Regular graphs are antimagic

Kristóf Bérczi, Attila Bernáth*, and Máté Vizer**

Abstract

In this note we prove - with a slight modification of an argument of Cranston et al. [2] - that k-regular graphs are antimagic for $k \geq 2$.

1 Introduction

Throughout the note graphs are assumed to be simple. Given an undirected graph $G = (V, E)$ and a subset of edges $F \subseteq E$, $F(v)$ denotes the set of edges in F incident to node $v \in V$, and $d_F(v) := |F(v)|$ is the **degree** of v in F. A **labeling** is an injective function $f : E \to \{1, 2, \dots, |E|\}$. Given a labeling f and a subset of edges F, let $f(F) = \sum_{e \in F} f(e)$. A labeling is **antimagic** if $f(E(u)) \neq f(E(v))$ for any pair of different nodes $u, v \in V$. A graph is said to be **antimagic** if it admits an antimagic labeling.

Hartsfield and Ringel conjectured [4] that all connected graphs on at least 3 nodes are antimagic. The conjecture has been verified for several classes of graphs (see e.g. [3]), but is widely open in general. In [2] Cranston et al. proved that every k-regular graph is antimagic if $k \geq 3$ is odd. Note that 1-regular graphs are trivially not antimagic. We have observed that a slight modification of their argument also works for even regular graphs, hence we prove the following.

Theorem 1. *For $k \geq 2$, every k-regular graph is antimagic.*

It is worth mentioning the following conjecture of Liang [5]. Let $G = (S, T; E)$ be a bipartite graph. A path $P = \{uv, vw\}$ of length 2 with $u, w \in S$ is called an S-**link**.

Conjecture 2. *Let $G = (S, T; E)$ be a bipartite graph such that each node in S has degree at most 4 and each node in T has degree at most 3. Then G has a matching M and a family $\mathcal{P}$ of node-disjoint S-links such that every node $v \in T$ of degree 3 is incident to an edge in $M \cup (\bigcup_{P \in \mathcal{P}} P)$.*

Liang showed that if the conjecture holds then it implies that every 4-regular graph is antimagic. The starting point of our investigations was proving Conjecture 2. As Theorem 1 provides a more general result, we leave the proof of Conjecture 2 for a forthcoming paper [1].

*MTA-ELTE Egerváry Research Group (EGRES), Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, Budapest, Hungary, H-1117. E-mail: {berkri,bernath}@cs.elte.hu

**MTA Alfréd Rényi Institute of Mathematics, P.O.B. 127, Budapest H-1364, Hungary. E-mail: vizermate@gmail.com

2 Proof of Theorem 1

A **trail** in a graph $G = (V, E)$ is an alternating sequence of nodes and edges $v_0, e_1, v_1, \ldots, e_t, v_t$ such that e_i is an edge connecting v_{i-1} and v_i for $i = 1, 2, \ldots, t$, and the edges are all distinct (but there might be repetitions among the nodes). The trail is **open** if $v_0 \neq v_t$, and **closed** otherwise. The **length** of a trail is the number of edges in it. A closed trail containing every edge of the graph is called an **Eulerian trail**. It is well known that a graph has an Eulerian trail if and only if it is connected and every node has even degree.

Lemma 3. *Given a connected graph $G = (V, E)$, let $T = \{v \in V : d_E(v)$ is odd$\}$. If $T \neq \emptyset$, then E can be partitioned into $|T|/2$ open trails.*

Proof. Note that $|T|$ is even. Arrange the nodes of T into pairs in an arbitrary manner and add a new edge between the members of every pair. Take an Eulerian trail of the resulting graph and delete the new edges to get the $|T|/2$ open trails. □

The main advantage of Lemma 3 is that the edge set of the graph can be partitioned into open trails such that at most one trail starts at every node of V. Indeed, there is a trail starting at v if and only if v has odd degree in G. This is how we see the Helpful Lemma of [2].

Corollary 4 (Helpful Lemma of [2])**.** *Given a bipartite graph $G = (U, W; E)$ with no isolated nodes in U, E can be partitioned into subsets $E^\sigma, T_1, T_2, \ldots, T_l$ such that $d_{E^\sigma}(u) = 1$ for every $u \in U$, T_i is an open trail for every $i = 1, 2, \ldots, l$, and the endpoints of T_i and T_j are different for every $i \neq j$.*

Proof. Take an arbitrary $E' \subseteq E$ with the property $d_{E'}(u) = 1$ for every $u \in U$. A component of $G - E'$ containing more than one node is called **nontrivial**. If there exists a nontrivial component of $G - E'$ that only contains even degree nodes then let $uw_1 \in E - E'$ be an edge in this component with $u \in U$ and $w_1 \in W$, and let $uw_2 \in E'$. Replace uw_2 with uw_1 in E'. After this modification, the component of $G - E'$ that contains u has an odd degree node, namely w_1. Iterate this step until every nontrivial component of $G - E'$ has some odd degree nodes. Let $E^\sigma = E'$ and apply Lemma 3 to get the decomposition of $E - E^\sigma$ into open trails. □

In what follows we prove that regular graphs are antimagic: for sake of completeness we include the odd regular case, too. We emphasize the differences from the proof appearing in [2].

Proof of Theorem 1. Note that it suffices to prove the theorem for connected regular graphs. Let $G = (V, E)$ be a connected k-regular graph and let $v^* \in V$ be an arbitrary node. Denote the set of nodes at distance exactly i from v^* by V_i and let q denote the largest distance from v^*. We denote the edge-set of $G[V_i]$ by E_i. Apply Corollary 4 to the induced bipartite graph $G[V_{i-1}, V_i]$ with $U = V_i$ to get E_i^σ and the trail decomposition of $G[V_{i-1}, V_i] - E_i^\sigma$ for every $i = 1, \ldots, q$. The edge set of $G[V_{i-1}, V_i] - E_i^\sigma$ is denoted by E_i'.

