Total domatic number of triangulated planar graphs

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Abstract

In [3], Goddard and Henning conjectured that every simple triangulated planar graph on at least four vertices has a 2-coloring such that every vertex has neighbors in both color classes. We give several equivalent and stronger reformulations of the conjecture and present some partial results that might serve as starting points for further investigations.

1 Introduction

Given an undirected graph $G = (V, E)$, a subset S of vertices is called a **dominating set** if every vertex of G has at least one neighbor both in S and in $V - S$. The **total dominating number** or **total domatic number** of G is the maximum number of pairwise disjoint total dominating sets in G. The domatic number of G is denoted by $d_t(G)$.

The starting point of our investigations is the following conjecture of Goddard and Henning [3].

Conjecture 1. *$d_t(G) \geq 2$ for every simple triangulated planar graph G on at least 4 vertices.*

The conjecture can be rephrased in terms of colorings: it states that the vertices of a simple triangulated planar graph on at least 4 vertices can be colored by two colors such that every vertex of G has neighbors in both color classes. Such colorings are usually called **2-coupon colorings** in the literature.

It is worth mentioning that the statement is not necessarily true if either simplicity or maximality is dropped, see Figure 1. Heggernes and Telle [5] showed that deciding if the total domatic number of a graph is at least 2 is NP-complete.

The conjecture was verified for several special classes of graphs. Nagy [4] gave a proof for Hamiltonian graphs, while Goddard and Henning [3] proved the conjecture for the dual of Hamiltonian graphs. They also gave a proof for graphs with only odd-degree vertices.

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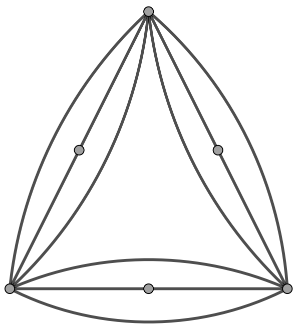
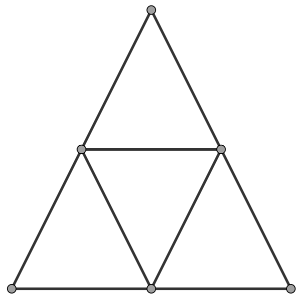


(a) Simplicity is needed



(b) Maximality is needed

Figure 1: Examples showing that the conditions of the conjecture are necessary

2 Equivalent forms and strengthenings

Let G be a triangulated planar graph. For a vertex v, each triangle containing v has an edge not containing v. We call the cycle consisting of these edges the **wheel defined by vertex** v.

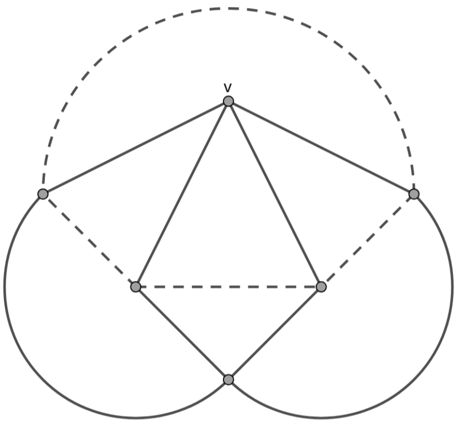


Figure 2: The dashed edges form the wheel defined by v

Statement 1. *Let $G = (V, E)$ be a simple triangulated graph on at least 4 vertices. Then there exists a bipartite subgraph $H = (V, F)$ of G such that F contains at least one edge from every wheel of G.*

Claim 1. *Conjecture 1 and Statement 1 are equivalent.*

Proof. Let G be a triangulated graph. Suppose first that it has a 2-coupon coloring. Then

$$F = \{uv \in E \mid u \text{ and } v \text{ are in different color classes}\}$$

defines a bipartite subgraph of G that contains at least one edge from each wheel.

