Triangle-free Eulerian planar graphs are friendly

Kristóf Bérczi⋆ and Zoltán Paulovics⋆⋆

Abstract

In [7], Shafique and Dutton proved that every simple triangle-free Eulerian graph has a friendly partition. We give a simple algorithmic proof for the statement in planar graphs. The proof also implies that every triangle-free planar graph has a weak internal partition.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph. For a set $A \subseteq V$ and vertex $v \in A$, the number of neighbours of v in A is denoted by $d_A(v)$. A partition of the vertex set into two disjoint non-empty parts V_1 and V_2 is **friendly** (also called **internal** or **satisfactory** in the literature) if every vertex has at least as many neighbors in its own part as in the other part, that is, $d_{V_i}(v) \geq \lceil d_V(v)/2 \rceil$ for every $v \in V$ and $i = 1, 2$. Consequently, a graph is called **friendly** if it has a friendly partition, otherwise it is **non-friendly** (note that this is different from the notion of unfriendly partitions).

The notion of friendly paritions was introduced by Gerber and Kobler [5]. Shafique and Dutton [7] showed that neither friendly nor non-friendly graphs have a forbidden subgraph characterization. Bazgan, Tuza and Vanderpooten [1] proved that the problem of deciding if a graph is friendly or not is NP-complete.

From the positive side, there are special graph classes for which the problem becomes tractable. Shafique and Dutton [7] characterized friendly $(3, 4)$-regular graphs. They also showed that if a graph G has a maximum-degree vertex not adjacent to any degree 2 vertex then its line graph is friendly if and only if G is not a star, and that the line graph of a graph having two non-adjacent vertices of maximum degree is always friendly. Gerber and Kobler [6] gave a polynomial time algorithm for graphs of bounded clique-width. Bazgan et al. [2] showed that the problem is solvable on graphs of bounded tree-width.

Regular graphs are also well-investigated. A conjecture due to DeVos [4] states that for every r, all but finitely many r-regular graphs have friendly partitions. The conjecture is known to be true for $r = 2$ (folklore), 3, 4 and 6 [7]. For a comprehensive overview of known results on friendly paritions, see [3].

The starting point of our investigations is the following result.

⋆Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, Budapest, Hungary, H-1117, MTA-ELTE Egerváry Research Group (EGRES). Email: `berkri@ cs.elte.hu`

⋆⋆Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, Budapest, Hungary, H-1117. Email: `zoli.paulovics@ gmail.com`

Theorem 1 (Shafique and Dutton [7]). *Every simple triangle-free Eulerian graph is friendly.*

There are only a few results about friendly paritions in planar graphs. For every positive integer n, there exist non-friendly planar graphs on n vertices. Indeed, let G be the graph obtained from a cycle of length n by adding all chords from one of its vertices. Then G is clearly planar, and it is not difficult to see that it has no friendly partition. However, the complexity of deciding if a planar graph is friendly is still open.

Given two functions $a, b : V \to \mathbb{Z}_+$, a partition if the vertex set into two non-empty parts V_a and V_b is an (a, b)-**partition** if $d_A(v) \geq a(v)$ for each $v \in V_a$ and $d_B(v) \geq b(v)$ for each $v \in V_b$.

Instead of deciding the existence of an (a, b)-partition, one might be interested in finding one that maximizes the number of vertices that have at least the required number of neighbours in their own part.

MAX SATISFYING DECOMPOSITION
Input: A simple undirected graph $G = (V, E)$ and two functions $a, b : V \to \mathbb{Z}_+$.
Output: A nontrivial partition (V_a, V_b) of V maximizing the number of satisfied vertices v, i.e. those with $d_{V_a}(v) \geq a(v)$ if $v \in V_a$ and $d_{V_b}(v) \geq b(v)$ if $v \in V_b$.

Theorem 2 (Bazgan, Tuza and Vanderpooten [2]). MAX SATISFYING DECOMPOSITION *admits a polynomial-time approximation scheme in planar graphs.*

An inclusionwise minimal cut of a planar graph corresponds to a cycle in the dual graph. Hence in a 2-edge-connected planar graph the problem of finding a friendly partition is equivalent to finding a cycle in its dual which contains at most half of the edges of every face.

Let $G = (V, E)$ be a simple Eulerian planar graph. Note that G is necesarily 2-edge-connected. The dual of G is a -not necessarily simple- bipartite planar graph in which every vertex has degree at least 3. If in addition G is triangle-free, then the minimum degree in the dual graph is at least 4.

By the above, Theorem 1 for planar graphs follows from the following theorem.

Theorem 3. *Let $G = (V, E)$ be a bipartite planar graph with minimum degree at least 4. Then G has a cycle which contains at most half of the edges of every face.*

Our aim is to give a simple algorithmic proof of Theorem 3. As we will see, the proof also implies the existence of weak internal partitions in simple triangle-free planar graphs.