Now we define the antimagic labeling f of G as follows. We reserve the $|E_q|$ smallest labels for labeling E_q, the next $|E_q^\sigma|$ smallest labels for labeling E_q^σ, the next $|E_q'|$ smallest labels for labeling E_q', the next $|E_{q-1}|$ smallest labels for labeling E_{q-1}, etc. There is an important difference here between our approach and that of [2] as we switched the order of labeling E_i^σ and E_i', and we don't yet define the labels, we only reserve the intervals to label the edge sets. Next we prove a claim that tells us how to label the edges in E_i'.

Claim 5. *Assume that we have to label the edges of E_i' from interval $s, s+1, \ldots, \ell$ (where $|E_i'| = \ell - s + 1$), and that we are given a trail decomposition of E_i' into open trails. We can label E_i' so that successive labels (in a trail) incident to a node $v_i \in V_i$ have sum at most $s+\ell$, and successive labels (in a trail) incident to a node $v_{i-1} \in V_{i-1}$ have sum at least $s+\ell$.*

Proof. Our proof of this claim is essentially the same as the proof in [2]: we merely restate it for self-containedness. Let T be the trail decomposition of E_i' into open trails. Take an arbitrary trail $T = u_0, e_1, u_1, \ldots, e_t, u_t$ of length t from T and consider the following two cases (see Figure 1 for an illustration).

- **Case A:** If $u_0 \in V_{i-1}$ then label $e_1, \ldots, e_t$ by $s, \ell, s+1, \ell-1, \ldots$ in this order. In this case the sum of 2 successive labels is $s+\ell$ at a node in V_i, and it is $s+\ell+1$ at a node in V_{i-1}.
- **Case B:** If $u_0 \in V_i$ then label $e_1, \ldots, e_t$ by $\ell, s, \ell-1, s+1, \ldots$ in this order. In this case the sum of 2 successive labels is $s+\ell-1$ at a node in V_i, and it is $s+\ell$ at a node in V_{i-1}.

We prove by induction on $|\mathcal{T}|$. The proof is finished by the following cases.

1. If T contains a trail of even length, then let T be such a trail (and again t denotes the length of T). If the endpoints of T fall in V_{i-1} then apply Case A. On the other hand, if the endpoints of T fall in V_i then apply Case B. In both cases we use $\frac{t}{2}$ labels from the lower end of the interval, and $\frac{t}{2}$ labels from the upper end, therefore we can label the edges of the trails in $\mathcal{T} - T$ from the (remaining) interval $s + \frac{t}{2}, s + \frac{t}{2} + 1, \ldots, \ell - \frac{t}{2}$, so that the lower bound $s + \frac{t}{2} + \ell - \frac{t}{2} = s + \ell$ holds for the sum of two successive labels at every $v_{i-1} \in V_{i-1}$, and the same upper bound holds at each node $v_i \in V_i$.
2. Every trail in T has odd length. If T contains only one trail then label it using either of the two cases above and we are done. Otherwise let T_1 and T_2 be two trails from T, and let t_i be the length of T_i for both $i = 1, 2$. Label first the edges of T_1 using Case A (starting at the endpoint of T_1 that lies in V_{i-1}). Note that the remaining labels form the interval $s+\frac{t_1+1}{2}, \ldots, \ell-\frac{t_1-1}{2}$. Next label the edges of T_2 using Case B (starting at the endpoint of T_2 that lies in V_i). Note that the sum of successive labels in the trail T_2 becomes $s + \frac{t_1+1}{2} + (\ell - \frac{t_1-1}{2}) - 1 = s + \ell$ at a node in V_i, and it is $s+\frac{t_1+1}{2}+(\ell-\frac{t_1-1}{2}) = s+\ell+1$ at a node in V_{i-1}, which is fine for us. Finally, the remaining labels form the interval $s + \frac{t_1+1}{2} + \frac{t_2-1}{2}, \ldots, \ell - \frac{t_1-1}{2} - \frac{t_2+1}{2}$,

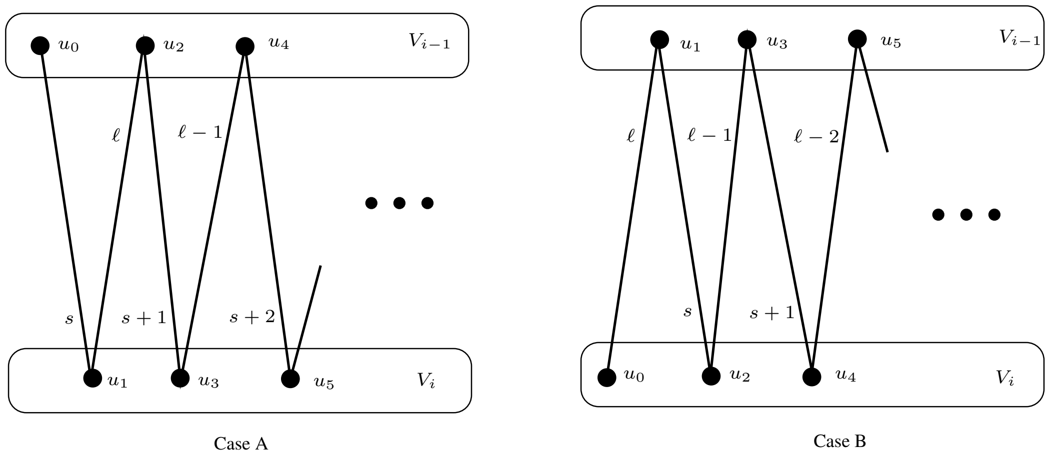

Figure 1: An illustration for labeling trails.

therefore we can label the edges of the trails in $\mathcal{T}-\{T_1,T_2\}$ from the remaining interval so that the lower bound $s+\frac{t_1+1}{2}+\frac{t_2-1}{2}+\ell-\frac{t_1-1}{2}-\frac{t_2+1}{2}=s+\ell$ holds for the sum of two successive labels at every node of V_{i-1}, and the same upper bound holds at every node of V_i.