Now suppose that there exists a bipartite subgraph that meets the requirement. Color the vertices in one of the classes to be black, and the vertices in the other class to be white. This way we get a 2-coupon coloring of the original graph. □

A natrual strengthening of Statement 1 would be the following.

Statement 2. *Let $G = (V, E)$ be a simple triangulated graph on at least 4 vertices. Then there exists a forest in G containing at least one edge from each wheel.*

Remark 2. If Statement 2 holds, then Statement 1 also holds.

We will need the following folklore observation.

Claim 3. *A connected planar graph is bipartite if and only if each of its faces have an even number of edges.*

Proof. Necessity is straightforward, we prove sufficiency. Suppose that the graph is not bipartite and thus there exists a cycle C of odd length. We show that there exists an odd face. The proof goes by induction on the number of faces in the interior of C. If C is a face, then we are done. If C is not a face, then there exists a face f in the inner side of C having at least one common edge with C. f does not contain every edge of C, since G is connected. Let C' be the symmetric difference of the edge sets of C and f. As C is odd, either f is an odd face or C' is an odd cycle containing less faces in its inner side than C. □

Statement 3. *Let $G = (V, E)$ be a simple triangulated graph on at least 4 vertices. Then there exists a subgraph $H' = (V, F')$ having the following two properties.*

1. *F' contains exactly one edge from each face of G.*
2. *There are no isolated vertices in H'.*

Claim 4. *If Statement 3 holds, then Statement 1 also holds.*

Proof. Let $H' = (V, F')$ be the subgraph required by Statement 3. We show that $H = (V, E - F')$ is a subgraph required by Statement 1. H is a bipartite graph by Lemma 3, as each of its faces have 4 edges. Take a wheel $v_1 v_2 \dots v_k$ defined by a vertex v. As v is not an isolated vertex in H', there exists a vertex v_i such that $vv_i \in F'$. As F' contains exactly one edge from each face, $v_i v_{i+1} \in E - F'$.

□

One can rephrase the Goddard-Henning conjecture in the dual graph as well.

Conjecture 2. *$G^* = (V^*, E^*)$ is the dual of a simple triangulated graph on at least 4 vertices if and only if G^* is a 3-regular 2-edge-connected planar graph on at least 4 vertices.*

Claim 5. *Conjectures 1 and 2 are equivalent.*

Proof. Clearly, G^* is 3-regular if and only if its dual is triangulated. It is also straightforward to verify that a cut consisting of one edge corresponds to a loop edge in the dual, and a cut consisting of two edges corresponds to a pair of parallel edges.

Finally, by 3-regularity and using Euler's formula,

$$f^* = m^* - n^* + 2 = 3n^*/2 - n + 2 = n^*/2 + 2,$$

where f^*, m^*, and n^* denote the number of faces, edges and vertices of G^*. Thus the dual of G^* has at least 4 vertices if and only if $4 \leq n^*/2 + 2$, i.e. G^* has at least 4 vertices. $\square$

Let f be a face of a planar graph and e be an edge not in f. We say that an edge e **leaves** f if e has at least one endpoint on f.

Statement 4. *Let $G^* = (V^*, E^*)$ be a 3-regular 2-edge-connected planar graph of on at least 4 vertices. Then there exists a subgraph $H^* = (V^*, F^*)$ in G^* with the following two properties.*

1. *H^* does not contain any odd cut of G^*.*
2. *For every face f of G^*, H^* contains an edge e leaving f.*

Claim 6. *Statements 1 and 4 are equivalent.*

Proof. We show that given a subgraph $H = (V, F)$ that meets the requirements of Statement 1, the edges corresponding to F in the dual of G form a subgraph H^* required by Statement 4, and vice versa. It is worth noting that H^* is not necessarily the same as the dual graph of H.

