A note on packing half-regular bipartite graphs

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Abstract

Necessary and sufficient conditions for packing half-regular subgraphs with given degree sequences in a complete bipartite graph were given in [1]. In this note, by using the Kőnig-Hall theorem on bipartite matchings, we give a simpler proof for this result.

1 Introduction

Let $G^* = (S, T; E^*)$ be the complete bipartite graph on colour classes S and T. A **degree-prescription** is a pair (d_T, d_S) where $d_T \in \mathbb{Z}_+^T$ and $d_S \in \mathbb{Z}_+^S$. We say that degree-prescriptions $(d_T^1, d_S^1), \ldots, (d_T^k, d_S^k)$ **pack** if G^* has edge-disjoint subgraphs $G_1, \ldots, G_k$ such that the degree-sequence of G_i is (d_T^i, d_S^i), and **fully pack** if in addition their edge-sets form a partition of E^*, that is, $\sum_{i=1}^k |E_i| = |E^*|$. If $d_T(t) = r$ for every $t \in T$ for some $r \in \mathbb{Z}_+$ then we call (d_T, d_S) **half-regular** and denote it by (r, d_S). Given a bipartite graph $H = (S, T; E)$, the **set of neighbours** of a set $X \subseteq T$ is denoted by $\Gamma_H(X)$.

Although the problem of deciding whether degree-prescriptions $(d_T^1, d_S^1), \ldots, (d_T^k, d_S^k)$ pack is NP-complete already for $k \geq 2$ [2], the following surprisingly simple characterization was given in [1] for the special case when each (d_T^i, d_S^i) is half-regular.

Theorem 1 (Aksen, Miklós, Zhou)**.** *Let $r_1, \ldots, r_k \in \mathbb{Z}_+$ and $d_1, \ldots, d_k \in \mathbb{Z}_+^S$. Then $(r_1, d_1), \ldots, (r_k, d_k)$ pack if and only if*

$$\sum_{i=1}^k d_i(u) \leq |T| \qquad \text{for each } u \in S, \text{ and}$$

$$\sum_{u \in S} d_i(u) = r_i |T| \qquad \text{for } i = 1, \ldots, k.$$

A special case of Theorem 1 is when the degree-prescriptions are required to fully pack. In this case $\sum_{i=1}^k d_i(u) = |T|$ for all $u \in S$. However, this version is equivalent to the general one: given $r_1, \ldots, r_k \in \mathbb{Z}_+$ and $d_1, \ldots, d_k \in \mathbb{Z}_+^S$ as in Theorem 1, define $r_{k+1} := |S| - \sum_{i=1}^k r_i$ and $d_{k+1}(u) := |T| - \sum_{i=1}^k d_i(u)$ for $u \in S$. Then $(r_1, d_1), \ldots, (r_k, d_k)$ pack if and only if $(r_1, d_1), \ldots, (r_{k+1}, d_{k+1})$ fully pack. Hence Theorem 1 can be reformulated as follows.

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Theorem 2. *Let $r_1,\dots,r_k \in \mathbb{Z}_+$ and $d_1,\dots,d_k \in \mathbb{Z}_+^S$. Then $(r_1,d_1),\dots,(r_k,d_k)$ fully pack if and only if*

$$\sum_{i=1}^{k} d_i(u) = |T| \qquad \text{for each } u \in S, \text{ and} \tag{1}$$

$$\sum_{u\in S} d_i(u) = r_i|T| \qquad \text{for } i = 1,\dots,k. \tag{2}$$

Necessity of the conditions is easy. From now on we concentrate on sufficiency and show that Theorem 2 is a direct consequence of the Kőnig-Hall theorem on bipartite matchings. We prove the following strengthening of the theorem.