□

Now we specify how the labels are determined to make sure $f(E(u))\neq f(E(v))$ for every $u\neq v$. We label the edges of every E_i arbitrarily from their dedicated intervals. Label the edges of every E'_i in the manner described by Claim 5. For any node $v\in V_i$ with $i>0$, let $\sigma(v)$ denote the unique edge of E^σ_i incident to v. Let $p(v)=f(E(v))-f(\sigma(v))$ for every $v\in V-v^*$. We label the edges in $E^\sigma_q,E^\sigma_{q-1},\dots,E^\sigma_1$ as in [2]: if we already labeled $E^\sigma_q,E^\sigma_{q-1},\dots,E^\sigma_{i+1}$ then $p(v_i)$ is already determined for every $v_i\in V_i$. So we order the nodes of V_i in an increasing order according to their p-value and assign the label to their σ edge in this order. This ensures that $f(E(u))\neq f(E(v))$ for an arbitrary pair $u,v\in V_i$.

We have fully described the labeling procedure. This labeling scheme ensures that $f(E(v_i))<f(E(v_j))$ if $v_i\in V_i, v_j\in V_j$ and $i\geq j+2$ since G is regular and the edges in $E(v_j)$ get larger labels than those in $E(v_i)$. Similarly, $f(E(v^*))>f(E(v))$ for every $v\in V-v^*$ for the same reason. It is only left is to show that $f(E(v_i))\neq f(E(v_{i-1}))$ for arbitrary $v_i\in V_i, v_{i-1}\in V_{i-1}$ and $i\geq 2$.

Claim 6. *For arbitrary $v_i\in V_i, v_{i-1}\in V_{i-1}$ and $i\geq 2$ we have*

(i) $p(v_i)\leq\frac{k-2}{2}(s+\ell)+\ell$ *and* $p(v_{i-1})\geq\frac{k-2}{2}(s+\ell)+s$, *if k is even, and*

(ii) $p(v_i)\leq\frac{k-1}{2}(s+\ell)$ *and* $p(v_{i-1})\geq\frac{k-1}{2}(s+\ell)$, *if k is odd.*

Proof. Assume first that k is even. In this case $p(v)$ is the sum of an odd number of labels. We pair up all but one of these labels using the trail decomposition of E'_i to get the bounds needed.

1. Take a node $v_i \in V_i$. Note that $f(e) < s$ for every $e \in E(v_i) - E'_i$. Let $t = d_{E'_i}(v_i)$.
   (a) If t is even then $\sum_{e \in E'_i \cap E(v_i)} f(e) \leq \frac{t}{2}(s + \ell)$ by Claim 5, giving $p(v_i) \leq \frac{t}{2}(s + \ell) + (k - 1 - t)s \leq \frac{k-2}{2}(s + \ell) + \ell$.
   (b) If t is odd then $\sum_{e \in E'_i \cap E(v_i)} f(e) \leq \frac{t-1}{2}(s+\ell)+\ell$ by Claim 5, giving $p(v_i) \leq \frac{t-1}{2}(s + \ell) + \ell + (k - 1 - t)s \leq \frac{k-2}{2}(s + \ell) + \ell$.
2. Now take a node $v_{i-1} \in V_{i-1}$. Note that $f(e) > \ell$ for every $e \in E(v_{i-1}) - E'_i$. Let again $t = d_{E'_i}(v_{i-1})$.
   (a) If t is even then $\sum_{e \in E'_i \cap E(v_{i-1})} f(e) \geq \frac{t}{2}(s+\ell)$ by Claim 5, giving $p(v_{i-1}) \geq \frac{t}{2}(s + \ell) + (k - 1 - t)\ell \geq \frac{k-2}{2}(s + \ell) + s$.
   (b) If t is odd then $\sum_{e \in E'_i \cap E(v_{i-1})} f(e) \geq \frac{t-1}{2}(s + \ell) + s$ by Claim 5, giving $p(v_{i-1}) \geq \frac{t-1}{2}(s + \ell) + s + (k - 1 - t)\ell \geq \frac{k-2}{2}(s + \ell) + s$.

This concludes the proof of (i).

Although the proof of (ii) can be found in [2], we also present it here to make the paper self contained. The proof is very similar to the even case. So assume that k is odd. In this case $p(v)$ is the sum of an even number of labels. We pair up these labels using the trail decomposition of E'_i to get the bounds needed.

1. Take a node $v_i \in V_i$. Note that $f(e) < s$ for every $e \in E(v_i) - E'_i$. Let $t = d_{E'_i}(v_i)$.
   (a) If t is even then $\sum_{e \in E'_i \cap E(v_i)} f(e) \leq \frac{t}{2}(s + \ell)$ by Claim 5, giving $p(v_i) \leq \frac{t}{2}(s + \ell) + (k - 1 - t)s \leq \frac{k-1}{2}(s + \ell)$.
   (b) If t is odd then $\sum_{e \in E'_i \cap E(v_i)} f(e) \leq \frac{t-1}{2}(s+\ell)+\ell$ by Claim 5, giving $p(v_i) \leq \frac{t-1}{2}(s + \ell) + \ell + (k - 1 - t)s \leq \frac{k-1}{2}(s + \ell)$.
2. Now take a node $v_{i-1} \in V_{i-1}$. Note that $f(e) > \ell$ for every $e \in E(v_{i-1}) - E'_i$. Let again $t = d_{E'_i}(v_{i-1})$.
   (a) If t is even then $\sum_{e \in E'_i \cap E(v_{i-1})} f(e) \geq \frac{t}{2}(s+\ell)$ by Claim 5, giving $p(v_{i-1}) \geq \frac{t}{2}(s + \ell) + (k - 1 - t)\ell \geq \frac{k-1}{2}(s + \ell)$.
   (b) If t is odd then $\sum_{e \in E'_i \cap E(v_{i-1})} f(e) \geq \frac{t-1}{2}(s + \ell) + s$ by Claim 5, giving $p(v_{i-1}) \geq \frac{t-1}{2}(s + \ell) + s + (k - 1 - t)\ell \geq \frac{k-1}{2}(s + \ell)$.