As cycles of a planar graph correspond to inclusionwise minimal cutsets in the dual graph, H is bipartite if and only if H^* does not contain any odd cut of G^*. Moreover, an edge from a wheel defined by v in G corresponds to an edge that leaves the face that corresponds to v in the dual graph of G. Hence H contains at least one edge from each wheel if and only if for every face of G^*, H^* contains at least one edge that leaves that face. $\square$

Statement 5. *Let $G^* = (V^*, E^*)$ be a 3-regular 2-edge-connected planar graph on at least 4 vertices. Then there exists a subgraph $\bar{H}^* = (V^*, \bar{F}^*)$ in G^* with the following two properties.*

1. *$\bar{H}^*$ intersects every odd cut of G^*.*
2. *For every face f of G^*, $\bar{H}^*$ does not contain all the edges leaving f.*

Claim 7. *Statements 4 and 5 are equivalent.*

Proof. If H^* meets the requirements of either of the statements, the complementer subgraph in G^* meets the requirements of the other. $\square$

A 2-factor of a graph $G = (V, E)$ consists of disjoint cycles covering V. We can formulate a sufficient condition for the Goddard-Henning conjecture with the help of 2-factors. The motivation for such a reformulation is the fact that the existence of 2-factors in which certain cycles (usually short cycles) are not allowed is a well-studied area in graph theory.

Statement 6. *Let $G^* = (V^*, E^*)$ be a 3-regular 2-edge-connected planar graph on at least 4 vertices. Then there exists a 2-factor not containing any of the faces.*

Claim 8. *If Statement 6 holds, then Statement 4 also holds.*

Proof. Let $H^* = (V^*, F^*)$ be the 2-factor containing none of the faces of G^*. Every cut of G^* has an even number of common edges with every cycle in H^*. Therefore H^* does not contain any odd cuts of G^*.

Let $f = v_1 v_2 \ldots v_l$ be a face of G^*. As F^* does not contain f, there must exist a vertex v_i such that $v_i v_{i+1} \notin F^*$. Moreover, every vertex has degree 2 in H^*, so there is an edge in F^* starting from v_i that leaves f. □

