{D} \max_{i \in X} w_d(i)$ is also a deep polymatroid function by Proposition 1. Thus f is the sum of a deep polymatroid function h and a modular function u, implying that f is indeed deep submodular. □

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