This concludes the proof of (ii), and we are done. □

The assignment of the labels implies $f(\sigma(v_i)) < s$ and $f(\sigma(v_{i-1})) > \ell$ for $v_i \in V_i$ and $v_{i-1} \in V_{i-1}$. Claim 6 yields $f(E(v_i)) < f(E(v_{i-1}))$, finishing the proof of Theorem 1. □

Remark 7. Observe that the proof of Theorem 1 does not really use the regularity of the graph, it merely relies on the fact that the degree of a node $v_i \in V_i$ is not smaller than that of a node $v_j \in V_j$ where $i < j$. Hence the following result immediately follows.

Theorem 8. *Assume that a connected graph $G = (V, E)$ ($|V| \geq 3$) has a node $v^* \in V$ of maximum degree such that $d_E(v_i) \geq d_E(v_j)$ whenever $v_i \in V_i, v_j \in V_j$ and $i < j$, where V_ℓ denotes the set of nodes at distance exactly ℓ from v^*. Then G is antimagic.*

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Erratum

Recently, Chang, Liang, Pan and Zhu observed that the proof of Theorem 1 is incorrect: in the proof of Claim 6 (page 5), Case 2 assumes that $f(e) > \ell$ for every $e \in E(v_{i-1}) - E'_i$. However, this assumption does not hold for edges in E_i^σ, thus the subsequent calculations are not valid. The aim of the present erratum is to fix the proof. It is important to mention that at the same time when the original paper appeared, regular graphs were proved to be antimagic by Chan et al. [2]. However, as our paper received several citations we felt that we should fix the problem appearing in the proof. Although the high level idea remained the same, the proof has changed significantly as we are relying on further results from matching theory (see the following subsection).

Cranston et al. [3] verified that regular graphs of odd degree are antimagic. In [1], the authors verified the conjecture for the case $k = 4$ by introducing a restricted path packing problem in bipartite graphs. As the case $k = 2$ is trivial, we concentrate on $k \geq 6$ and k being even.

Tools

Let us recall the following folklore result from matching theory.

Theorem 9. *In a bipartite graph there exists a matching that covers every node of maximum degree.*

We will also build upon the following theorem.

Theorem 10. *Let $G = (S, T; E)$ be a bipartite graph and $T = T_1 \cup T_2$ be a partition of T. For a set $X \subseteq S$ let $N_i(X)$ denote the neighbours of X in T_i ($i = 1, 2$). If $\lceil |N_1(X)|/2\rceil + |N_2(X)| \geq |X|$ for all $X \subseteq S$, then there exists a matching covering S that covers at most $\lceil |T_1|/2\rceil$ nodes from T_1.*

Proof. Extend the graph by adding a set S' of new nodes to S with $|S'| = \lfloor |T_1|/2\rfloor$ together with a complete bipartite graph between T_1 and S'. We claim that the resulting bipartite graph has a matching covering $S \cup S'$. This would prove the theorem as deleting the newly added edges from such a matching results in a matching covering S that covers at most $|T_1| - \lfloor |T_1|/2\rfloor = \lceil |T_1|/2\rceil$ nodes of T_1.

By Hall's theorem it is enough to show that for every set $Y \subseteq S \cup S'$, $|N(Y)| \geq |Y|$ holds where $N(Y)$ denote the neighbours of Y. It suffices to verify the inequality for Y's satisfying either $Y \subseteq S$ or $S' \subseteq Y$. Indeed, if $Y \cap S' \neq \emptyset$ then for $Y' = Y \cup S'$ we have $N(Y') = N(Y)$ and $|Y'| \geq |Y|$, thus giving a more strict constraint.

If $Y \subseteq S$, then the inequality holds by the assumptions of the theorem. If $S' \subseteq Y$, then $Y = S' \cup X$ for some $X \subseteq S$, and $|N(Y)| = |N(S' \cup X)| = |T_1| + |N_2(X)| = |S'| + \lceil |T_1|/2\rceil + |N_2(X)| \geq |S'| + \lceil |N_1(X)|/2\rceil + |N_2(X)| \geq |S'| + |X| = |Y|$, concluding the proof. □

Another tool that our proof relies on is a theorem that appeared in [Corollary 9][1] in a more general form (formulated using hypergraph terminology).

Theorem 11. *Let $G = (U, W; E)$ be a bipartite graph and k be a positive even integer. Assume that each node in W has degree $k - 1$ and $d_G(u) \leq k$ for every $u \in U$. Then there exists a family of pairwise node-disjoint stars $(w_1, U_1; F_1), \ldots, (w_q, U_q; F_q)$ such that $w_i \in W$, $|U_i|$ is either even or $k - 1$, and each node $u \in U$ of degree k is covered by one of the stars.*

Let $G = (U, W; E)$ be a bipartite graph. A path $P = \{u'w, wu''\}$ of length 2 with $u', u'' \in U$ is called a U-**link**. The **center node** of the U-link is w. Based on Theorem 11, we prove the following.