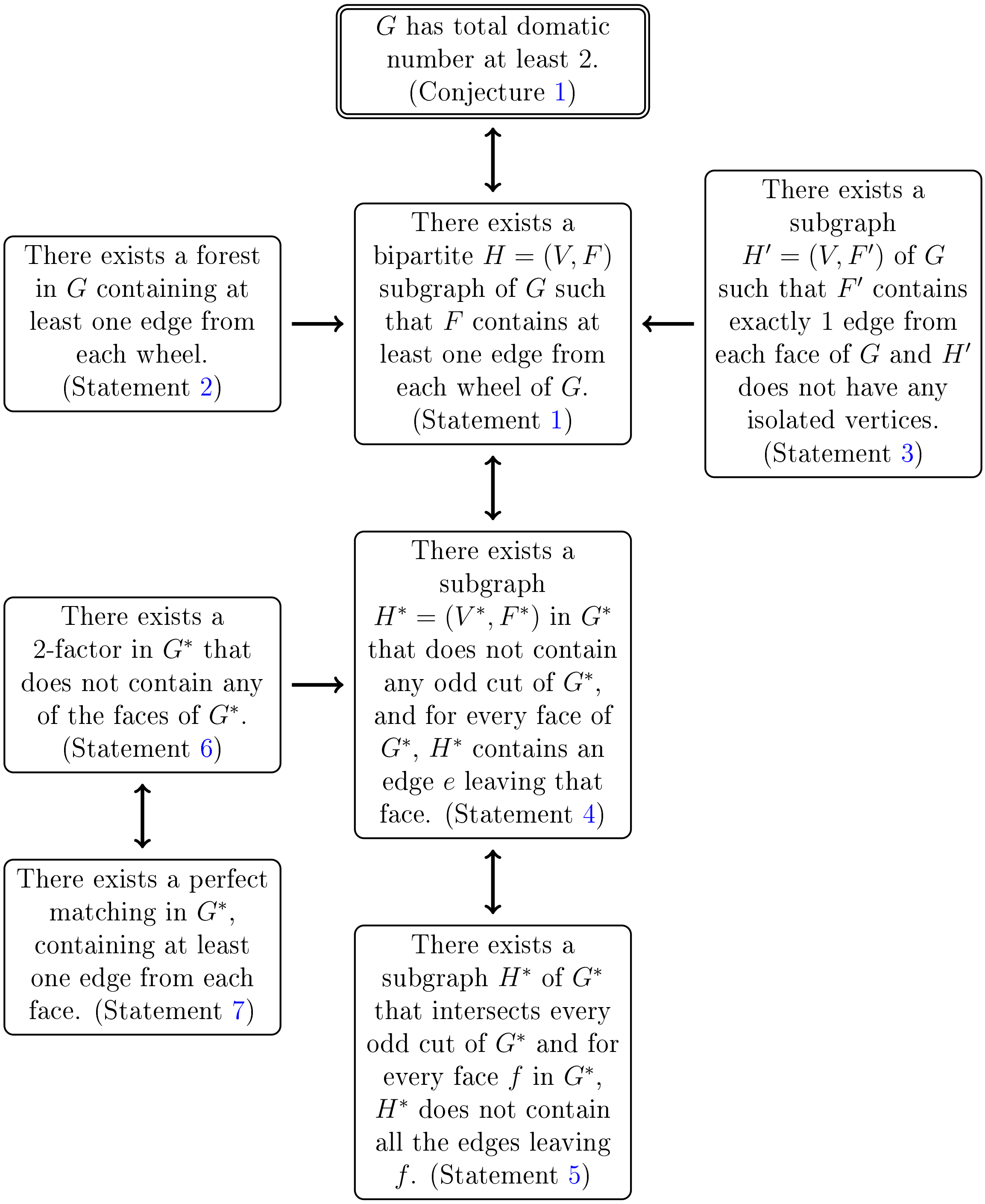
Statement 6 can easily be rephrased as a statement about perfect matchings.

Statement 7. *Let $G^* = (V^*, E^*)$ be a 3-regular 2-edge-connected planar graph on at least 4 vertices. Then there exists a perfect matching containing at least one edge from every face.*

Claim 9. *Statements 6 and 7 are equivalent.*

Proof. As G^* is 3-regular, a subgraph is a 2-factor if and only if the complementer subgraph is a perfect matching. Clearly, a subgraph contains none of the faces if and only if the complementer subgraph contains at least one edge from every face. □

The following figure summarizes the statements of this section. For this, let G be a simple triangulated planar graph on at least 4 vertices and let G^* denote its dual. Note that G^* is a 3-regular 2-edge-connected planar graph on at least 4 vertices.


G has total domatic number at least 2. (Conjecture 1)
There exists a forest in G containing at least one edge from each wheel. (Statement 2)
There exists a bipartite $H = (V, F)$ subgraph of G such that F contains at least one edge from each wheel of G. (Statement 1)
There exists a subgraph $H' = (V, F')$ of G such that F' contains exactly 1 edge from each face of G and H' does not have any isolated vertices. (Statement 3)
There exists a 2-factor in G^* that does not contain any of the faces of G^*. (Statement 6)
There exists a subgraph $H^* = (V^*, F^*)$ in G^* that does not contain any odd cut of G^*, and for every face of G^*, H^* contains an edge e leaving that face. (Statement 4)
There exists a perfect matching in G^*, containing at least one edge from each face. (Statement 7)
There exists a subgraph H^* of G^* that intersects every odd cut of G^* and for every face f in G^*, H^* does not contain all the edges leaving f. (Statement 5)

3 An approach based on quadrangulated subgraphs

Let H be a hypergraph. The **incidence graph** of H is a bipartite graph with one of its classes corresponding to the vertices of H, and the other class corresponding to the hyperedges of H. Then ve is an edge in the incidence graph if and only if the hyperedge e contains vertex v in H. A hypergraph is called **planar** if its incidence graph is planar. A vertex coloring of a hypergraph by two colors is **proper** if all of its hyperedges contain vertices from both color classes.