2 Proof of Theorem 3

Let $G = (V, E)$ be a bipartite planar graph with minimum degree at least 4. Take a cycle $C = v_0 v_1 \dots v_q$ where $v_i v_{i+1}$ is the second edge around v_i from $v_{i-1} v_i$ in a counterclockwise order for $i = 1, \dots, q - 1$. Such a cycle can be found algorithmically by starting from an arbitrary vertex v and edge vv', and always continuing through the second edge from the edge through which we reached the actual vertex in a counterclockwise order. As the number of vertices is finite, at some point we will reach a vertex that was already visited before. Let v_0 be this vertex. Then the segment of our walk between the two occurrences of v_0 form a cycle satisfying the above condition (see Figure 1 for an example).

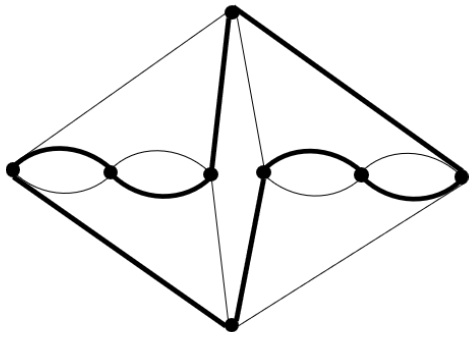

Figure 1: Output of the algorithm

We claim that C contains at most half of the edges of every face.

Claim 4. *If C contains two consecutive edges xy and yz of a face F then $y = v_0$.*

Proof. Recall that the graph has minimum degree at least 4. If $y \neq v_0$, then by the choice of C the edges xy and yz are non-neighboring edges on y, that is, there is at least one further edge between them in both directions. This contradicts the fact that xy and yz are edges of the same face F. □

Claim 5. *C contains at most one pair of consecutive edges of every face F.*

Proof. Assume x, y, z and x', y', z' are two triples of consecutive vertices of F for which all of the edges $xy, yz, x'y'$ and $y'z'$ are contained in C. By Claim 4, this is only possible if $y = y' = v_0$, that is, the two triples coincide. □

The following claim is the key observation of the proof.

Claim 6. *Let F be a face of the graph. Then*

$$|C \cap F| \leq \begin{cases} \lfloor \frac{|F|}{2} \rfloor & \text{if } v_0 \notin F, \\ \lfloor \frac{|F|+1}{2} \rfloor & \text{if } v_0 \in F. \end{cases}$$

Proof. Assume first that $|F| = 2$, that is, F consists of a pair of parallel edges between two vertices, say x and y. If C contains both edges of F, then by Claim 4 $x = y = v_0$, a contradiction. Hence $|C \cap F| \leq 1$.

For $|F| \geq 3$, Claim 5 implies that $C \cap F$ consists of independent edges if $v_0 \notin F$, and of independent edges plus probably a single path of length two if $v_0 \in F$ (see Figure 2). Hence the claim follows. □

The graph is bipartite, hence $|F|$ is even. By Claim 6, $|C \cap F| \leq \lfloor \frac{|F|+1}{2} \rfloor = \frac{|F|}{2}$, concluding the proof of the theorem.

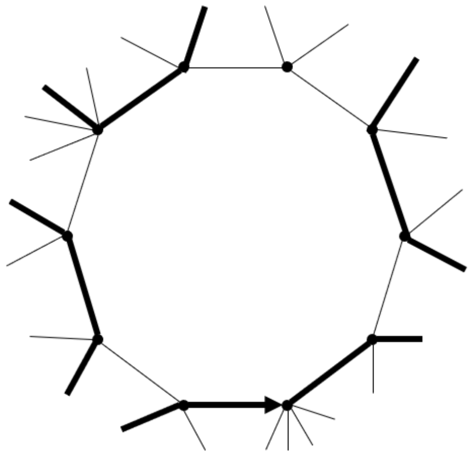

Figure 2: Illustration of Claim 6

3 Weak internal partitions

A partition of the vertex set into two disjoint non-empty parts V_1 and V_2 is a **weak internal partition** if $d_{V_1}(v) \geq \lfloor d_V(v) \rfloor$ for $v \in V_1$ and every $v \in V$ and $d_{V_2}(v) \geq \lceil d_V(v) \rceil$ for $v \in V_2$.

Assume now that $G = (V, E)$ is a 2-edge-connected triangle-free, but not necessarily Eulerian planar graph. For such graphs, the algorithm of Section 2 does not necessarily provide a friendly partition: if F is a face with $|F|$ being odd and $v_0 \in F$ then $|C \cap F|$ might be $\lfloor \frac{|F|+1}{2} \rfloor$.

However, if F is such a face then C contains two consecutive edges of F and these edges are both incident to v_0. That is, there is at most one odd face violating the condition $|C \cap F| \leq \frac{|F|}{2}$, and for such a face we have $|C \cap F| = \frac{|F|+1}{2}$. Thus we get the following.

Theorem 7. *Every simple triangle-free planar graph has a weak internal partition.*

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