Theorem 12. *Let $G = (U, W; E)$ be a bipartite graph and k be a positive even integer. Assume that each node in U has degree at most k and each node in W has degree at most $k - 1$. Then G has a matching M and a family $\mathcal{P}$ of node-disjoint U-links with center nodes having degree $k-1$ such that every node $w \in W$ of degree $k-1$ is incident to an edge in $M \cup (\bigcup_{P\in\mathcal{P}} P)$.*

Proof. Observe that it suffices to verify the theorem for the special case when each node in W has degree exactly $k-1$ as we can simply delete nodes of degree less than $k-1$. Let $U' \subseteq U$ denote the set of nodes having degree k. Consider a family of stars provided by Theorem 11. The union of the edges of the stars is denoted by $F = \bigcup_{i=1}^{q} F_i$. Let W' be the set of nodes in W not covered by F. As $d_{E-F}(u) \leq k-1$ for each $u \in U$, W' can be covered by a matching M disjoint from F, by Theorem 9.

Now we trim each star either into a matching edge or into an U-link. If M covers at most one node from U_i, then keep only one edge $w_i u \in F_i$ where u is not covered by M (such an edge exists as $|U_i| \geq 2$). If M covers at least two nodes from U_i, then keep two edges $w_i u', w_i u'' \in F_i$ where both u' and u'' are covered by M. This way we get a matching and a family of U-links whose union together covers W. □

As a consequence, we can give a special partition of the edges of a bipartite graph.

Theorem 13. *Let $G = (U, W; E)$ be a bipartite graph and k be a positive even integer. Assume that $1 \leq d_G(u) \leq k$ for each node $u \in U$ and each node in W has degree at most $k-1$. Then E can be partitioned into three pairwise disjoint parts $E = E' \cup E^\sigma \cup E^L$ satisfying the following conditions:*

(i) each node in U has degree one in E^σ, that is, E^σ is the union of pairwise node-disjoint stars with center nodes in W together covering U,

(ii) E^L is the union of pairwise node-disjoint U-links with center nodes having degree $k-1$ in G,

(iii) $E^\sigma \cup E^L$ covers each node in W of degree $k-1$.

Proof. Take a matching M and a family $\mathcal{P}$ of node-disjoint U-links provided by Theorem 12. Add M to E^σ, and for each node $u \in U$ not covered by $M \cup (\bigcup_{P \in \mathcal{P}} P)$ add an arbitrary edge incident on u to E^σ. Let E^L consist of the edges of those U-links in $\mathcal{P}$ whose center nodes are not covered by E^σ. Finally, set $E' = E \setminus (E^\sigma \cup E^L)$. The partition $E = E' \cup E^\sigma \cup E^L$ thus obtained satisfies the conditions of the theorem. □

Recall the definition of an open or closed trail $v_0, e_1, v_1, \dots, e_t, v_t$. We will say that e_1 and e_t are the **terminal edges** of the trail, while v_0 and v_t are the **terminal nodes**. Besides Lemma 3, we will use the following.

Lemma 14. *If each node of a connected graph $G = (V, E)$ has even degree, then E is a closed trail.*

Proof. A closed trail containing every edge of the graph is basically an Eulerian trail. It is well known that a graph has an Eulerian trail if and only if it is connected and every node has even degree. □

The main advantage of Lemmas 3 and 14 is that the edge set of the graph can be partitioned into open and closed trails such that the closed trails form connected components of the graph, while at most one open trail starts at every node of V.

Corollary 15. *Given a bipartite graph $G = (S, T; E)$ with no isolated nodes, E can be partitioned into trails $T_1, \dots, T_\ell$ such that T_i forms a connected component of G if it is closed, and the endpoints of odd trails T_i and T_j are different if $i \neq j$.*

Proof of Theorem 1

Recall that the odd regular case was settled in [3], the case $k = 2$ is trivial, and the case $k = 4$ was solved in [1]. Hence we assume that k is even and is at least 6.

It suffices to prove the theorem for connected regular graphs. Let $G = (V, E)$ be a connected k-regular graph and let $v^* \in V$ be an arbitrary node. Denote the set of nodes at distance exactly i from v^* by V_i and let q denote the largest distance from v^*. We denote the edge-set of $G[V_i]$ by E_i. Apply Theorem 13 and Corollary 15 to the induced bipartite graph $G[V_{i-1}, V_i]$ with $W = V_{i-1}$ and $U = V_i$ to get a partition E'_i, E^σ_i and E^L_i together with a trail decomposition of E'_i for every $i = 1, \dots, q$. Note that the BFS tree we started with makes sure that there are no isolated nodes in U and the degree of a node $w \in W$ is at most $k - 1$ in $G[V_{i-1}, V_i]$.

We call a connected component C of E'_i **critical**, if C is $(k-2)$-regular and every node in $C \cap V_i$ is covered by E^L_i. Note that a critical component forms a closed trail.

Claim 16. *We can assign a V_i-link $\{u'v, vu''\}$ to each critical component C with $u' \in C \cap V_i$ in such a way that the following holds.*

1. *Different critical components get different V_i-links.*
2. *No open trail ends in the center nodes of two different V_i-links assigned to critical components.*
3. *If n_o denotes the number of odd open trails in E'_i, then at most $\lceil \frac{n_o}{2} \rceil$ of the odd open trails end in the set of center nodes of V_i-links assigned to critical components.*

Proof. We construct a bipartite graph as follows. One of the color classes, denoted by S, corresponds to the critical components of E'_i. The other color class, denoted by T, corresponds to the V_i-links of E^L_i modulo open trails, that is, if the center nodes of two V_i-links form the terminal nodes of the same open trail then they are represented by the same node in the bipartite graph. We add an edge between a node corresponding to a critical component C and a node representing a V_i-link $\{u'v, vu''\}$ if $u' \in C$.

Let $T = T_1 \cup T_2$ where T_1 corresponds to those V_i-links whose center nodes are terminal nodes of odd open trails. Let X be a subset of the nodes representing the critical components. We claim that the assumption of Theorem 10 is satisfied, that is, $\lceil |N_1(X)|2 \rceil + |N_2(X)| \geq |X|$ holds.