The following theorem is due to Dvorák and Král [1].

Theorem 10. *Let $H = (V, E)$ be a planar hypergraph with at most 2 hyperedges of size 2. Then H has a proper vertex coloring with two colors.*

By using Theorem 10, we could verify the following.

Theorem 11. *Let G be a simple triangulated graph. If there are at most two vertices of degree at most 4, then $t_d(G) \geq 2$.*

Proof. Recall that the dual G^* is a 3-regular 2-edge-connected planar graph. Thus, by Petersen's theorem, there exists a perfect matching M in G^*. By deleting the edges corresponding to M from G, we get a graph G' such that all of its faces contain 4 vertices. We call such graphs quadrangulated. Note that for each vertex v, we deleted at most half of the edges starting from v, thus there are at most two vertices in G' of degree 2 and none of the vertices has less than two neighbors.

By Claim 3, G' is a bipartite graph. Let S_1 and S_2 be the two classes of G'. G' is the incidence graph of two hypergraphs: let H_1 be the hypergraph defined on the vertex set S_1 with hyperedges S_2, and H_2 be the hypergraph defined on the vertex set S_2 with hyperedges S_1. As the original graph contains at most two vertices of degree at most 4, both of these hypergraphs have at most two hyperedges of size 2. Hence by Theorem 10, H_1 and H_2 has proper 2-colorings. Take the union of these colorings c. I.e. on the vertices of S_1, c is defined by a proper coloring of H_1, whereas on the vertices of S_2, c is defined by a proper coloring of H_2. The coloring c thus obtained is a 2-coupon coloring of G', and so it is also a 2-coupon coloring of the original graph G. □

Not every simple quadrangulated graph has a 2-coupon coloring, see Figure 3 for an example. Theorem 11 shows that Conjecture 1 holds whenever the triangulated planar graph in question has a spanning quadrangulated subgraph containing at most two vertices of degree 2. Unfortunately, it is not always possible to find such a quadrangulated subgraph. However, the proof of the theorem only uses the fact that both hypergraphs H_1 and H_2 determined by the quadrangulation have proper vertex colorings. Hence it is natural to formulate the following stregthening of Conjecture 1.

Conjecture 3. *Every simple triangulated graph G on at least 4 vertices has a spanning quadrangulated subgraph G' for which $d_t(G') \geq 2$.*

The approach sketched above raises the following question: given a hypergraph $H = (S, \mathcal{E})$ whose incidence graph is quadrangulated and planar, when does H admit

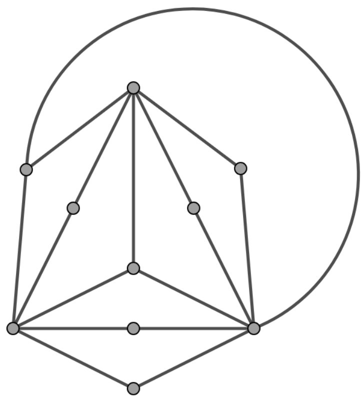


Figure 3: A quadrangulated graph without two disjoint dominating sets

a proper vertex coloring? For answering this question, we define a new graph whose set of vertices corresponds to $\mathcal{E}$ as follows: two vertices -corresponding to two hyperedges of H- are connected if they are opposite vertices of a face -which is a square- of the incidence graph of H. We denote the graph thus obtained by $G_H = (\mathcal{E}, F)$ (by abuse of notation, the vertex set of the new graph is also denoted by $\mathcal{E}$). Note that F may contain parallel edges. It can be verified that H has a proper vertex coloring if and only if G_H has a spanning even subgraph, that is, a subgraph in which every vertex has a positive even degree.

