PPAD-completeness of polyhedral versions of Sperner's Lemma

Tamás Király and Júlia Pap

Abstract

We prove that some polyhedral versions of Sperner's Lemma, where the colouring is explicitly given in the input, are PPAD-complete.

Sperner's Lemma on the existence of a multicoloured triangle in a suitable colouring of a triangulation has many versions and generalizations. The following is a version that concerns colourings of an n-dimensional polytope, see [5]. Given a colouring of the vertices of a polytope by n colours, a facet is *multicoloured* if it contains vertices of each colour.

Theorem 1. *Let P be an n-dimensional polytope, with a simplex facet F_0. Suppose we have a colouring of the vertices of P with n colours such that F_0 is multicoloured. Then there is another multicoloured facet.*

This theorem leads naturally to a computational problem where the task is to find a multicoloured facet different from F_0.

POLYTOPAL SPERNER Input: vectors $v^i \in \mathbb{Q}^n$ $(i = 1, \dots, m)$ whose convex hull is a full-dimensional polytope P; a colouring of the vertices by n colours; a multicoloured simplex facet F_0 of P. Output: n affine independent vectors $v^{i_1}, \dots, v^{i_n}$ with different colours which lie on a facet of P different from F_0.

The complexity class PPAD was introduced by Papadimitriou [6], who proved among other results that a computational version of 3D Sperner's lemma is PPAD-complete. Later Chen and Deng [1] proved that the 2 dimensional problem is also PPAD-complete. The input of these computational versions is a polynomial algorithm that computes a legal colouring, while the number of vertices to be coloured is exponential in the input size. This is conceptually different from POLYTOPAL SPERNER, where the input explicitly contains the vertices and the colouring. In POLYTOPAL SPERNER the difficulty lies not in the large number of vertices but in that the structure is encoded as a polytope. We note that in fixed dimension POLYTOPAL SPERNER is solvable in polynomial time since then the number of facets is polynomial in the number of vertices.

In this note we prove that POLYTOPAL SPERNER is PPAD-complete. Our proof of PPAD-hardness is essentially the same as the proof by Kintali [3] of PPAD-hardness of the computational problem SCARF which is related to Scarf's Lemma. Before describing the details, let us show that the problem is in PPAD.

Theorem 2. POLYTOPAL SPERNER *is in PPAD.*

Proof. We reduce it to the problem END OF THE LINE, see e.g. [2]. We can compute in polynomial time a perturbation of the vertices in the input such that every facet becomes a simplex, and every facet (as a vertex set) is a subset of an original facet. Assume that $[n]$ is the set of colours. We define a digraph whose nodes are the facets that contain all colours in $[n-1]$ (formally, we may associate a node to each n-tuple of vertices, all other nodes being isolated). Each $(n-2)$-dimensinal face with all colours in $[n-1]$ is in exactly two facets, and we can say that one of them is on the left side of the face and the other is on the right side, with respect to a fixed orientation. For each such $(n-2)$-dimensinal face, the digraph contains an arc from the node corresponding to the facet on the left side to the node corresponding to the facet on the right side.

The obtained digraph has in-degree and out-degree at most 1 in every node, and the neighbours of a node can be computed in polynomial time. A node has degree 1 if and only if the corresponding facet is multicoloured. We may assume w.l.o.g. that the node corresponding to F_0 is a source, so the solution of END OF THE LINE for this digraph corresponds to finding a multicoloured facet different from F_0. □

In order to prove PPAD-completeness, we use a problem introduced in [5] that is polynomially reducible to POLYTOPAL SPERNER. The following theorem in [5] is obtained from Theorem 1 by taking a bounded polar of a polyhedron with n independent extreme rays, and adding a cut that cuts off the vertex corresponding to the original facet at infinity.

Theorem 3. *Let P be an n-dimensional pointed polyhedron whose characteristic cone is generated by n linearly independent vectors. If we colour the facets of the polyhedron by n colours such that facets containing the i-th extreme direction do not get colour i, then there is a multicoloured vertex.*

The corresponding computational problem, which we call EXTREME DIRECTION SPERNER, is polynomially reducible to POLYTOPAL SPERNER because a bounded polar and a cut that cuts off a certain simplicial vertex can be computed in polynomial time.

EXTREME DIRECTION SPERNER Input: matrix $A \in \mathbb{Q}^{m\times n}$ and vector $b \in \mathbb{Q}^m$ for which $P = \{x : Ax \le b\}$ is a pointed polyhedron whose characteristic cone is generated by n linearly independent vectors; a colouring of the facets by n colours such that facets containing the i-th extreme direction do not get colour i. Output: a multicoloured vertex of P.

Theorem 4. Extreme Direction Sperner *is PPAD-complete.*

Proof. The proof is analogous to the proof of PPAD-completeness of Scarf by Kintali [3], who proves that the problem 3-Strong Kernel defined below is PPAD-complete, and reduces it to Scarf. A digraph $D = (V, E)$ is *clique-acyclic* if for every directed cycle either the reverse of one of the arcs is also in E, or there are two nodes of the cycle that are not connected by an arc in E. A *strong fractional kernel* of D is a vector $x : V \to \mathbb{R}_+$ such that $x(K) \leq 1$ for every clique K, and for each node v there is at least one clique K in the out-neighbourhood of v such that $x(K) = 1$.

3-Strong Kernel
Input: A clique-acyclic digraph $D = (V, E)$ with maximum clique size at most 3
Output: A strong fractional kernel of D.

To reduce 3-Strong Kernel to Extreme Direction Sperner, let $n = |V|$, and let us consider the polyhedron

$$P = \{x \in \mathbb{R}^n : x(K) \leq 1 \text{ for every clique } K \text{ of } D\}.$$

Since every clique has size at most 3, the number of cliques is polynomial in n. The extreme directions of P are $-e_j$ ($j \in [n]$). As a set of colours, we use the nodes of V. Let the colour of the facet $x(K) = 1$ be a source node of K. This colouring satisfies the criterion in Theorem 3, so we have a valid input for Extreme Direction Sperner. Let x^* be a mulitcoloured vertex. For each node $v \in V$, there is a clique K such that the facet $x(K) = 1$ contains x^* and has colour v, hence v is a source of K, i.e. K is in the out-neighbourhood of v. This means that x^* is a strong fractional kernel. □

Corollary 5. Polytopal Sperner *is PPAD-complete.*

Proof. As we have seen, Extreme Direction Sperner is polynomially reducible to Polytopal Sperner. □

References

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