A note on V-free 2-matchings

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Abstract

Motivated by a conjecture of Liang [Y.-C. Liang. *Anti-magic labeling of graphs*. PhD thesis, National Sun Yat-sen University, 2013.], we introduce a restricted path packing problem in bipartite graphs that we call a V-free 2-matching. We verify the conjecture through a weakening of the hypergraph matching problem. We close the paper by showing that it is NP-complete to decide whether one of the color classes of a bipartite graph can be covered by a V-free 2-matching.

1 Introduction

Throughout the paper, graphs are assumed to be simple. Given an undirected graph $G = (V, E)$ and a subset $F \subseteq E$ of edges, $F(v)$ denotes the set of edges in F incident to a node $v \in V$, and $d_F(v) := |F(v)|$ is the **degree** of v in F. We say that F **covers** a subset of nodes $X \subseteq V$ if $d_F(v) \geq 1$ for every $v \in X$. Let $b : V \to \mathbb{Z}_+$ be an upper bound function. A subset $N \subseteq E$ of edges is called a ***b*-matching** if $d_N(v)$ is at most $b(v)$ for every node $v \in V$. For some integer $t \geq 2$, by a ***t*-matching** we mean a b-matching where $b(v) = t$ for every $v \in V$. If $t = 1$, then a t-matching is simply called a **matching**.

A **hypergraph** is a pair $H = (V, \mathcal{E})$ where V is a finite set of nodes and $\mathcal{E}$ is a collection of subsets of V. The members of $\mathcal{E}$ are called **hyperedges**, and for a hyperedge $e \in \mathcal{E}$ let $|e|$ denote its cardinality (as a subset of V). In hypergraphs – unlike in graphs– we will allow hyperedges of cardinality 1 in this paper. A **matching** in a hypergraph is a collection of pairwise disjoint hyperedges, and the matching is said to be perfect if the union of the hyperedges in the matching contains every node. The **hypergraph matching problem** is to decide whether a given hypergraph has a perfect matching. Given a hypergraph $H = (V, \mathcal{E})$, we can represent it as a bipartite graph $G_H = (U_V, U_{\mathcal{E}}; E)$, where nodes of U_V correspond to nodes in V, nodes in $U_{\mathcal{E}}$ correspond to hyperedges in $\mathcal{E}$, and there is an edge in G between a node $u_v \in U_V$

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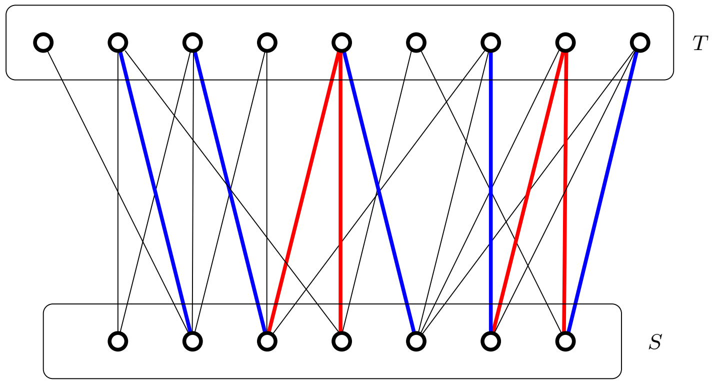

Figure 1: An illustration for Liang's conjecture. Nodes in T have degree at most 3, and those in S have degree at most 4. The matching is highlighted with blue, the family of S-links is highlighted with red.

(corresponding to $v \in V$) and a node $u_e \in U_{\mathcal{E}}$ (corresponding to $e \in \mathcal{E}$) if and only if $v \in e$ (G_H is also called the Levi graph of H).

Let $G = (S, T; E)$ be a bipartite graph. A path $P = (\{u, v, w\}, \{uv, vw\})$ of length 2 with $u, w \in S$ is called an **S-link**, and a **T-link** can be defined analogously. In [13], Liang proposed the following conjecture and showed that, if it is true, the conjecture implies that 4-regular graphs are antimagic (where a simple graph $G = (V, E)$ is said to be **antimagic** if there exists a bijection $f : E \to \{1, 2, \ldots, |E|\}$ such that $\sum_{e \in E(v_1)} f(e) \neq \sum_{e \in E(v_2)} f(e)$ for every pair $v_1, v_2 \in V$).