The existence of such subgraphs can be characterized by using a result of Frank, Sebő and Tardos [2] on ℓ-congruent orientations. Given a graph $G = (V, E)$ and a function $\ell : V \to \mathbb{Z}_+$, an orientation of G is called **ℓ-congruent** if $\varrho(v) \equiv \ell(v)$ $(mod\ 2)$ holds for the degree $\varrho(v)$ of every vertex $v \in V$.

Theorem 12 (Theorem 5b, [2])**.** *A simple graph $G = (V, E)$ has an ℓ-congruent orientation for which $\varrho(v) \geq \ell(v)$ for every $v \in V$ if and only if $e_G(X) - o_G(X) \geq \ell(X)$ for $X \subseteq V$, where $e_G(x)$ denotes the number of edges with at least one endpoint in X and $o_G(X)$ denotes the number of components C of $G - X$ for which $e(C) \not\equiv \ell(C)$ $(mod\ 2)$.*

Theorem 13. *A simple graph $G = (V, E)$ has a spanning even subgraph if and only if $2i_G(X) + d_G(X) \geq 2|X| + \bar{o}_G(X)$ holds for every set $X \subseteq V$, where $i_G(X)$ denotes the number of edges with both endpoints in X, $d_G(X)$ denotes the number of edges with exactly one endpoint in X and $\bar{o}_G(X)$ denotes the number of components C of $G - X$ for which $d_G(C)$ is odd.*

Proof. Subdivide each edge e by a new vertex v_e into two edges e' and e''. The sets of new nodes and edges are denoted by V_E and E^+. We define $V^+ := V \cup V_E$. The graph thus arising is denoted by $G^+ = (V^+, E^+)$. Let $\ell(v)$ be 2 for $v \in V$ and let $\ell(v_e)$ be 0 for $e \in E$. We claim that G has a spanning even subgraph if and only if G^+ has an ℓ-congruent orientation with $\varrho \geq \ell$.

Indeed, given a spanning even subgraph of G, define an orientation of G^+ by orienting e' and e'' away from v_e if e is contained in the subgraph and toward v_e otherwise.

As the subgraph is even and spanning, we get an ℓ-congruent orientation of G^+ with $\varrho \geq \ell$. To see the other direction, observe that in each ℓ-congruent orientation of G^+, e' and e'' are both oriented either away from v_e or toward v_e. The subgraph of G corresponding to edges e for which both e' and e'' are oriented away from v_e is a spanning even subgraph of G.

By Theorem 12, G^+ has a proper orientation if and only if $e_{G^+}(Y) - o_{G^+}(Y) \geq \ell(Y)$ for $Y \subseteq V^+$. If Y containts exactly one endpoint of an original edge $e \in E$, then we may assume that $v_e \notin Y$ as leaving it out from Y will only result in a more strict inequality. Let $X := Y \cap V$. Then $e_{G^+}(Y) = 2i_G(X) + d_G(X)$, and the definition of ℓ implies $o_{G^+}(Y) = \bar{o}_G(X)$ and $\ell(Y) = 2|X|$. This concludes the proof of the theorem. □

If the minimum degree in the graph is at least 2, then it suffices to require the inequality for stable subsets.

Theorem 14. *A simple graph $G = (V, E)$ with minimum degree at least 2 has a spanning even subgraph if and only if $d_G(X) \geq 2|X| + \bar{o}_G(X)$ holds for every stable set $X \subseteq V$, where $d_G(X)$ denotes the number of edges with exactly one endpoint in X and $\bar{o}_G(X)$ denotes the number of components C of $G - X$ for which $d_G(C)$ is odd.*

Proof. By Theorem 13, a requested subgraph exists if and only if $2i_G(X) + d_G(X) \geq 2|X| + \bar{o}_G(X)$ holds for every set $X \subseteq V$.