Recall that a critical component C corresponds to $(k-2)$-regular subgraphs in which every node in $C \cap V_i$ is covered by a V_i-link. As $k - 2 \geq 4$ and a V_i-link uses two edges, there are at least $2|X|$ many V_i-links incident to the critical components in X. Due to the construction of the bipartite graph, some of these V_i-links might be represented by the same node in T (if the center nodes of two V_i-links form the terminal nodes of the same open trail). Let m_1 denote the number of V_i-links whose center node is the terminal node of an odd open trail, and let m_2 be the number of the remaining ones. Then $\lceil |N_1(X)|/2 \rceil + |N_2(X)| \geq \lceil m_1/2 \rceil + m_2/2 \geq (m_1 + m_2)/2 \geq |X|$ as requested.

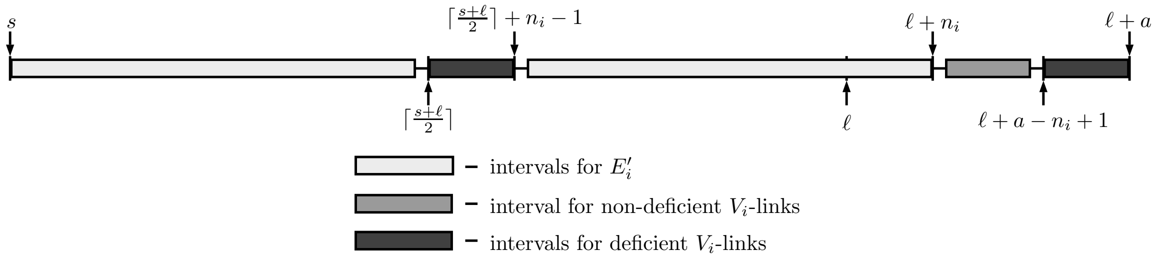

Figure 2: Assigning the intervals to E'_i and E^L_i.

By applying Theorem 10 to the bipartite graph constructed above, we get a matching which corresponds to an assignment satisfying the conditions of the theorem, concluding the proof. □

V_i-links assigned to critical components are called **deficient**, and we will refer to their center nodes also as **deficient nodes**. The node u' and edge $u'v$ appearing in Claim 16 are called the **core node** and the **core edge** of the critical component C, respectively.

The **starting node** of a closed trail is defined as follows. If the trail is a critical component, then the starting node is set to be the core node of the component. If the trail is not a critical component and has a node $v \in V_i$ with $d_{E^L_i}(v) = 0$, then set the starting node to be such a node. Otherwise, set the starting node to be an arbitrary node of the trail with degree at most $k-3$.

In what follows, we state the algorithm that provides a labeling of the graph. We reserve the $|E_q|$ smallest labels for labeling E_q, the next $|E'_q| + |E^L_q|$ smallest labels for labeling $E'_q \cup E^L_q$, the next $|E^\sigma_q|$ smallest labels for labeling E^σ_q, the next $|E_{q-1}|$ smallest labels for labeling E_{q-1}, etc. We assume that we are given a trail decomposition of E'_i into a set $\mathcal{T}$ of trails together with V_i-links assigned to critical trails as in Claim 16 for $i = 1, \dots, q$. We label the edge-sets in order

$$E_q \to E'_q \to E^L_q \to E^\sigma_q \to E_{q-1} \to \dots \to E^\sigma_2 \to E_1 \to E'_1 \to E^L_1 \to E^\sigma_1.$$

For $i > 0$, assume that $|E^L_i| = a$, the number of critical components in E'_i is n_i, and that the edges of $E'_i \cup E^L_i$ are labeled using the interval $[s, \ell + a]$ (that is, $|E'_i| = \ell - s + 1$). We will use the intervals $[\lceil \frac{s+\ell}{2} \rceil, \lceil \frac{s+\ell}{2} \rceil + n_i - 1] \cup [\ell + a - n_i + 1, \ell + a]$ for labeling the deficient V_i-links of E^L_i. The edges of the non-deficient V_i-links are labeled by using labels from $[\ell + n_i + 1, \ell + a - n_i]$ (note that $a \geq 2n_i$). The edges of the trails appearing in the decomposition of E'_i are labeled by using labels from $[s, \lceil \frac{s+\ell}{2} \rceil - 1] \cup [\lceil \frac{s+\ell}{2} \rceil + n_i, \ell + n_i]$ (see Figure 2).

Step 1. Labeling the edges in E_i.
We label the edges of E_i arbitrarily from its dedicated interval.

Step 2. Labeling trails.
We initialize $I_1 = \{s, s+1, \dots, \lceil \frac{s+\ell}{2} \rceil - 1\}$ and $I_2 = \{\lceil \frac{s+\ell}{2} \rceil + n_i, \lceil \frac{s+\ell}{2} \rceil + n_i + 1, \dots, \ell + n_i\}$. Notice that $|I_1| \leq |I_2| \leq |I_1| + 1$ holds for the initial setup. We will use the subroutine LabelOneTrail (see Algorithm 1) for labeling one trail.

Algorithm 1: LabelOneTrail$(v_0, e_1, v_1, \dots, e_t, v_t)$

Input : A trail $T = v_0, e_1, v_1, \dots, e_t, v_t$ with a designated starting node v_0.
Output: A labeling of T.

```
Assume that I_1 = {a_1, a_1 + 1, ..., b_1} and I_2 = {a_2, a_2 + 1, ..., b_2} are the
available intervals for labeling.
if v_0 ∈ V_{i−1} then
  | label e_1, e_2, ..., e_t with the labels a_1, b_2, a_1 + 1, b_2 − 1, ...;
else
  | label e_1, e_2, ..., e_t with the labels b_2, a_1, b_2 − 1, a_1 + 1, ...;
end
Remove the used labels from I_1 and I_2.
```

When labeling the trails, we want to make sure that *deficient nodes do not get a small label*. This means the following: if v is deficient then the trail T that ends in v will be labeled such that v is the final node, and not the starting one, thus the terminal edge of T at v will get a label from I_2. The labeling of the trails is done as follows.