Conjecture 1. *Assume that $G = (S, T; E)$ is a bipartite graph such that each node in S has degree at most 4 and each node in T has degree at most 3. Then G has a matching M and a family $\mathcal{F}$ of node-disjoint S-links such that every node $v \in T$ of degree 3 is covered by an edge in $M \cup (\cup_{P \in \mathcal{F}} P)$.*

Observe that it suffices to verify the conjecture for the special case when each node in T has degree exactly 3, as we can simply delete nodes of degree less than 3. Although it was recently proved that regular graphs are antimagic [1], we prove the conjecture in Section 3 as it is interesting in its own. The proof is based on a weakening of the hypergraph matching problem.

While working on the proof of the conjecture, an interesting restricted path factor problem came to our attention. For simplicity, we will call a T-link a **V-path** (the name comes from the shape of these paths when T is placed 'above' S, see Figure 1 for an illustration). It is easy to see that a 2-matching consists of pairwise node-disjoint paths and cycles. We call a 2-matching **V-free** if it does not contain a V-path as a connected component.

Consider the problem of finding a matching M and a family $\mathcal{F}$ of node-disjoint S-links such that $M \cup (\cup_{P \in \mathcal{F}} P)$ covers T. We can assume that M does not contain any edge of $\bigcup \mathcal{F}$, as such edges can be simply deleted from M. Furthermore, we may assume that each node $v \in T$ has degree at most 2 in $M \cup (\cup_{P \in \mathcal{F}} P)$. Indeed, if a node $v \in T$ has degree 3 in $M \cup (\cup_{P \in \mathcal{F}} P)$ then it is covered by both M and $(\cup_{P \in \mathcal{F}} P)$, so the edge in M incident to v can be deleted (see Figure 1). It is not difficult to see that $M \cup (\cup_{P\in\mathcal{F}} P)$ is a V-free 2-matching covering T in this case.

Conversely, given an arbitrary V-free 2-matching N that covers T, edges can be left out from N in such a way that the resulting V-free 2-matching N' still covers T and consists of paths of length 1 and 4, the latter having both end-nodes in T. Then N' can be partitioned into a matching and a family of node-disjoint S-links.

By the above, the problem of finding a matching M and a family $\mathcal{F}$ of node-disjoint S-links whose union covers T is equivalent to finding a V-free 2-matching N that covers T. The proof of Conjecture 1 shows that these problems can be solved when nodes in S have degree at most 4, and those in T have degree at most 3. However, in Section 4 we show that the problem of finding a V-free 2-matching in a bipartite graph $G = (S, T; E)$ covering T is NP-complete in general.

Let us now recall some well known results from matching theory that will be used below.

Theorem 2. *In a bipartite graph there exists a matching that covers every node of maximum degree.*

Theorem 3 (Dulmage and Mendelsohn [3])**.** *Given a bipartite graph $G = (S, T; E)$ and subsets $X \subseteq S$, $Y \subseteq T$, if there exist two matchings M_X and M_Y in G such that M_X covers X and M_Y covers Y then there exists a matching M in G that covers $X \cup Y$.*

Theorem 4 (Gallai-Edmonds Decomposition Theorem for graphs, see eg. [15])**.** *Given a graph $G = (V, E)$, let D be the set of nodes which are not covered by at least one maximum matching of G, A be the set of neighbours of D and $C := V - (D \cup A)$. Then **(a)** the components of $G[D]$ are factor-critical, **(b)** $G[C]$ has a perfect matching, and **(c)** G has a matching covering A.*

The paper is organized as follows. Section 2 gives a brief overview of earlier results on restricted path packing problems. In Section 3, we introduce a variant of the hypergraph matching problem and prove a general theorem which in turn implies the conjecture. The paper is closed with a complexity result on V-free 2-matchings in a bipartite graph $G = (S, T, E)$ covering T, see Section 4.