Theorem 3. *Let $r_1,\dots,r_k \in \mathbb{Z}_+$ and $d_1,\dots,d_k \in \mathbb{Z}_+^S$ be given satisfying (1) and (2). Let $T' \subseteq T$ and assume that pairwise edge-disjoint graphs $G'_1,\dots,G'_k$ are already given so that $d_{G'_i}(u) \le d_i(u)$ for $u \in S$, $d_{G'_i}(v) = r_i$ for $v \in T'$ and $d_{G'_i}(v) = 0$ for $v \in T - T'$. Then there exists a full packing $G_1,\dots,G_k$ of $(r_1,d_1),\dots,(r_k,d_k)$ such that G'_i is a subgraph of G_i.*

By setting $T' = \emptyset$ and $G'_1,\dots,G'_k$ to be the empty graphs we get back Theorem 2.

2 Proof of Theorem 3

It suffices to show that the edges in E^* incident to a node $v \in T - T'$ can be distributed among $G'_1,\dots,G'_k$ while maintaining the conditions of the theorem. Indeed, by applying this step iteratively, we get a proper packing of the degree-prescriptions.

By (1) and (2),

$$\sum_{i=1}^{k} r_i|T| = \sum_{i=1}^{k}\sum_{u\in S} d_i(u) = \sum_{u\in S}\sum_{i=1}^{k} d_i(u) = |S||T|,$$

hence $\sum_{i=1}^{k} r_i = |S|$ holds. This, together with $d_{G'_i}(v) = r_i$ for $v \in T'$ implies $\sum_{i=1}^{k} d_{G'_i}(u) = |T'|$ for all $u \in S$.

Take an arbitrary node $v \in T - T'$. We construct a bipartite graph $H = (S, C; E)$ as follows. S is the same as before, $C = \{c_1,\dots,c_k\}$, and we add $d_i(u) - d_{G'_i}(u)$ parallel edges between $u \in S$ and c_i for $i = 1,\dots,k$. Then the edges incident to v can be distributed among $G'_1,\dots,G'_k$ properly if and only if H has a subgraph in which each node $u \in S$ has degree 1, while c_i has degree r_i for $i = 1,\dots,k$. As $\sum_{i=1}^{k} r_i = |S|$, such a subgraph exists if and only if $|\Gamma_H(X)| \ge \sum_{i:c_i\in X} r_i$ for every $X \subseteq C$. (This can be derived from the Kőnig-Hall theorem by taking r_i copies of c_i for each $i = 1,\dots,k$ and looking for a matching covering this new set of nodes in the extended graph.)

Take an arbitrary set $X \subseteq C$. The degree of a node $u \in S$ in H is $\sum_{i=1}^{k}[d_i(u) - d_{G'_i}(u)]$, hence, by (1) and by $\sum_{i=1}^{k} d_{G'_i}(u) = |T'|$, the total number of edges between X and $\Gamma_H(X)$ is bounded from above by

$$\sum_{u\in\Gamma_H(X)}\sum_{i=1}^{k}[d_i(u) - d_{G'_i}(u)] = \sum_{u\in\Gamma_H(X)}[|T| - |T'|] = |\Gamma_H(X)||T - T'|.$$

On the other hand, the degree of a node $c_i \in C$ is $\sum_{u\in S}[d_i(u) - d_{G'_i}(u)] = r_i|T - T'|$. That is, the number of edges between X and $\Gamma_H(X)$ is exactly

$$\sum_{i:c_i\in X} r_i|T - T'|.$$

By the above, $|\Gamma_H(X)||T - T'| \geq \sum_{i:c_i\in X} r_i|T - T'|$, implying $|\Gamma_H(X)| \geq \sum_{i:c_i\in X} r_i$. This concludes the proof of the theorem.

References

[1] M. Aksen, I. Miklós, and K. Zhou. Half-regular factorizations of the complete bipartite graph. *ArXiv e-prints*, Feb. 2016.

[2] C. Dürr, F. Guinez, and M. Matamala. Reconstructing 3-colored grids from horizontal and vertical projections is NP-hard. In *European Symposium on Algorithms*, pages 776–787. Springer, 2009.