Step 2a. While there is a not yet labeled closed trail $T = v_0, e_1, v_1, \dots, e_{2t}, v_{2t}$ with starting node v_0, label it by calling $\mathsf{LabelOneTrail}(v_0, e_1, v_1, \dots, e_{2t}, v_{2t})$. Notice that $|I_1| \le |I_2| \le |I_1| + 1$ is maintained after this call.

Step 2b. While there exists a not yet labeled open even trail, take one such trail $T = v_0, e_1, v_1, \dots, e_{2t}, v_{2t}$. By Claim 16, we can assume that v_0 is not deficient. Label T by calling $\mathsf{LabelOneTrail}(v_0, e_1, v_1, \dots, e_{2t}, v_{2t})$. Again notice that $|I_1| \le |I_2| \le |I_1| + 1$ is maintained after this call.

Step 2c. If all even trails are labeled then create pairs of the odd trails in an arbitrary manner with the only restriction that at most one terminal node of the members of the pair can be deficient. This can be done since $n_n \ge n_d - 1$ by Claim 16, where n_d denotes the number of odd open trails having a deficient terminal node, while n_n denotes the number of odd open trails having no deficient terminal node. If the number of odd trails is odd then one trail will have no pair, and if $n_d = n_n + 1$ then this trail can have a deficient terminal node. Label first the pairs as follows. Let $T = v_0, e_1, v_1, \dots, e_{2t+1}, v_{2t+1}$ and $T' = v'_0, e'_1, v'_1, \dots, e'_{2t'+1}, v'_{2t'+1}$ be an arbitrary pair with $v_0 \in V_i$ and $v'_0 \in V_{i-1}$ where we assume that v'_0 is not deficient (that is, v_{2t+1} might be deficient). Call first $\mathsf{LabelOneTrail}(v_0, e_1, v_1, \dots, e_{2t+1}, v_{2t+1})$ and next $\mathsf{LabelOneTrail}(v'_0, e'_1, v'_1, \dots, e'_{2t'+1}, v'_{2t'+1})$ for labeling this pair. Notice that $|I_1| \le |I_2| \le |I_1| + 1$ is maintained after these two calls. Finally, if there is a single trail $T = v_0, e_1, v_1, \dots, e_{2t+1}, v_{2t+1}$ that is not yet labeled then label it by calling $\mathsf{LabelOneTrail}(v_0, e_1, v_1, \dots, e_{2t+1}, v_{2t+1})$ where $v_0 \in V_i$ is assumed (and v_{2t+1} is either deficient or non-deficient).

Step 3. Labeling deficient V_i-links.
Recall that deficient links are labeled using the intervals $[\lceil \frac{s+\ell}{2} \rceil, \lceil \frac{s+\ell}{2} \rceil + n_i - 1] \cup [\ell + a - n_i + 1, \ell + a]$. In an arbitrary order, take the next deficient V_i-link $\{u'v, vu''\}$ and assume that the core edge is $u'v$. Label $u'v$ with the smallest available label, and vu''

with the largest available label. This scheme makes sure that the sum of the labels on the link is $\lceil \frac{s+\ell}{2} \rceil + \ell + a$.

Step 4. Labeling non-deficient V_i-links.
The edges of the non-deficient V_i-links are labeled by using labels from $[\ell+n_i+1, \ell+a-n_i]$ (note that $a \geq 2n_i$). In an arbitrary order, take the next non-deficient V_i-link $\{u'v, vu''\}$ and label $u'v$ with the smallest available label, and vu'' by the largest available label. This scheme makes sure that the sum of the labels on the link is $2\ell + a + 1$.

Step 5. Labeling the edges in E_i^σ.
For any node $v \in V_i$ ($i > 0$), let $\sigma(v)$ denote the unique edge of E_i^σ incident to v and let $p(v) = f(E(v)) - f(\sigma(v))$. Note that we have already labeled $E_q, E'_q, E_q^L, E_q^\sigma, \ldots, E_i, E'_i, E_i^L$, hence $p(v_i)$ is already determined for every $v_i \in V_i$. So we order the nodes of V_i in an increasing order according to their p-value and assign the label to their σ edge in this order. This ensures that $f(E(u)) \neq f(E(v))$ for an arbitrary pair $u, v \in V_i$.

We have fully described the labeling procedure. This labeling scheme ensures that $f(E(v_i)) < f(E(v_j))$ if $v_i \in V_i, v_j \in V_j$ and $i \geq j+2$ since G is regular and the edges in $E(v_j)$ get larger labels than those in $E(v_i)$. Similarly, $f(E(v^*)) > f(E(v))$ for every $v \in V - v^*$ for the same reason. It is only left to show that $f(E(v_i)) \neq f(E(v_{i-1}))$ for arbitrary $v_i \in V_i$, $v_{i-1} \in V_{i-1}$ and $i \geq 2$.

To prove this, first we collect the observations that are true for this labeling and will be used later. For the subsequent proofs we introduce the following notation. If $v \in V_{i-1} \cup V_i$ then let $p^L(v) = \sum_{e \in E_i^L \cap E(v)} f(e)$, $p'(v) = \sum_{e \in E'_i \cap E(v)} f(e)$ and $p(v) = \sum_{e \in E(v) - \sigma(v)} f(e)$.

Observation 17. Let $v \in V_{i-1}$.

(a) Successive labels on any trail incident to v have sum at least $s + \ell + n_i$.

(b) If $d_{E'_i}(v)$ is odd then $f(e) \geq s + n_i$ for the edge $e \in E(v) \cap E'_i$ that is the terminal edge of a trail. (This holds because we first labeled the closed trails, that includes all the critical trails.)