2 Previous work

For a set $\mathcal{F}$ of connected graphs, a spanning subgraph M of a graph G is called an **$\mathcal{F}$-factor** of G if every component of M is isomorphic to one of the members of $\mathcal{F}$. The **path** and **cycle** having n nodes are denoted by P_n and C_n, respectively. The **length** of P_n is $n - 1$, the number of its edges.

The problem of packing $\mathcal{F}$-factors is widely studied. Kaneko presented a Tutte-type characterization of graphs admitting a $\{P_n | n \geq 3\}$-factor [8]. Kano, Katona and Király [9] gave a simpler proof of Kaneko's theorem and also a min-max formula for the maximum number of nodes that can be covered by a 2-matching not containing a single edge as a connected component. Such a 2-matching is often called 1**-restricted**. These results were further generalized by Hartvigsen, Hell and Szabó [6] by introducing the so-called k**-piece packing** problem, where a k-piece is a connected graph with highest degree exactly k. In contrast with earlier approaches, their result is algorithmic, and so it provides a polynomial time algorithm for finding a 1-restricted 2-matching covering a maximum number of nodes. Later Janata, Loebl and Szabó [7] described a Gallai-Edmonds type structure theorem for k-piece packings and proved that the node sets coverable by k-piece packings have a matroidal structure.

In [5], Hartvigsen considered the edge-max version of the 1-restricted 2-matching problem, that is, when a 1-restricted 2-matching containing a maximum number of edges is needed. He gave a min-max theorem characterizing the maximum number of edges in such a subgraph, and he also presented a polynomial algorithm for finding one. The notion of 1-restricted 2-matchings was generalized by Li [12] by introducing j**-restricted** k**-matchings** that are k-matchings with each connected component having at least $j+1$ edges. She considered the node-weighted version of the problem of finding a j-restricted k-matching in which the total weight of the nodes covered by the edges is maximal and presented a polynomial algorithm for the problem as well as a min-max theorem in the case of $j < k$. She also proved that the problem of maximizing the number of nodes covered by the edges in a j-restricted k-matching is NP-hard when $j \geq k \geq 2$.

A graph is called **cubic** if each node has degree 3. Cycle-factors and path-factors of cubic graphs are well-studied. The fundamental theorem of Petersen states that each 2-connected cubic graph has a $\{C_n|n \geq 3\}$-factor [14]. From Kaneko's theorem it follows that every connected cubic graph has a $\{P_n|n \geq 3\}$-factor. Kawarabayashi, Matsuda, Oda and Ota proved that every 2-connected cubic graph has a $\{C_n|n \geq 4\}$-factor, and if the graph has order at least six then it also has a $\{P_n|n \geq 6\}$-factor [11]. For bipartite graphs, these results were improved by Kano, Lee and Suzuki by showing that every connected cubic bipartite graph has a $\{C_n|n \geq 6\}$-factor, and if the graph has order at least eight then it also has a $\{P_n|n \geq 8\}$-factor [10].

Although the V-free 2-matching problem shows lots of similarities to these problems, it does not seem to fit in the framework of earlier approaches.

3 Extended matchings

While working on Conjecture 1, we arrived at a relaxation of the hypergraph matching problem that we call the **extended matching problem**. An **extended matching** of a hypergraph $H = (V, \mathcal{E})$ is a disjoint collection of hyperedges and pairs of nodes where a pair (u, v) may be used only if there exists a hyperedge $e \in \mathcal{E}$ with $u, v \in e$. An extended matching is **perfect** if it covers the node-set of H. Note that one can decide in polynomial time if a hypergraph has a perfect extended matching by the results of [2] (see also Theorem 4.2.16 in [16]). Indeed, given a hypergraph $H = (V, \mathcal{E})$, consider its bipartite representation $G_H = (U_V, U_\mathcal{E}; E)$. Then a perfect extended matching in H corresponds to a subgraph in G_H in which nodes of U_V have degree one, and a node $u_e \in U_\mathcal{E}$ corresponding to $e \in \mathcal{E}$ has degree $|e|$, or any even number not greater than $|e|$.

However, we have found a simple proof of the following result, a special case of the extended matching problem, which implies Conjecture 1, as we show below.