Assume that X spans an edge e and let v be one of the end vertices of e. By leaving out v from X, the left side decreases by $d_G(v)$. Regarding the right side, there are at most $d_G(v)-1$ components C of $G-X$ with $d_G(C)$ being odd such that there is an edge from v to C. If there are at most $d_G(v)-2$ of them, then $\bar{o}(X-v) \geq \bar{o}(X) - d_G(v) + 2$ clearly holds. If there are exactly $d_G(v) - 1$ such components, these components and v will form a single component C' of $G - (X - v)$ with $d_G(C')$ being odd, hence $\bar{o}(X-v) \geq \bar{o}(X) - d_G(v) + 2$ again. That is, the right side decreases by at most $d_G(v)$, hence the inequality for $X - v$ implies the one for X. This shows that it suffices to require the inequality for stable sets. □

Let us now return to the case of triangulated planar graphs. According to Theorem 14, we can decide if the hypergraphs H_1 and H_2 have proper vertex colorings. Hence a possible way of verifying Concejture 3 (and so Conjecture 1) would be to show that a triangulated planar graph always has a quadrangulated subgraph for which G_{H_1} and G_{H_2} satisfy the conditions of Theorem 14.

Closing this section, we mention the following stronger conjecture formulated by Goddard and Henning [3] on planar triangulations without degree 3 vertices.

Conjecture 4. *If G is a simple triangulated graph with all its vertices having degree at least 4, then G admits three disjoint total dominating sets.*

4 An approach based on 2-factors

Theorem 15. *Let G be a simple triangulated graph of order $n \geq 4$. If G has a 2-factor with none of its cycles having length congruent to 2 modulo 4, then the total*

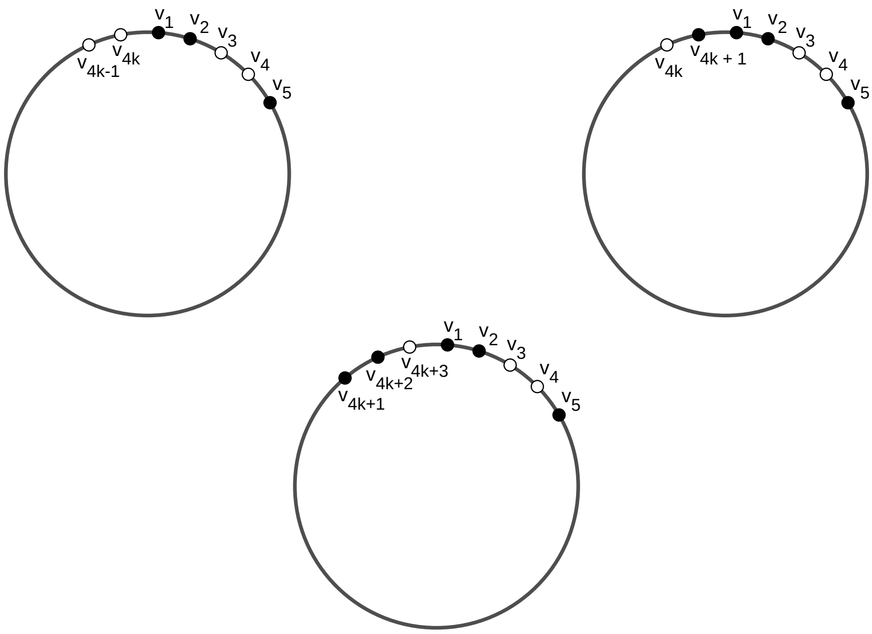


Figure 4: Alternating colors in pairs on cycles of length not congruent to 2 modulo 4

dominating number of G is at least 2.

Proof. If the 2-factor consists of one cycle, then Theorem 3 in [4] proves the claim. Suppose that there are at least two cycles in the 2-factor. First note that by alternating colors in pairs, any cycle of length not congruent to 2 modulo 4 can be colored in a way such that there is at most one vertex with a monochromatic neighborhood, see Figure 4.