(c) If v is deficient (in which case $d_{E'_i}(v) = k-3$) then $f(e) \geq \frac{s+\ell}{2} + n_i$ for the edge $e \in E(v) \cap E'_i$ that is the terminal edge of a trail.

Observation 18. Let $v \in V_i$.

(a) Successive labels on any trail incident to v have sum at most $s + \ell + n_i$.

(b) If v is the starting node of a closed trail then the sum of the labels on the terminal edges of the trail is at most $s + \ell + n_i + \frac{\ell - s}{2}$.

(c) If v is a core node then $p^L(v) \leq \frac{s+\ell}{2} + n_i$.

Lemma 19. *For arbitrary $v \in V_{i-1}$ and $i \geq 2$ we have $p(v) \geq \frac{k-2}{2}(s + \ell + n_i) + \ell + a$.*

Proof. The idea of the proof is the following. Since $p(v) = \sum_{e\in E(v)-\sigma(v)} f(e)$ is the sum of $k-1$ edge-labels, we will pair the edges in this sum (except for one) such that the sum of the labels in each pair is $\geq s+\ell+n_i$, while the bound $f(e) \geq \ell + a$ will be applied for the remaining edge that does not have a pair. This idea will work in almost all of the cases below.

The edges in E_i' that are subsequent on a trail are naturally paired with each other by Observation 17a. Furthermore, if two edges both get a label $\geq \ell + a$ then they can be paired with each other.

Notice that $d_{E_i'}(v) \leq k-2$ holds for $v \in V_{i-1}$.

Case 1: There is no V_i-link at v. Notice that the edges in $E(v) - \sigma(v)$ either fall into E_i' or get a label $\geq l+a$. If $d_{E_i'}(v) = k-2$ then our rule for choosing the starting node of a closed trail will not choose v, that is, all edges of $E_i \cap E(v)$ are paired by the trail. So assume that $d_{E_i'}(v) < k-2$. In this case at least two edges get a label $\geq l+a$. If $d_{E_i'}(v)$ is odd then let e be the only edge at v that is not paired by a trail: we will pair it with an edge that has label $\geq \ell + a$ and apply the trivial lower bound $f(e) \geq s$. If $d_{E_i'}(v)$ is even then it is at most $k-4$, so even if v is the starting node of a closed trail, the two edges e, e' that are not paired by the trail (terminal edges) can be paired by edges having labels $\geq \ell + a$.

Case 2: There is a V_i-link at v. In this case $d_{E_i'}(v) = k-3$. If v is not deficient then $p^L(v) = 2l+a+1$ and $p'(v) \geq s+n_i+\frac{k-4}{2}(s+l+n_i)$, by Observation 17b. On the other hand, if v is deficient then $p^L(v) = \lceil\frac{s+\ell}{2}\rceil + \ell + a$ and $p'(v) \geq \frac{s+\ell}{2} + n_i + \frac{k-4}{2}(s+\ell+n_i)$ by Observation 17c, finishing the proof. □

Lemma 20. *For arbitrary $v \in V_i$ and $i \geq 1$, we have $p(v) \leq \frac{k-2}{2}(s+\ell+n_i) + \ell + a$.*

Proof. The idea of the proof is the the same as it was in Lemma 19 with the only exception that we aim for an upper bound. That is, we pair all but one of the $k-1$ edges that appear in the formula for $p(v)$ such that the sum of the labels in each pair is $\leq s+l+n_i$, while the trivial bound $f(e) \leq \ell + a$ will be applied for the remaining edge that does not have a pair.

The edges in E_i' that are subsequent on a trail are naturally paired with each other by Observation 18a. Furthermore, if two edges both get a label less than s then they can be paired with each other.

Case 1: There is no V_i-link at v. Notice that the edges in $E(v) - \sigma(v)$ either fall into E_i' or get a label $< s$. If $d_{E_i'}(v)$ is odd then there is nothing to do: we apply $f(e) \leq \ell + a$ for the edge $e \in E(v)$ that is the terminal edge of a trail, and the remaining edges are either paired by the trails or have label $< s$. If $d_{E_i'}(v)$ is even then it is at most $k-2$ and there is at least one edge $h \in E(v)$ having label $< s$. If v is not the starting node of a trail then all the edges at v are either paired by the trails or have label $< s$. If v happens to be the starting node of a closed trail then let e and e' be the first and the last edge of the trail and observe that $f(e) + f(h) \leq s + \ell + n_i$ while we can apply the trivial bound $f(e') \leq \ell + a$.

Case 2: There is a V_i-link at v. If v is a core node then apply Observation 18c to get $p^L(v) \leq \frac{s+\ell}{2} + n_i$ and Observation 18b to get $p'(v) \leq \frac{k-2}{2}(s+\ell+n_i) + \frac{l-s}{2}$ giving $p(v) \leq \frac{k-2}{2}(s+\ell+n_i) + \ell + n_i \leq \frac{k-2}{2}(s+\ell+n_i) + \ell + a$. If v is not a core node then the trivial bound $p^L(v) \leq \ell + a$ can be applied for the V_i-link, since v is either not a starting node in a trail (in which case all edges in $E'_i \cap E(v)$ are paired by the trails and $f(e) < s$ holds for every other edge $e \in E(v) - \sigma(v)$). On the other hand if v is the starting node of a trail then either $d_{E'_i}(v)$ is odd and the terminal edge of the trail can be paired with an edge with label $< s$, or $d_{E'_i}(v)$ is even, in which case there are at least 2 edges with label $< s$: pair those with the terminal edges of the trail. □

The fact that $f(\sigma(v_i)) < f(\sigma(v_{i-1})$ and Lemmas 19 and 20 together yield $f(E(v_i)) < f(E(v_{i-1}))$, finishing the proof of Theorem 1.

□

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