Theorem 5. *In a 3-uniform hypergraph $H = (V, \mathcal{E})$ there exists an extended matching that covers the nodes of maximum degree in H.*

Theorem 5 is the special case of a more general result (Corollary 9) that we introduce below. Before doing so, we show that Theorem 5 implies Conjecture 1.

Proof of Conjecture 1. Recall that it suffices to verify the conjecture for graphs $G = (S, T; E)$ with $d_E(v) = 3$ for every $v \in T$. Such a G is the incidence graph (or Levi graph) of a 3-uniform hypergraph $H = (S, \mathcal{E})$ in which each node has degree at most 4.

Let $S' \subseteq S$ denote the set of nodes having degree 4 in H. By Theorem 5, H has an extended matching covering S'. That is, S' can be covered by pairwise node-disjoint S-links and S-claws of G, where an S-claw is a star with 3 edges having its center node in T. We denote the edge-set of these S-links and claws by N.

Let T' be the set of nodes in T not covered by N. As $d_{E-N}(v) \leq 3$ for each $v \in S$, T' can be covered by a matching M disjoint from N, by Theorem 2. By leaving out an edge from each S-claw of N, we get a matching M and a family of S-links whose union together covers T. □

Let us now introduce and prove a generalization of Theorem 5. We call a hypergraph $H = (V, \mathcal{E})$ **oddly uniform** if every hyperedge has odd cardinality. The **quasi-degree** of a node $v \in V$ is defined as $d^-(v) := \sum[|e| - 1 : \ v \in e \in \mathcal{E}]$, and the hypergraph is **Δ-quasi-regular** (or **quasi-regular** for short) if $d^-(v) = \Delta$ for each $v \in V$ where $\Delta \in \mathbb{Z}_+$. Note that a uniform regular hypergraph is quasi-regular.

Theorem 6. *Every oddly uniform quasi-regular hypergraph has a perfect extended matching.*

Proof. Assume that $H = (V, \mathcal{E})$ is an oddly uniform Δ-quasi-regular hypergraph, and let $G = (V, E)$ denote the graph obtained by replacing each hyperedge $e \in \mathcal{E}$ with a complete graph on node-set $e \subseteq V$. That is, there are as many parallel edges between u and v in E as the number of hyperedges containing both u and v. Note that the quasi-regularity of H is equivalent to the regularity of G.

If G admits a perfect matching M, then M is a perfect extended matching of H and we are done.

Assume that G does not have a perfect matching. Take the Gallai-Edmonds decomposition of G into sets D, A and C (see Theorem 4). Let D_1 be the union of those connected components of $G[D]$ that span a hyperedge $e \in \mathcal{E}$ in H, and $D_2 := D - D_1$.

Claim 7. *Every component K of $G[D_1]$ has a perfect extended matching in H.*

Proof. As K is factor-critical, it has a perfect matching after deleting the nodes of any of its odd cycles (including the case when the cycle consists of a single node). Let $e \in \mathcal{E}$ be a hyperedge spanned by K. By the above, $G[K - e]$ has a perfect matching, which together with e form a perfect extended matching of K, proving the claim. □

Claim 8. *For every component K of $G[D_2]$ we have $d_G(K) \geq \Delta$.*

Proof. Let $u \in K$ be an arbitrary node. K does not span a hyperedge in H, hence for every hyperedge e containing u we have $e \cap K \neq \emptyset, e \cap A \neq \emptyset$ and $e \subseteq K \cup A$. By the definition of G, there are at least $\sum[|e \cap K| \cdot |e \cap A| : u \in e \in \mathcal{E}] \geq \sum[|e| - 1 : u \in e \in \mathcal{E}] = \Delta$ edges between K and A, thus concluding the proof of the claim. □