Contract each cycle to a single vertex. G is connected, hence there exists a tree T in the contracted graph. Choose T to have a minimal number of degree one vertices. Let $E(T)$ denote the edges of the original graph that were mapped to T. We show a 2-coupon coloring in the subgraph defined by the union of the 2-factor and $E(T)$. (See Figure 5 for an example.) Choose a root vertex r from the degree one vertices in T.

We color the cycle C_0 corresponding to r first. Choose a vertex v in C_0 such that there exists a uv edge in $E(T)$. We color C_0 such that only v may have a monochromatic neighborhood and assign u the missing color.

After this, we iteratively color a child of a cycle that is already colored. We start by coloring cycles that do not correspond to leaves in T. Suppose that C_1 is such a cycle, and let C_2 be a child of C_1 and v_1v_2 be the edge in $E(T)$ such that $v_1 \in C_1$ and $v_2 \in C_2$. Let c_1 be a coloring of C_1 such that only v_1 may have a monochromatic neighborhood. There might be a vertex (but only one) in C_1 that already has a fixed color, but this does not cause any problem as the role of the two colors can be interchanged in c_1 if necessary. Color v_2 in a way that provides v_1 the missing color.

Now we color cycles that correspond to leaves but have at least 4 vertices. Let C be such a cycle. By Theorem 2 in [4] there exists a 2-coupon coloring c of C. Again, if there is a vertex in C that already has a color, then we might need to flip the colors

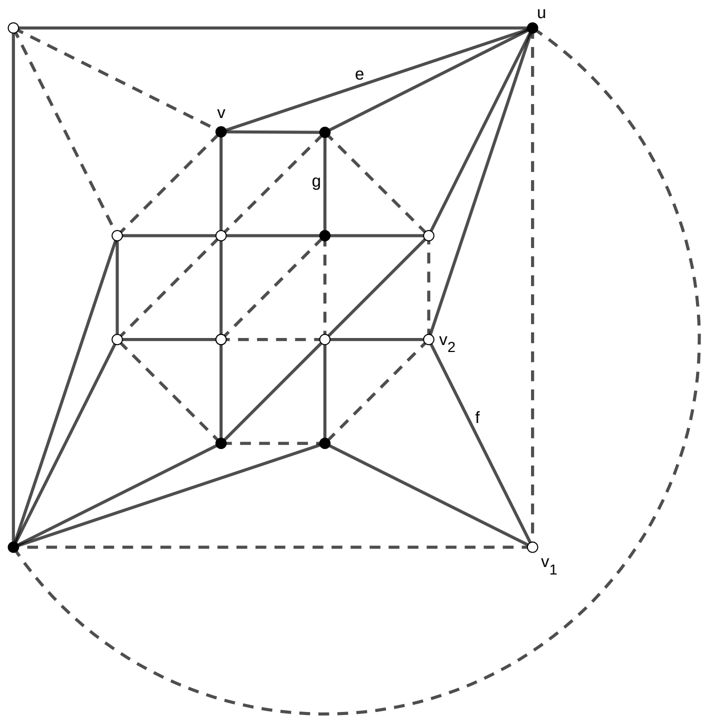


Figure 5: 2-coupon coloring corresponding to a 2-factor: the dashed edges form a 2-factor, E(T) consists of the edges of e, f and g.

of c.

Finally, we need to color cycles corresponding to leaves of T and having only 3 vertices. Let uvw be such a cycle, where v is the only vertex that may already have a color. Suppose it is colored to black. There exists a face vwx where $x \neq u$. If x is colored to black, then color w and u to white. If x is colored to white, then color w to white and u to black. The only remaining case is when x does not have a color yet. In this case, there must be a face xyz corresponding to a leaf of T. These two leaves have a closest common ancestor t. As t is of degree at least 2 in T, we have $t \neq r$. So t must have degree at least 3. By adding the edge corresponding to vx to T and removing the first edge of the tv path, we get a tree T' having fewer degree 1 vertices than T, contradicting the choice of T. □

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