List colourings with restricted lists

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Abstract

We prove an extension of Galvin's theorem, namely that any graph is χ'-edge-choosable if no odd cycle has a common colour in the lists of its edges.

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Let $G = (V, E)$ be a finite loopless graph. For each edge $e \in E$, let $L(e) \subset \mathbb{N}$ be a set of available colours for e. We say that G is *L-edge-choosable* if G has an *L-edge-colouring*, that is, a proper edge-coloring $c : E \to \mathbb{N}$ such that $c(e) \in L(e)$ holds for each edge e of E. Graph G is called *k-edge-choosable* if G is L-edge-choosable for any $L : E \to \binom{\mathbb{N}}{k}$. The famous list colouring conjecture states that any finite loopless graph G is $\chi'(G)$-edge-choosable, where chromatic index $\chi'(G)$ denotes the minimum number of colours needed to properly colour the edges of G. By proving the Dinitz conjecture in [2], Galvin essentially justified the list colouring conjecture for (complete) bipartite graphs. In this note, we prove that G is L-edge-choosable whenever $L : E \to \binom{\mathbb{N}}{\chi'(G)}$ and $L^{-1}(i)$ is bipartite for each colour i, that is, if the edges of no odd cycle of G contain a common colour in their lists. Our main tool to achieve this goal is an extension of Galvin's method. Unlike Galvin, here we shall lean on the terminology of stable matchings.

Assume that $G = (V, E)$ is a loopless finite graph and for each vertex v of V, a linear order $\preceq_v$ on the set $E(v)$ of edges incident to v is given. A *matching* of G is a set M of disjoint edges of G and matching M is *stable* if for each edge e of G, there is a vertex v and an edge m of M such that $m \preceq_v e$ holds. The well-known stable marriage theorem states the following.

Theorem 1 (Gale-Shapley [1])**.** *If $G = (V, E)$ is a finite bipartite graph and $\preceq_v$ is a linear order on $E(v)$ for each vertex v of G then there is a stable matching of G.* □

Our main result is the following.

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Theorem 2. *Let $G = (V, E)$ be a finite loopless graph and $c : E \to \{1, 2, \ldots, k\}$ be a proper edge-colouring of G. If $L(e) \subset N$ is a list of at least k colours for each edge e of G and $\bigcap\{L(e) : e \in C\} = \emptyset$ for each odd cycle C of G then G is L-edge-choosable.*

Proof. For $i = 1, 2, \ldots$ define $E_i := \{e \in E : 2i - 1 \leq c(e) \leq 2i\}$. Clearly, $E = E_1 \cup E_2 \cup \ldots \cup E_{\lceil k/2 \rceil}$. As the maximum degree in $G_i = (V, E_i)$ is not more than 2, each component of G_i is a path or a cycle. Orient the edges of G such that each component of each G_i becomes a directed path or a directed cycle. For edge $e = uv \in E_i$ define

$$r(e, v) = \begin{cases} i & \text{if } v \text{ is the head of the arc that corresponds to } e \\ k + 1 - i & \text{if } v \text{ is the tail of the arc that corresponds to } e \end{cases}$$

Observe that if $r(e, v) = r(f, v)$ then e and f must belong to the same set E_i and orientations of e and f either both enter or both leave v. Hence $r(e, v) = r(f, v)$ implies $e = f$ and consequently $\preceq_v$ is a linear order on $E(v)$ where $e \preceq_v f$ means that $r(e, v) \leq r(f, v)$. Assume now that $e = uv$ is the oriented version of edge $e \in E_i$. From $r(e, u) = i$ and $r(e, v) = k + 1 - i$ we get that

$$|\{f \in E(u) : f \preceq_u e\}| + |\{f \in E(v) : f \preceq_v e\}| \leq i - 1 + (k + 1 - i) - 1 = k - 1\ . \quad (1)$$

The above observations enable us to employ Galvin's method to finish the proof. Define $E^i := \{e \in E : i \in L(e)\}$ as the set of i-colourable edges and let $G^i := (V, E^i)$. As none of the G^is contain an odd cycle by the assumption, each G^i is bipartite. For $i = 0, 1, 2, \ldots$ define M^i as a stable matching of graph $G^i \setminus (M^0 \cup \ldots \cup M^{i-1})$ with restricted linear orders $\preceq_v$. Such matching exists by Theorem 1.

To show that G is L-edge-choosable, give colour i to edges of M^i. Clearly, no two edges of the same colour share a vertex and each coloured edge receives its colour from its list. The only thing left is to show that each edge of G receives some colour.

Observe that if edge $e = uv$ of G^i does not receive colour i, (i.e. if $e \notin M^i$) then either $e \in M^j$ for some $j < i$ (hence e received colour j before M^i was defined) or M^i contains an edge f such that $f \preceq_u e$ or $f \preceq_v e$. So if e does not receive any colour, that is, if $e \notin \bigcup\{M^j : j \in L(e)\}$ then there is an $f^j \in M^j$ for each $j \in L(e)$ with $f^j \preceq_u e$ or $f^j \preceq_v e$. As $|L(e)| \geq k$, this is impossible by (1) and this contradiction proves that the above algorithm finds a proper L-edge-colouring of G. □

References

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