Let $G' = (D', A; F)$ denote the bipartite graph obtained from G by deleting the nodes of C and the edges induced by A, and by contracting each component of $G[D]$ to a single node (the set of new nodes is denoted by D'). Nodes of D' are partitioned into sets D'_1 and D'_2 accordingly. As $d_{G'}(v) \leq \Delta$ for each $v \in A$, Claim 8 and Theorem 2 imply that G' has a matching covering D'_2. By Theorem 4 (c), G' has a matching covering A, hence the result of Dulmage and Mendelsohn (Theorem 3) implies that G' has a matching M' covering A and D'_2 simultaneously. Considering M' as a matching in G and using Theorem 4 (a) and (b), M' can be extended to a matching M of G that covers every node that is in $C \cup A$ or in a component of $G[D]$ that is incident to an edge in M'. By Claim 7, there is an extended matching covering the nodes of the remaining components of $G[D]$, since they fall in D_1. The union of M and this extended matching forms a perfect extended matching of H. This completes the proof of the theorem. □

As a consequence, we get the following result.

Corollary 9. *Every oddly uniform hypergraph has an extended matching that covers the set of nodes having maximum quasi-degree.*

Proof. Let $H = (S, \mathcal{E})$ be an oddly uniform hypergraph and let Δ denote the maximum quasi-degree in H. The **deficiency** of a node $v \in S$ is $\gamma(v) := \Delta - d^-(v)$. A node $v \in S$ is called **deficient** if $\gamma(v) > 0$. As H is oddly uniform, $\gamma(v)$ is even for every node v.

It suffices to show that H can be extended to a Δ-quasi-uniform hypergraph $H' = (V', \mathcal{E}')$ by adding further nodes and hyperedges. Indeed, by Theorem 6, H' admits a perfect extended matching whose restriction to the original hypergraph gives an extended matching covering each node having quasi-degree Δ.

If there is no deficient node in H, then we are done. Otherwise consider the hypergraph obtained by taking the disjoint union of three copies of H, denoted by H_1, H_2 and H_3, respectively. For each deficient node $v \in S$, add $\gamma(v)$ copies of the hyperedge $\{v_1, v_2, v_3\}$ to the hypergraph, where v_i denotes the copy of v in H_i. The hypergraph H' thus obtained is clearly Δ-quasi-regular. □

4 Complexity result

In what follows we show that deciding the existence of a V-free 2-matching covering T is NP-complete in general. We will use reduction from the following problem (see [4, (SP2)]).

Theorem 10 (3-dimensional matching)**.** *Let $H = (X, Y, Z; \mathcal{E})$ be a tripartite 3-regular 3-uniform hypergraph, meaning that each node $v \in X \cup Y \cup Z$ is contained in exactly 3 hyperedges, and each hyperedge $e \in \mathcal{E}$ contains exactly one node from all of X, Y and Z. It is NP-complete to decide whether H has a perfect matching, that is, a 1-regular sub-hypergraph.*

Our proof is inspired by the construction of Li for proving the NP-hardness of maximizing the number of nodes covered by the edges in a 2-restricted 2-matching [12].

Theorem 11. *Given a bipartite graph $G = (S, T; E)$ with maximum degree 4, it is NP-complete to decide whether G has a* V*-free 2-matching covering T.*

Proof. We prove the theorem by reduction from the 3-dimensional matching problem. Take a 3-uniform 3-regular tripartite hypergraph $H = (X, Y, Z; \mathcal{E})$. For a hyperedge $e \in \mathcal{E}$, we use the following notions: $x_e := e \cap X$, $y_e := e \cap Y$ and $z_e := e \cap Z$.

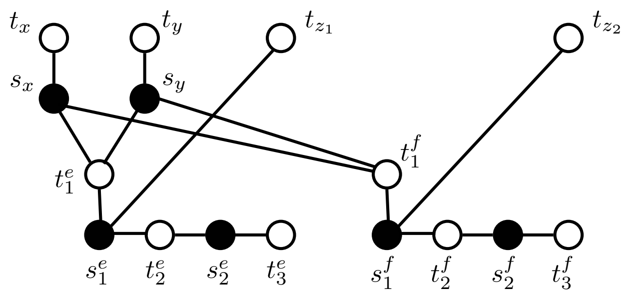

Figure 2: Gadgets corresponding to hyperedges $e = \{x, y, z_1\}$ and $f = \{x, y, z_2\}$

We construct an undirected bipartite graph as follows. For each node $x \in X$ and $y \in Y$, add a pair of nodes s_x, t_x and s_y, t_y to G, respectively, with $s_x, s_y \in S$ and $t_x, t_y \in T$. For each node $z \in Z$, add a single node t_z to T. Furthermore, for each $x \in X$ and $y \in Y$ add the edges $s_x t_x$ and $s_y t_y$ to E.

We assign a path P_e with node set $V(P_e) = \{t_1^e, s_1^e, t_2^e, s_2^e, t_3^e\}$ and edge set $E(P_e) = \{t_1^e s_1^e, s_1^e t_2^e, t_2^e s_2^e, s_2^e t_3^e, t_3^e\}$ of length four to each hyperedge $e \in \mathcal{E}$ and add edges $s_{x_e} t_1^e$, $s_{y_e} t_1^e$ and $t_{z_e} s_1^e$ to E (see Figure 2). It is easy to check that the graph thus arising is bipartite and has maximum degree 4 (here we use that every node $v \in X \cup Y \cup Z$ is contained in exactly 3 hyperedges of H).

We claim that H admits a perfect matching if and only if G has a V-free 2-matching covering T, which proves the theorem. Assume first that H has a perfect matching and let $\mathcal{M} \subseteq \mathcal{E}$ be the set of matching hyperedges. Then

$$M := \bigcup_{e \in \mathcal{M}} \{s_{x_e} t_{x_e}, s_{y_e} t_{y_e}, s_{x_e} t_1^e, s_{y_e} t_1^e, t_{z_e} s_1^e, E(P_e) - t_1^e s_1^e\} \cup \bigcup_{e \notin \mathcal{M}} E(P_e)$$

is a V-free 2-matching covering T (see Figure 3).

For the other direction, take a V-free 2-matching M of G covering T. Observe that $s_x t_x, s_y t_y \in M$ for each $x \in X$ and $y \in Y$ as M covers T. Moreover, M is V-free

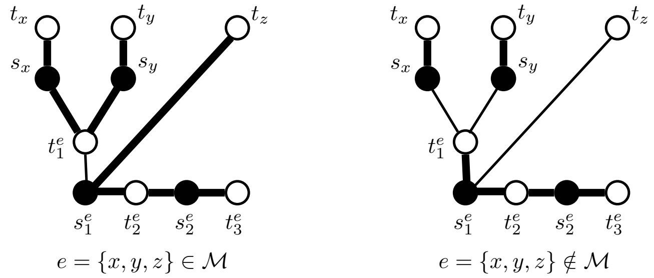

Figure 3: Edges included in M depending on whether $e \in \mathcal{M}$ or not

hence $t_1^e s_1^e \notin M$ implies $s_{e_x} t_1^e, y_{e_x} t_1^e \in M$. We may assume that $E(P_e) - t_1^e s_1^e \subseteq M$ for each $e \in \mathcal{E}$. Indeed, M has to cover t_2^e and t_3^e, hence the V-freeness of M implies $s_1^e t_2^e, s_2^e t_3^e \in M$. Consequently, $t_2^e s_2^e \in M$ can be assumed.

We claim that $d_M(t_z) = 1$ for each $z \in Z$. Indeed, if $t_{z_e} s_1^e \in M$ for some $e \in \mathcal{E}$ then $s_{x_e} t_1^e, s_{y_e} t_1^e \in M$. In other words, if $t_{z_e} s_1^e \in M$ then e 'reserves' nodes s_{x_e}, s_{y_e} and t_{z_e} for M being a V-free 2-matching. On the other hand, for each $x \in X$ there is at most one $e \in \mathcal{E}$ such that $s_{x_e} t_1^e \in M$, and the same holds for each $y \in Y$. As the hypergraph is 3-uniform and 3-regular, we have $|X| = |Y| = |Z|$. Hence the number of edges of form $t_{z_e} s_2^e$ in M can not exceed the cardinality of these sets. Let

$$\mathcal{M} := \{e \in \mathcal{E} : \ t_{z_e} s_2^e \in M\}.$$

By the above, $\mathcal{M}$ is a 1-regular subhypergraph, thus concluding the proof. □

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